Characteristic Cauchy data
of positive-frequency solutions of the wave equation

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INTRODUCTION

- Let (M_0, g_0) — globally hyperbolic *(Lorentzian, time-oriented, with Cauchy surface)*. Conformal wave operator

$$P := -\Box_{g_0} + \frac{1}{6}R \in \mathrm{Diff}^2(M_0).$$

- $\mathrm{Char}(P) = \mathcal{N}_+ \cup \mathcal{N}_-$, so by [Duistermaat, Hörmander '72] distinguished parametrices $G_+, G_-, G_{\mathrm{F}}, G_{\overline{\mathrm{F}}}$
- In QFT, exact inverses needed + positivity property

Amounts to **Two-point functions** $\Lambda_\pm : C_0^\infty(M) \to C^\infty(M)$ s.t.

(1) $P\Lambda_\pm = \Lambda_\pm P = 0$

(2) $\Lambda_\pm \geq 0, \quad \Lambda_+ - \Lambda_- = \mathrm{i}(G_+ - G_-)$

(3′) $\mathrm{WF}'(\Lambda_\pm) \subset \mathcal{N}_\pm \times \mathcal{N}_\pm$

- then $G_{\mathrm{F}}(t,t') := \Lambda_+(t,t')\mathbb{1}_{t>t'} - \Lambda_-(t,t')\mathbb{1}_{t<t'}$
- QFT interpretation — $\Lambda_+ + \Lambda_-$ gives one-particle Hilbert space, $\Lambda_\pm$ selects particles/anti-particles

If M_0 **asymptotically flat** one expects 'distinguished' $\Lambda_{\pm}$.

- Define G_{F} as 'P^{-1}' in Fredholm setup [Gell-Redman, Haber, Vasy '14]
- Or conformally embed M_0 in manifold with null boundary C in an unphysical spacetime M.

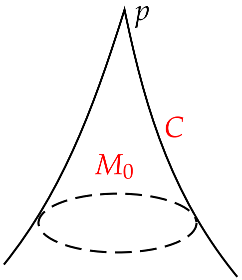
  Specify $\Lambda_{\pm}$ on C [Dappiagi,Moretti,Pinamonti '06-'11]. *In this talk*: solution of resulting Goursat problem [Gérard, W. '14].

$$\begin{cases} Pu = 0, \\ u\restriction_{C} = \varphi \end{cases} \qquad N^{*}C \subset \mathrm{Char}(P)$$

? Full QFT information on C?
(∼ density of C^{∞}_{sc} solutions in $\Lambda_{+} + \Lambda_{-}\restriction_{C}$)

Geometric setup I

- (M, g) globally hyperbolic spacetime
- Klein-Gordon operator $P = -\Box_g + V$, $V \in C^\infty(M, \mathbb{R})$
- Fix $p \in M$, set $M_0 := I^-(p)$ and $C := \partial J^-(p) \setminus \{p\}$ (backward lightcone from p)



GEOMETRIC SETUP II

Hypothesis

Assume there exists $f \in C^\infty(M)$ *s.t.:*

1) $C \subset f^{-1}(\{0\})$, $\nabla_a f \neq 0$ *on* C,
2) $\nabla_a \nabla_b f(p) = -2g_{ab}(p)$,
3) *the vector field* $\nabla^a f$ *is complete on* C.

(C is null hence $\nabla^a f$ is tangent to C).

- 1st consequence: (M_0, g) is a globally hyperbolic spacetime. If $K \subset M_0$ is compact, then [Moretti '06]:

$$J^-(K, M_0) = J^-(K, M), \quad J^+(K, M_0) = J^+(K, M) \cap M_0.$$

- We will see that C is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{d-1}$.

The main device are coordinates s.t.

$$g\restriction_C = -2dfds + h_{ij}(s,\theta)d\theta^i d\theta^j,$$

where $h_{ij(s,\theta)}d\theta^i d\theta^j$ Riemannian metric on $\mathbb{S}^{d-1}$.

- Introduce normal coordinates (t, y) at p, so $C = \{t^2 = y^2,\ t < 0\}, f = (t^2 - y^2)(1 + O(t, y))$, set $\psi = \frac{y}{|y|} \in \mathbb{S}^{d-1}$.
- $C \cap \{|y| = R\}$ for $0 < R \ll 1$ is *transversal* to $\nabla^a f$.
- global coordinates $s \in \mathbb{R}$, $\theta \in \mathbb{S}^{d-1}$ on C by solving:

$$\begin{cases} \nabla^a f \nabla_a s = -1, \\ s\restriction_{\{y_0+|y|=\varepsilon\}} = 0, \end{cases} \qquad \begin{cases} \nabla^a f \nabla_a \theta = 0, \\ \theta\restriction_{\{y_0+|y|=\varepsilon\}} = \psi \end{cases}$$

  so that $s \sim \ln(|y| - t),\ \theta \sim \psi$ near p.
- One extends s, θ to a neighborhood of C in M by keeping them constant along the null geodesics normal to $C \cap \{s = s_0\}$ and transversal to C.

