Quantum fields
on asymptotically de Sitter spacetimes
and their extension across the conformal horizon

joint work w. András Vasy

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Introduction

Let (M, g) Lorentzian spacetime. **Quantum fields**:

$$(\Box_g - m^2)\hat{\psi}(x) = 0, \quad [\hat{\psi}(x), \hat{\psi}(y)] = \mathrm{i}G(x, y),$$

where $G = P_+^{-1} - P_-^{-1}$ — difference of adv./ret. propagator.

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To construct and study $\hat{\psi}(x)$ one needs:

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$$\mathrm{Sol}(\Box_g - m^2) = G\mathcal{C}_\mathrm{c}^\infty(M).$$

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$$\mathrm{Sol}(\Box_g - m^2) = G\mathcal{C}_c^\infty(M).$$
- Splitting of solution space into particles and anti-particles:
  - **Minkowski vacuum** state: $\mathrm{e}^{\mathrm{i}\sqrt{-\Delta_x + m^2}}$ vs. $\mathrm{e}^{-\mathrm{i}\sqrt{-\Delta_x + m^2}}$ equivalently (if $m = 0$):
  $\Lambda^+(x,y) = \lim_{\epsilon\searrow 0} \frac{1}{4\pi^2} \frac{1}{(x-y)^2 + \mathrm{i}\epsilon(x_0 - y_0) + \epsilon^2}$ vs.
  $\Lambda^-(x,y) = \lim_{\epsilon\searrow 0} \frac{1}{4\pi^2} \frac{1}{(x-y)^2 - \mathrm{i}\epsilon(x_0 - y_0) + \epsilon^2}$.

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  - On globally hyperbolic (M,g):

$$\Lambda^+(x,y) - \Lambda^-(x,y) = iG(x,y), \quad \Lambda^\pm \geq 0$$

+ **Hadamard condition** [Kay, Wald]:

$$\Lambda^+(x,y) = \lim_{\epsilon\searrow 0} \frac{1}{8\pi^2} \frac{1}{\sigma(x-y) + i\epsilon(t(x)-t(y)) + \epsilon^2} + \log\text{term} + \mathcal{C}^\infty(M\times M)$$

- On **asymptotically de Sitter** spacetimes one expects canonical splitting
  - In 'cosmological patch', construction by *Goursat problem at cosmological horizon* state in [Dappiaggi, Moretti, Pinamonti]

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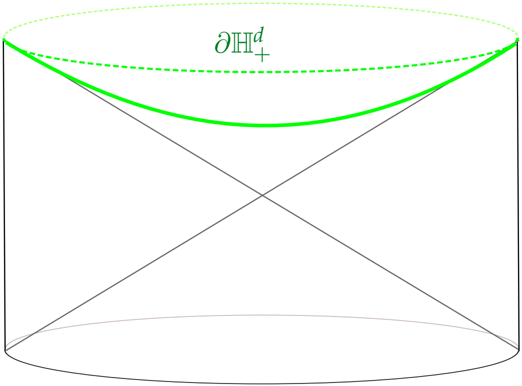
- On **asymptotically de Sitter** spacetimes one expects canonical splitting
  - To do so globally, we extend $\mathrm{Sol}(\Box_g - m^2)$ *across the conformal horizon*.

Hyperbolic space

In Minkowski space $\mathbb{R}^{1,d}$, $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$,

$$\mathbb{H}^d_+ = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = 1,\ z_0 > 0\}$$

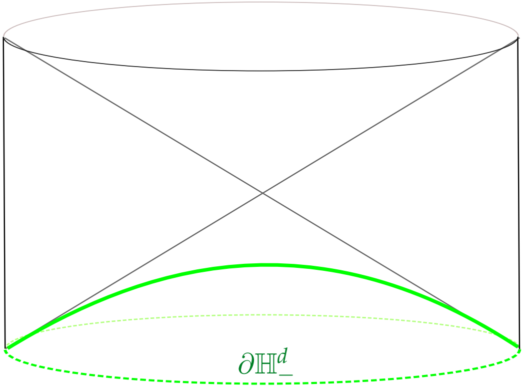


- Plane waves: $\phi_{\mathbb{H}^d_\pm,\xi} = |\xi \cdot z|^{i\nu-(d-1)/2}\restriction_{\mathbb{H}^d_\pm}$
- **Spectral projection**: $E(x,y) \propto \int \overline{\phi_{\mathbb{H}^d_\pm,\xi}(x)}\phi_{\mathbb{H}^d_\pm,\xi}(y)d\mu(\xi)$

