The classical phase space in the BRST formalism

on curved spacetime

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INTRODUCTION

Consider (M, g) — globally hyperbolic spacetime,
and $P \in \mathrm{Diff}(M; V_1)$ — linearized EOM.

BRST framework replaces P with $L \in \mathrm{Diff}(M; V)$ hyperbolic and gives codifferential $\gamma \in \mathrm{Diff}(M; V)$.

Resulting perturbative interacting theory now understood on a general basis [Dütsch, Fredenhagen], [Hollands], [Fredenhagen, Rejzner].

Yet in the linear theory...

- *"The physical space is* $\mathrm{Ker}\gamma/\mathrm{Ran}\gamma$ *(at ghost number zero)"*
- The physical space is isomorphic to a subspace of $\mathrm{Ker}L|_{\Gamma_{\mathrm{sc}}}$?

Other natural question:

- Construction of Hadamard states in the BRST/BV framework
  - ✓ Maxwell/Yang-Mills linearized around 0, Feynman gauge, compact Cauchy surface Σ [Hollands]
  - ⚠ Difficult in general due to infrared problem.

CLASSICAL FIELD THEORY

'Hyperbolic' classical field theory, typically:

- (M, g) with Cauchy surface Σ, $V \to M$ 'hermitian' bundle
  (hermitian $(\cdot|\cdot)_V$ on fibers)
- $P \in \mathrm{Diff}(M; V)$ hyperbolic, $P^* = P$
  (causal propagator $G_P^* = -G_P$)
- A well-posed Cauchy problem

$$\begin{cases} Pu = 0, & u \in \Gamma(M, V) \\ \rho u = v, & v \in \Gamma_c(\Sigma, V_\Sigma). \end{cases}$$

- **Phase space**:

$$\frac{\Gamma_c(M, V)}{\mathrm{Ran}\, P|_{\Gamma_c}} \xrightarrow{[G_P]} \mathrm{Ker}\, P|_{\Gamma_{sc}} \xrightarrow{\rho} \Gamma_c(\Sigma, V_\Sigma)$$

Similar structure wanted for BRST! Should agree with expressions in 'subsidiary condition framework' (eg. [Hack & Schenkel])

BRST AS INCOMPREHENSIBLE BLACK BOX

Input:

- $P \in \mathrm{Diff}(M; V_1)$ coming from Lagrangian $\mathscr{L}$, $P^* = P$
- $K \in \mathrm{Diff}(M; V_0, V_1)$ s.t. $PK = 0$ (*gauge transf.: $f \to f + K\chi$*)

Output *(typically!)*:

- hermitian graded bundle $V = \bigoplus_i V_{[i]}$

($V_{[0]}$ is V_1 and *Lagrange multipliers*, $V_{[1]}$ *ghosts*, $V_{[-1]}$ *anti-ghosts*)

- $\gamma \in \mathrm{Diff}(M; V)$ s.t.

$$\text{a)} \quad \gamma^2 = 0, \ \ \gamma\Gamma(M; V_{[i]}) \subset \Gamma(M; V_{[i-1]})$$

$$\text{b)} \quad H_{-1,\mathrm{c}}(\gamma) := \left.\frac{\mathrm{Ker}\gamma|_{\Gamma_\mathrm{c}}}{\mathrm{Ran}\gamma|_{\Gamma_\mathrm{c}}}\right|_{[-1]} = \{0\};$$

- hyperbolic operator $L \in \mathrm{Diff}(M; V)$, $L = L^*$, s.t.

$$\text{a)} \quad \gamma^* L = L\gamma$$

$$\text{b)} \quad L\Gamma(M; V_{[i]}) \subset \Gamma(M; V_{[-i]}), \quad i \in \{-1, 0, 1\}.$$

CLASSICAL PHASE SPACE IN BRST

Theorem ([Zahn, W. '14])

Assuming **Output**, well-definiteness and isomorphisms of

$$\frac{\mathrm{Ker}\gamma^*|_{\Gamma_c}}{(\mathrm{Ran}\gamma^*|_{\Gamma_c} + \mathrm{Ran}L|_{\Gamma_c} \cap \mathrm{Ker}\gamma^*|_{\Gamma_c})}\Bigg|_{[0]} \xrightarrow{[G_L]} \frac{\mathrm{Ker}L|_{\Gamma_{sc}} \cap \mathrm{Ker}\gamma|_{\Gamma_{sc}}}{\mathrm{Ran}\gamma G_L|_{\Gamma_c}}\Bigg|_{[0]}$$

$$\downarrow \rho_L$$

$$\frac{\mathrm{Ker}\gamma_\Sigma|_{\Gamma_c}}{\mathrm{Ran}\gamma_\Sigma|_{\Gamma_c}}\Bigg|_{[0]}$$

where $\gamma_\Sigma = \rho_L \gamma U_L \in \mathrm{Diff}(\Sigma; V_\Sigma)$.

Moreover, non-degeneracy $\Leftrightarrow$ injectivity of arrows (at $|_{[0]}$):

$$H^0_c(\gamma^*_\Sigma) \xrightarrow{\imath} H^0(\gamma^*_\Sigma) \xrightarrow{\jmath} \big(H_{0,c}(\gamma_\Sigma)\big)^*.$$

Generalizes $H^1_c(d_\Sigma) \xrightarrow{\tilde{\imath}} H^1(d_\Sigma)$ [Dappiaggi, Hack, Sanders].