CHARACTERISTIC CAUCHY PROBLEM

Proposition

For appropriate choice of $\beta \in C^\infty(M)$ ($\beta^{-1} \sim \mathrm{e}^s$)*, the map*

$$\rho : \phi \mapsto \beta^{-1}\phi\restriction_C(s,\theta)$$

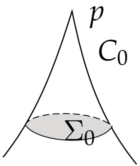
maps $\mathrm{Sol}_{\mathrm{sc}}(P)$ ($C^\infty_{\mathrm{sc}}(M_0)$*-solutions of* P) *to*

$\mathcal{H}(C) := \bigcap_{m\in\mathbb{N}} H^m(C)$, $H^m(C)$ *Sobolev space of order* m.

characteristic Cauchy problem:

$$\begin{cases} Pu = 0, \\ \rho u = \varphi, \quad \varphi \in H^1_0(C_0). \end{cases}$$

Known results: analytic setting or high regularity/compact support [Hörmander '90], [Bär, Wafo '14].



Σ_0 — Cauchy surface $\cap\, C_0$, so $\partial\Sigma_0 = \partial C_0$

Theorem

Let U_{Σ_0} solve the usual Cauchy problem. The map

$$T : H^1_0(\Sigma_0) \oplus L^2(\Sigma_0) \to H^1_0(C_0)$$
$$\varphi \mapsto (U_{\Sigma_0}\varphi)_{|C_0}$$

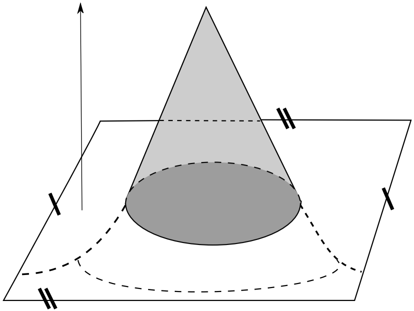
is a homeomorphism. Moreover, if $\dim M \geq 4$ *then*

$$T(C^\infty_0(\Sigma_0) \oplus C^\infty_0(\Sigma_0)) \subset |D_s|^{-\frac{1}{2}} L^2(C)$$

is dense.

- Consequence: $\rho\mathrm{Sol}_{\mathrm{sc}}(P)$ dense in $|D_s|^{-\frac{1}{2}} L^2(C)$.

PROOF OF EXISTENCE

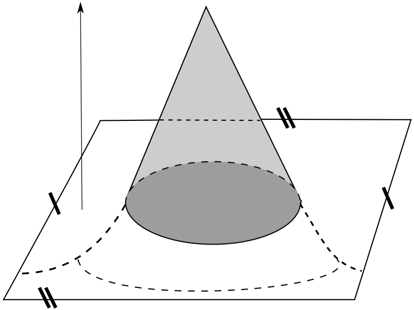


- reduction to a compact Cauchy surface Σ: embedding $\Sigma_0 \subset \tilde{\Sigma}$ s.t.

$$J^+(\tilde{\Sigma} \backslash \overline{\Sigma}_0; \mathbb{R} \times \tilde{\Sigma}) \cap \bar{C}_0 = \emptyset,$$

$$\tilde{C} \text{ weakly space-like in } \mathbb{R} \times \tilde{\Sigma}.$$

PROOF OF EXISTENCE



- $\tilde{C} = \{(F(\mathrm{x}), \mathrm{x}) : \ \mathrm{x} \in \tilde{\Sigma}\}$, F Lipschitz, is *weakly space-like* if

$$\sup_{\mathrm{x}} \sum_{ab} g^{ab}(F(\mathrm{x}), \mathrm{x}) \partial_a F(\mathrm{x}) \partial_b F(\mathrm{x}) \leq 1,$$

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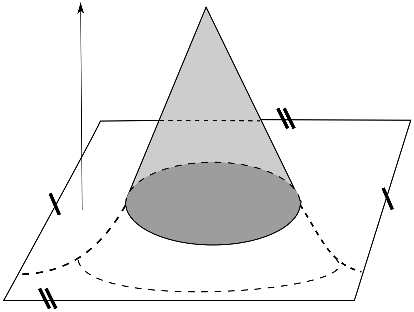


- then we use [Hörmander '90] + Huyghens principle.
  - C_0 is at best Lipschitz when Σ compact
  - deform $-\Box_\alpha := \partial_t^2 - \alpha\Delta$ and derive energy estimates uniform in α