Hyperbolic space

In Minkowski space $\mathbb{R}^{1,d}$, $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$,

$$\mathbb{H}^d_- = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = 1,\ z_0 < 0\}$$



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de Sitter space

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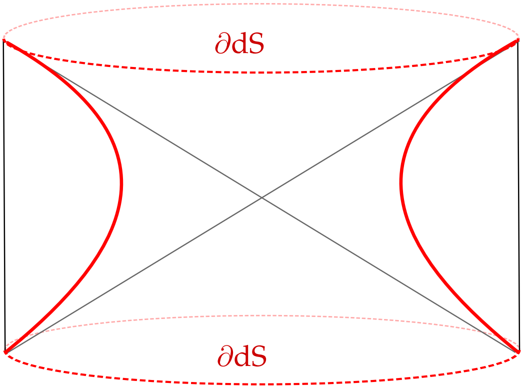

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  $\Lambda^{\pm}(x,y) \propto \int \overline{\phi^{\pm}_{\mathrm{dS},\xi}(x)}\phi^{\pm}_{\mathrm{dS},\xi}(y)d\mu(\xi)$

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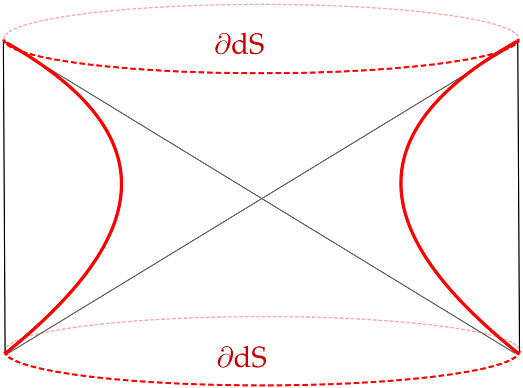
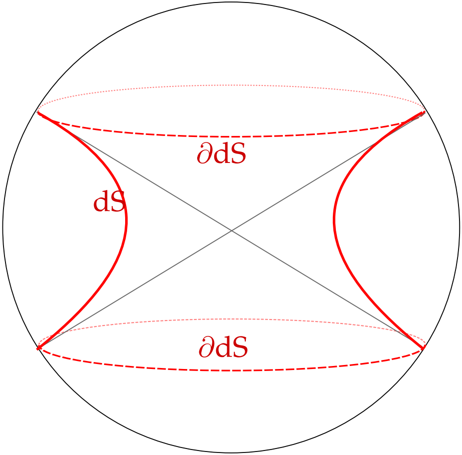


- Plane waves: $\phi^{\pm}_{\mathrm{dS},\xi} = (\xi \cdot z \pm \mathrm{i}0)^{\mathrm{i}\nu - (d-1)/2}\restriction_{\mathrm{dS}}$

? How to distinguish $\phi^{+}_{\mathrm{dS},\xi}$ vs. $\phi^{-}_{\mathrm{dS},\xi}$ *asymptotically*?

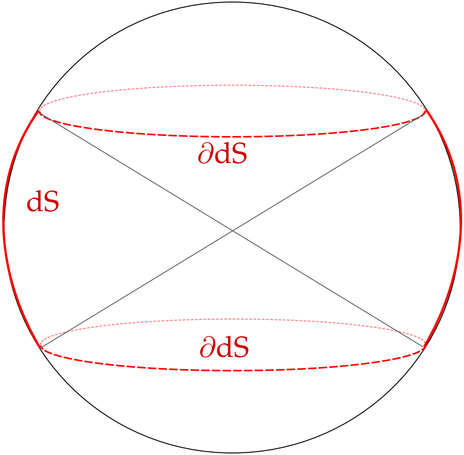
EXTENDED DE SITTER SPACE

Now in *radially compactified* Minkowski space:



Extended de Sitter space

- Identification $\mathrm{dS} \subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathrm{dS}}, f = \left|\frac{z_0^2 - (z_1^2 + \cdots + z_d^2)}{z_0^2 + z_1^2 + \cdots + z_d^2}\right|^{\frac{1}{2}}$.