HADAMARD STATES

Definition: Two-point functions $\lambda_L^{\pm}$ of **Hadamard state** on CCR or CAR *-algebra:

$$
\begin{aligned}
&i) && \lambda_L^{\pm} : \Gamma_c(M;V) \to \Gamma(M;V) \\
&ii) && \lambda_L^{\pm} = \lambda_L^{\pm *} \text{ for } (\cdot|\cdot)_V, \\
&iii) && \lambda_L^{\pm} L = 0, \\
&iv) && \lambda_L^{\pm} \Gamma_c(M; V_{[i]}) \subset \Gamma_c(M; V_{[-i]}), \quad i \in I, \\
&v) && \lambda_L^{+} \mp (-1)^{\mathrm{gh}} \lambda_L^{-} = \mathrm{i}^{-1} G_L,
\end{aligned}
$$

$$
\begin{aligned}
&(\text{g.i.}) && \lambda_L^{\pm} : \ \mathrm{Ran}\gamma^* |_{\Gamma_c(M;V_{[0]})} \to \mathrm{Ran}\gamma |_{\Gamma_c'(M;V_{[0]})}, \\
&(\text{pos}) && \lambda_L^{\pm} \geq 0 \ \text{ on } \ \mathrm{Ker}\gamma^* |_{[0]}, \\
&(\mu\text{sc}) && \mathrm{WF}'(\lambda_L^{\pm}) = (\mathcal{N}^{\pm} \times \mathcal{N}^{\pm}) \cap \mathrm{WF}'(G_L),
\end{aligned}
$$

Cheap alternative: require *iii*) & *v*) mod. operators that map to $\mathrm{Ran}\gamma$

HADAMARD STATES IN PRACTICE

Reduction to technically simplier ('subsidiary condition') framework ($Pu = 0 \Leftrightarrow (P + TK^*)u = 0$ & $K^*u = 0$):

Maxwell/Yang-Mills case:

- $P = \delta^{A_0} d^{A_0} - *[*F_0, \cdot], \quad K = d^{A_0}.$

$$L = \begin{pmatrix} P & K & 0 & 0 \\ K^* & -\alpha\mathbf{1} & 0 & 0 \\ 0 & 0 & K^*K & 0 \\ 0 & 0 & 0 & K^*K \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix}$$

- Two-point functions for $\alpha = 1$ [Hollands]

$$\lambda_L^+ := \begin{pmatrix} \lambda^+_{\Box_1} & K\lambda^+_{\Box_0} & 0 & 0 \\ \lambda^+_{\Box_0} K^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^+_{\Box_0} \\ 0 & 0 & \lambda^+_{\Box_0} & 0 \end{pmatrix},$$

using $P + KK^* = \Box_1$ and $K^*K = \Box_0$.

HADAMARD STATES IN PRACTICE

Reduction to technically simplier ('subsidiary condition') frameworks:

Rarita-Schwinger case: (here Ricci-flat (M,g), $n \geq 3$)

- $(P\psi)_\mu = \nabla\!\!\!/\, \psi_\mu + \frac{1}{n-2}\gamma_\mu \nabla\!\!\!/\, \gamma^\nu \psi_\nu$ (instead of $(P\psi)^\mu := \gamma^{\mu\nu\lambda}\nabla_\nu\psi_\lambda$),
- $(K\phi)_\mu = \nabla_\mu \phi - \frac{1}{2}\gamma_\mu \nabla\!\!\!/\, \phi$, $(T\phi)_\mu := -\gamma_\mu\phi$

$$L = \begin{pmatrix} P & TK^*T & 0 & 0 \\ T^*KT^* & \frac{\alpha}{2}K^*T & 0 & 0 \\ 0 & 0 & 0 & K^*TK^*T \\ 0 & 0 & K^*TK^*T & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix}.$$

- Two-point functions for $\alpha = (n-2)^2$ [Zahn, W.]:

$$\lambda_L^+ := \begin{pmatrix} \lambda^+_{\nabla\!\!\!/} & \frac{4}{(n-2)^2}K\lambda^+_\square & 0 & 0 \\ \frac{4}{(n-2)^2}\lambda^+_\square K^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{(n-2)^2}\lambda^+_\square \\ 0 & 0 & \frac{4}{(n-2)^2}\lambda^+_\square & 0 \end{pmatrix}$$

HADAMARD STATES IN PRACTICE

Then one uses constructions from 'subsidiary condition' framework:

EOM linearized around $A_0 = 0$:

- *Maxwell/Y-M* — Σ compact, trivial cohomology [Fewster & Pfenning], [Hollands]
- *Maxwell/Y-M* — Σ well behaved at ∞ [Finster & Strohmaier]
- *Maxwell/Y-M* — (M, g) asymptotically flat [Dappiaggi, Siemssen]
- *Linearized gravity* — (M, g) asymptotically flat, Geroch-Xanthopoulos gauge [Benini, Dappiaggi, Murro]

EOM linearized around non-trivial solution A_0:

- *Maxwell/Y-M* — $\dim M \leq 4$, $\Sigma \cong \mathbb{R}^d$ or Σ compact, $\mathfrak{g}$ compact (Killing form $\kappa > 0$), background solution $A_0 \in \Lambda^1_{\mathrm{sc}}(M, \mathfrak{g})$ [Gérard, W.]

💡 better $1 + d$ reduction, ΨDOs of bounded geometry, scattering theory; global methods (based on Melrose's b-calculus and [Hintz, Vasy])

OUTLOOK

The new:

- ✓ Rigorous and working definition of **classical phase space** in **BRST framework**
- ✓ For *Maxwell*, *Yang-Mills* and *Rarita-Schwinger*, isomorphic to phase space in subsidiary condition framework
  - ✓ two-point functions can be 'transported' from one framework to the other

Questions not fully solved:

- ? Construction of Hadamard states for *Rarita-Schwinger*
- ? 'Generalized Poincaré duality' for *Yang-Mills*, *Rarita-Schwinger*, etc.
- ? BV formalism and Hadamard states for *linearized gravity*?

Thank you for your attention!