CHARACTERISTIC DATA OF TWO-POINT FUNCTIONS

- Let $G := G_+ - G_-$. Then $\mathrm{Sol}_{\mathrm{sc}}(P) = GC_0^\infty(M_0)$. Moreover

$$\int_C \overline{(\rho G\psi_1)}\partial_s(\rho G\psi_2)|m|^{\frac{1}{2}}ds\,d\theta = \int_{M_0} \overline{\psi_1}(G\psi_2)dx, \;\; \psi_1,\psi_2 \in C_0^\infty(M_0)$$

- We will find suitable $\lambda_\pm : \mathcal{H}'(C) \to \mathcal{H}'(C)$ and set

$$\Lambda_\pm := (\rho\circ G)^* \circ \lambda_\pm \circ (\rho \circ G),$$

Theorem

It suffices that $\lambda_\pm : \mathcal{H}'(C) \to \mathcal{H}'(C)$ satisfies

$$\lambda_+ - \lambda_- = -2D_s, \quad (D_s := \mathrm{i}^{-1}\partial_s)$$

$$\lambda_\pm \geq 0,$$

$$\mathrm{WF}(\lambda_\pm)' \cap \{(Y_1,Y_2) : \pm\sigma_1 > 0 \text{ or } \pm\sigma_2 > 0\} = \emptyset,$$

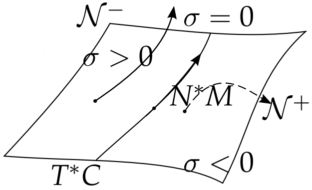
$$\mathrm{WF}(\lambda_\pm)'_C =_C \mathrm{WF}(\lambda_\pm)' = \emptyset.$$

WAVE FRONT SET ON BOUNDARY

Notation $(f, s, \theta) = (r, s, y)$, dual variables (ρ, σ, η), so

$$C = \{r = 0\},\ N^*M = \{r = \sigma = \eta = 0\},$$
$$p\restriction_C = -2\rho\sigma + h(s, y, \eta), \quad h \textit{ elliptic.}$$

- a point $Y_1 = ((s, y), (\sigma, \eta)) \in T^*C$ can be lifted to a unique $X_1 = ((0, s, y), (\rho, \sigma, \eta)) \in T^*M$ with $X_1 \in \mathcal{N}$, i.e. $-2\rho\sigma + h(s, y, \eta) = 0$.



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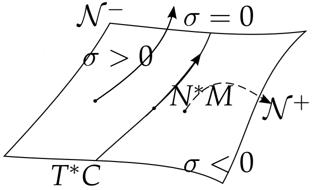

  - if $\sigma = 0$, then $h(0, y, \eta) = 0$ hence $\eta = 0$, (h elliptic), so $X_1 \in N^*M$, hence the bicharacteristic from X_1 does not leave C.
  - if $\pm\sigma > 0$ then $X_1 \in \mathcal{N}_{\mp}$.
  - the singularity of C at p plays no role: no null bicharacteristic curve from M_0 can reach p [Moretti '09]

$$\mathrm{WF}(\lambda_\pm)' \cap \{(Y_1, Y_2) : \ \pm\sigma_1 > 0 \text{ or } \pm\sigma_2 > 0\} = \emptyset,$$
$$\mathrm{WF}(\lambda_\pm)'_C =_C \mathrm{WF}(\lambda_\pm)' = \emptyset.$$

- $\lambda_\pm$'s as product type ΨDOs:
- $\Psi^{p_1,p_2}(C) \sim \Psi^{p_1}(\mathbb{R}) \otimes \Psi^{p_2}(\mathbb{S}^{d-1})$
- Useful to allow operators which are pseudodifferential only in θ, by replacing $\Psi^p(\mathbb{R})$ by $B^p(\mathbb{R}) := \{A : H^l(\mathbb{R}) \to H^{l-p}(\mathbb{R})\ \forall l \in \mathbb{Z}\}$,
- $$B^{p_1,p_2}(C) := B^{p_1}(\mathbb{R}) \otimes \Psi^{p_2}(\mathbb{S}^{d-1}),$$ $$\tilde{\Psi}^{p_1,p_2}(C) = \Psi^{p_1,p_2}(C) + B^{-\infty,p_2}(C).$$
- many choices of $\lambda_\pm \in \pm 2D_s \mathbb{1}_{\mathbb{R}^\pm}(D_s) + \tilde{\Psi}^{-\infty,p}(C)$.
- Key fact in applications: $(\cdot|(\lambda_+ + \lambda_-)\cdot)_{L^2(C)} \sim |D_s|^{-\frac{1}{2}} L^2(C)$.

SUMMARY

- ✓ Solution of characteristic Cauchy problem for data in appropriate Sobolev spaces
- ✓ Characteristic data of two-point functions in QFT
  - ✓ Density result solves a conjecture by [Moretti '06]
- ? Relation to [Gell-Redman, Haber, Vasy '14]?
- ? Scattering theory instead of conformal compactification?
- ? Extension to more involved geometric situations, like bifurcate Killing horizons?

Thank you for your attention !