$\{f = 0\} = \partial\mathrm{dS}$

Extended de Sitter space

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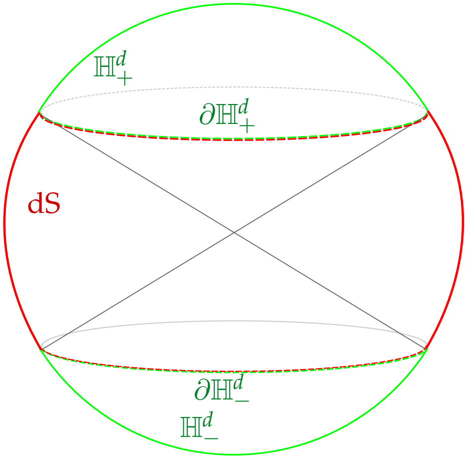
- Identification $\mathbb{H}^d_\pm \subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathbb{H}}$



$\mathbb{S}^d = \mathbb{H}^d_+ \cup \mathrm{dS} \cup \mathbb{H}^d_-$

$\{f = 0\} = \partial\mathrm{dS}$
$= \partial\mathbb{H}^d_+ \cup \partial\mathbb{H}^d_-$

Extended de Sitter space

- Identification dS $\subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathrm{dS}}, f = \left|\frac{z_0^2-(z_1^2+\cdots+z_d^2)}{z_0^2+z_1^2+\cdots+z_d^2}\right|^{\frac{1}{2}}$.
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- $\mathbb{S}^d = \mathbb{H}^d_+ \cup \mathrm{dS} \cup \mathbb{H}^d_-$; $\{f=0\} = \partial\mathrm{dS} = \partial\mathbb{H}^d_+ \cup \partial\mathbb{H}^d_-$
- **Plane waves**

$$\phi^\pm_\xi = (\xi\cdot z \pm \mathrm{i}0)^{\mathrm{i}\nu-(d-1)/2}\restriction_{\mathbb{S}^d} = \begin{cases} f^{\mathrm{i}\nu-(d-1)/2}\phi^\pm_{\mathrm{dS},\xi} & \text{on dS} \\ f^{\mathrm{i}\nu-(d-1)/2}\phi_{\mathbb{H},\xi} & \text{on } \mathbb{H}^d_+ \end{cases}$$

Setting $v := -f^2$ on dS, $v := f^2$ on $\mathbb{H}^d_\pm$, $\phi^\pm_\xi \sim (v\pm\mathrm{i}0)^{-\mathrm{i}\nu}$!

- These are solutions of $P\phi = 0$, $P = 4v\partial_v^2 + \ldots \in \mathrm{Diff}(\mathbb{S}^d)$

$$P = \begin{cases} f^{\mathrm{i}\nu-(d-1)/2-2}(\Box_{\mathrm{dS}} - (\frac{d-1}{2})^2 - \nu^2) f^{-\mathrm{i}\nu+(d-1)/2} \text{ on dS}, \\ f^{\mathrm{i}\nu-(d-1)/2-2}(-\Delta_{\mathbb{H}_\pm} + (\frac{d-1}{2})^2 + \nu^2) f^{-\mathrm{i}\nu+(d-1)/2} \text{ on } \mathbb{H}^d_\pm, \end{cases},$$

Extended asymptotically de Sitter spacetimes

- Same structure extending **even asymptotically de Sitter spacetimes** (M, g):
  - $g = df^2 - h(f^2, y, dy)$ in $v < 0$ (f^2 times as. dS metric) ,
    $g = df^2 + h_\pm(f^2, y, dy)$ in $v > 0$ (f^2 times as. $\mathbb{H}^d_\pm$ metric)
    (close to conformal horizon $\{v = 0\} = \{f = 0\} =: S_+ \cup S_-$).
  - Non-trapping assumption.
- The **Vasy operator**

$$P = \begin{cases} f^{i\nu-(d-1)/2-2}(\Box_{f^2 g} - (\frac{d-1}{2})^2 - \nu^2) f^{-i\nu+(d-1)/2} \text{ on } \{v < 0\}, \\ f^{i\nu-(d-1)/2-2}(-\Delta_{f^2 g} + (\frac{d-1}{2})^2 + \nu^2) f^{-i\nu+(d-1)/2} \text{ on } \{v > 0\}, \end{cases}$$

  - Solutions in $\mathrm{Sol}(P) := \{Pu = 0,\ \mathrm{WF}(u) \subset N^*\{v = 0\}\}$ can be written as:

$$u = (v + i0)^{-i\nu} a^+ + (v - i0)^{-i\nu} a^- + a, \quad a^+, a^-, a \in C^\infty(M).$$

  - P fits into *Fredholm framework* of Vasy

RADIAL ESTIMATES

A toy example:

- $M = \mathbb{R}^2$, coordinates $(x, y) \in \mathbb{R}^2$, dual coordinates (ξ, η)
- $Pu = xu$ — multiplication operator
- characteristic set $\mathcal{N} = \{x = 0\}$, Hamiltonian v. field $-\partial_\xi$
- radial set $N^*\{x = 0\} = \{x = 0, \eta = 0\}$ with components

$$\mathcal{R}^+ \cup \mathcal{R}^- := \{x = 0, \eta = 0, \xi > 0\} \cup \{x = 0, \eta = 0, \xi < 0\}$$

- near $\mathcal{R}^\pm$ change of coordinates on $T^*M \setminus o$, $\theta = \eta/\xi$, $\tilde{\rho} = \pm\xi^{-1}$ gives Hamiltonian v. field proportional to

$$\tilde{\rho}\partial_{\tilde{\rho}} + \theta\partial_\theta + x\partial_x.$$

  Bicharacteristics flow from source at $\mathcal{R}^-$ to sink at $\mathcal{R}^+$!
- Two inverses $(x \pm \mathrm{i}0)^{-1}$ correspond to high regularity at $\mathcal{R}^\mp$ and low regularity at $\mathcal{R}^\pm$.

INVERSES OF P

The **Vasy operator**

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- Solutions in $\mathrm{Sol}(P) := \{Pu = 0,\ \mathrm{WF}(u) \subset N^*\{v=0\}\}$ can be written near S_+ as:

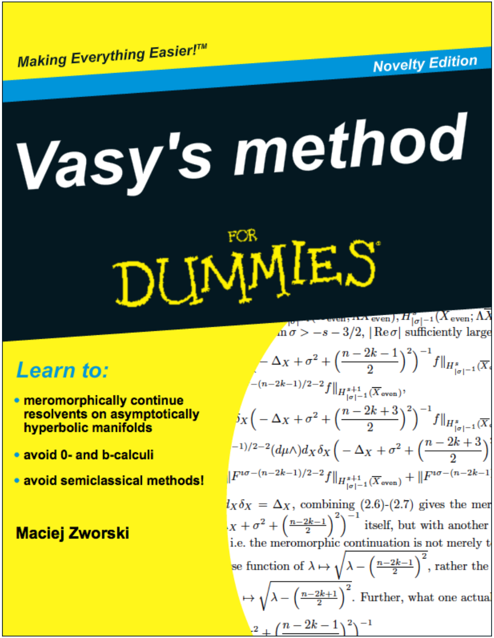
$$u = (v+\mathrm{i}0)^{-\mathrm{i}\nu} a^+ + (v-\mathrm{i}0)^{-\mathrm{i}\nu} a^- + a,\ \ a^+, a^-, a \in \mathcal{C}^\infty(M).$$

- P fits into *Fredholm framework* of Vasy [Vasy '12-'16]

  $\Rightarrow$ P has **inverses** $P_\pm^{-1}$ (as meromorphic functions in $\nu \in \mathbb{C}$)
- $P_\pm^{-1}$ *conformally related* related to ret./adv. propagators and meromorphic continuations of resolvent

ADVERTISEMENT

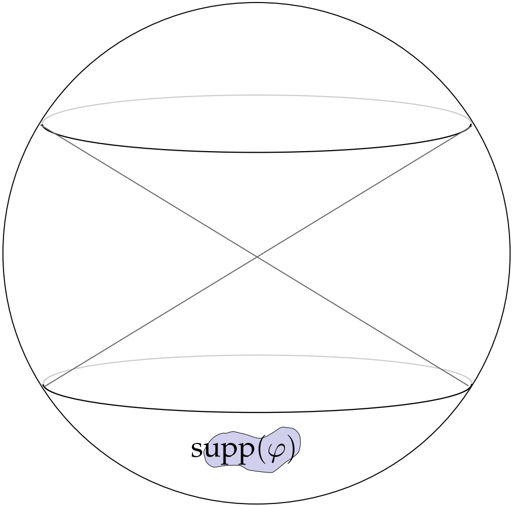
- Advantageous even purely from perspective of as. hyperbolic spaces, see [Vasy '13] and:



+ Conjectures on Vasy's operator by [Lebeau, Zworski '16]

INVERSES OF P

- Support properties of P_+^{-1} [Baskin, Vasy, Wunsch '12]



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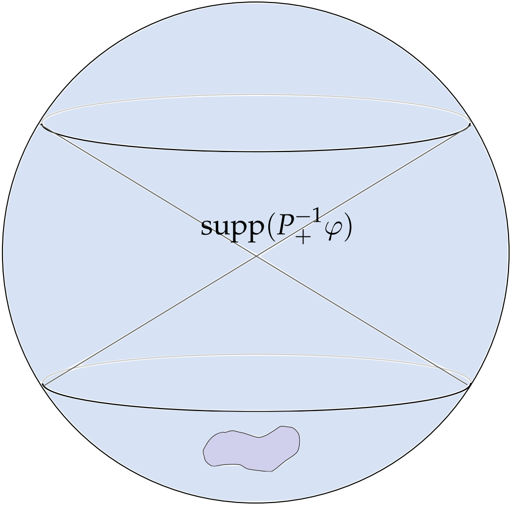

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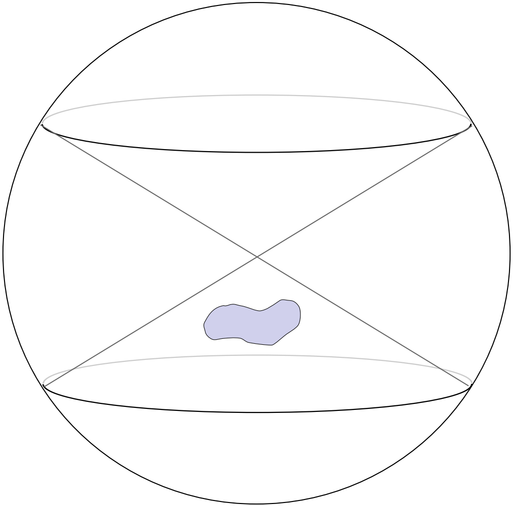

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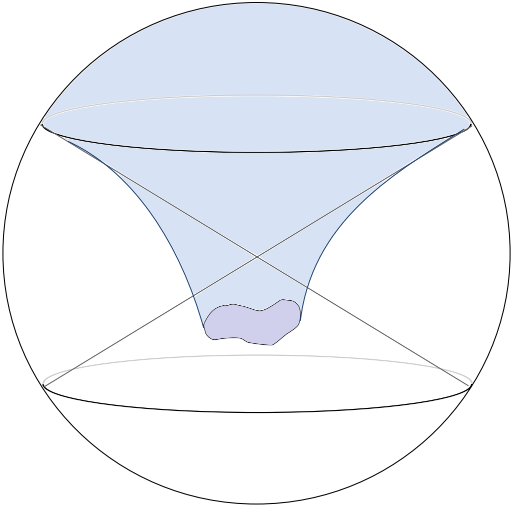

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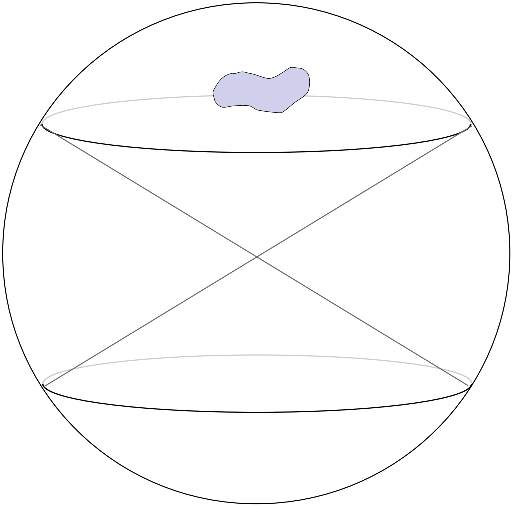

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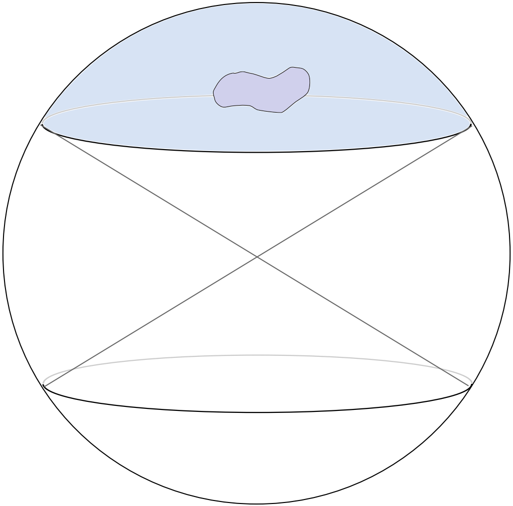

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- P fits into *Fredholm framework* of Vasy

  $\Rightarrow$ P has **inverses** $P_\pm^{-1}$ (as meromorphic functions in $\nu \in \mathbb{C}$)
- $P_\pm^{-1}$ *conformally related* related to ret./adv. propagators and meromorphic continuations of resolvent
- **Asymptotic data** of u: $\varrho_{S_\pm} u := (a^+, a^-)\restriction_{S_\pm}$

  Poisson/Møller operator $\varrho_{S_\pm}^{-1}$ constructed using $P_\pm^{-1}$

Main result I

Set $G := P_+^{-1} - P_-^{-1}$.

Theorem ([Vasy,W.])

*For ν not a pole of $P_\pm(\nu)^{-1}$, **isomorphisms**:*

$$\frac{\mathcal{C}^\infty(M)}{P\mathcal{C}^\infty(M)} \xrightarrow{G} \mathrm{Sol}(P) \xrightarrow{\restriction_{\{v<0\}} \circ f^{\mathrm{i}\nu+(d-1)/2}} \mathrm{Sol}(\Box - m^2).$$

- $\Rightarrow$ solutions of $(\Box - m^2)u = 0$ on asymptotically dS region have canonical *weighted extensions* to M.
- $\Rightarrow$ same conclusion for *quantum fields*.

Main result II

Set $G := P_+^{-1} - P_-^{-1}$. Recall $\varrho_{S_+} u$ — asymptotic data of u at S_+.

Theorem ([Vasy,W.])

Hadamard two-point functions *induced from data at conformal boundary S_+:*

$$\Lambda^+ = G^* \varrho_{S_+}^* \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \varrho_{S_+} G, \quad \Lambda^- = G^* \varrho_{S_+}^* \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \varrho_{S_+} G,$$

in particular $\Lambda^+ - \Lambda^- = \mathrm{i}G$

- gives Bunch-Davies two-point functions in exact dS case
- extends spectral projection from as. hyperbolic region
- Hadamard condition by *propagation of singularities* (inc. *radial sets* version)
- $\Lambda^+ - \Lambda^- = \mathrm{i}G$ from "pairing formula"

More isomorphisms

Let H_+ be one of the two as. hyperbolic regions.

Theorem ([Vasy,W.])

*For ν not a pole of $P_\pm(\nu)^{-1}$, **isomorphisms**:*

$$\left(\frac{\mathcal{C}^\infty(H_+)}{(-\Delta+m^2)\mathcal{C}^\infty(H_+)}\right)^{\oplus 2} \longrightarrow \left(\mathrm{Sol}(-\Delta+m^2)\right)^{\oplus 2} \longrightarrow \mathrm{Sol}(P).$$

$\Rightarrow$ (Linear) fields on as. dS $\longleftrightarrow$ *pair* of fields on as. $\mathbb{H}^d_+$.

Summary & outlook

The new:

- ✓ Canonical Hadamard two-point functions (and hence quantum fields) from asymptotic S_+ or S_- data
  - Similar results for wave equation on asymptotically Minkowski spacetimes.
- ✓ Extension of *non-interacting* QFT across the conformal horizon

Some questions:

- ? Consequences for Strominger's dS/CFT correspondence?
- ? Extension across the boundary for non-interacting (non-linear) theories?

Thank you for your attention!