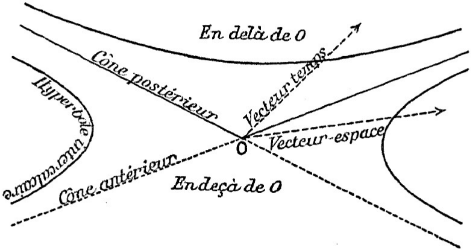
Quantum fields on curved spacetime:

the microlocal point of view

(part II)



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TWO-POINT FUNCTIONS

Let (M,g) — globally hyperbolic, $V \in C^\infty(M,\mathbb{R})$. Wave operator

$$P := -\Box_g + V \in \mathrm{Diff}^2(M).$$

Two-point functions $\Lambda_\pm : C_c^\infty(M) \to C^\infty(M)$

(1) $P\Lambda_\pm = \Lambda_\pm P = 0$

(2) $\Lambda_\pm \geq 0, \quad \Lambda_+ - \Lambda_- = \mathrm{i}(G_+ - G_-)$

(3′) $\mathrm{WF}'(\Lambda_\pm) \subset \mathcal{N}_\pm \times \mathcal{N}_\pm$

- Construct **distinguished** $\Lambda_\pm$ with specified symmetries/additional properties
- Parametrize **large class** of $\Lambda_\pm$'s
  - ✓ today method of parametrix for Cauchy problem [Gérard, W. '14]

$M = \mathbb{R} \times S$. In Gaussian normal coordinates wrt. S,
$g = -dt^2 + h_t$

$$
\begin{aligned}
P = \partial_t^2 + a(t) + r(t)\partial_t, \\
a(t) := -\Delta_{h_t} + V(t) \in \mathrm{Diff}^2(S) \\
r(t) := |h_t|^{-\frac{1}{2}} \partial_t(|h_t|^{\frac{1}{2}})(t) \in \mathrm{Diff}^0(S).
\end{aligned}
$$

Here $a^* = a$ wrt. $L^2(S, |h_t|^{\frac{1}{2}} dx)$. Rewrite $(\partial_t^2 + a + r\partial_t)\phi(t) = 0$ as

$$
\mathrm{i}^{-1}\partial_t \psi(t) = A(t)\psi(t), \quad \psi(t) = \begin{pmatrix} \phi(t) \\ \mathrm{i}^{-1}\partial_t \phi(t) \end{pmatrix},
$$

with $A(t) = \begin{pmatrix} 0 & \mathbf{1} \\ a(t) & \mathrm{i}r(t) \end{pmatrix}$ — 'scattering theorist's Laplacian'.

$$\mathrm{i}^{-1}\partial_t\psi(t) = A(t)\psi(t), \quad A(t) = \begin{pmatrix} 0 & \mathbf{1} \\ a(t) & \mathrm{i}r(t) \end{pmatrix},$$

Notation: For $b(t) \in C^\infty(\mathbb{R}, \Psi^\infty)$, $\mathcal{U}_b(t,s)$ solution of (if exists!)

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{U}_b(t,s) = \mathrm{i}b(t)\mathcal{U}_b(t,s), \\ \frac{\partial}{\partial s}\mathcal{U}_b(t,s) = -\mathrm{i}\mathcal{U}_b(t,s)b(s), \\ \mathcal{U}_b(t,t) = \mathbf{1}. \end{cases}$$

Recall $G = G_+ - G_-$. Consequence of Stoke's formula

$$\mathcal{U}_A(t,s) = \rho_t G^* \rho_s^* \sigma, \quad \rho_s\phi := \begin{pmatrix} \phi\restriction_{t=s} \\ \mathrm{i}^{-1}\partial_t\phi\restriction_{t=s} \end{pmatrix}, \quad \sigma := \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

Now setting

$$\mathcal{U}_A^\pm(t,s) := \pm\mathrm{i}\rho_t\Lambda_\pm\rho_s\sigma$$

(1)-(2) mean $\mathcal{U}_A^+(t,s) + \mathcal{U}_A^-(t,s) = \mathcal{U}_A(t,s)$, $\sigma\mathcal{U}^\pm(t,t) \geq 0$

💡 *Suppose* we have $b(t) \in C^\infty(\mathbb{R}, \Psi^1(S))$ s.t.

$$(\partial_t + \mathrm{i}b(t) + r(t)) \circ (\partial_t - \mathrm{i}b(t)) = \partial_t^2 + a(t) + r(t)\partial_t$$

- Hence also $(\partial_t - \mathrm{i}b^* + r) \circ (\partial_t + \mathrm{i}b^*) = \partial_t^2 + a + r\partial_t$

Set $\tilde{\psi}(t) := \begin{pmatrix} \partial_t + ib^*(t) \\ \partial_t - ib(t) \end{pmatrix} \phi(t)$. Now $(\partial_t^2 + a + r\partial_t)\phi(t) = 0 \Leftrightarrow$

$$\mathrm{i}^{-1}\partial_t\tilde{\psi}(t) = B(t)\tilde{\psi}(t), \quad B(t) = \begin{pmatrix} b^*(t) + \mathrm{i}r(t) & 0 \\ 0 & -b(t) + \mathrm{i}r(t) \end{pmatrix}.$$

Of course, $\mathcal{U}_B(t,s) = \begin{pmatrix} \mathcal{U}_{-b^*+\mathrm{i}r}(t,s) & 0 \\ 0 & \mathcal{U}_{b+\mathrm{i}r}(t,s) \end{pmatrix}$ and
$\mathcal{U}_A(t,s) = S(t)\mathcal{U}_B(t,s)S^{-1}(s)$ where $\psi(t) = S(t)\tilde{\psi}(t)$.

- Now $\mathcal{U}_A^+(t,s) := S(t) \begin{pmatrix} \mathcal{U}_{-b^*+\mathrm{i}r}(t,s) & 0 \\ 0 & 0 \end{pmatrix} S^{-1}(s)$.

Properties of

$$\mathcal{U}_A^+(t,s) = S(t)\begin{pmatrix} \mathcal{U}_{-b^*+\mathrm{i}r}(t,s) & 0 \\ 0 & 0 \end{pmatrix} S^{-1}(s),$$

$$\mathcal{U}_A^-(t,s) = S(t)\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{U}_{b+\mathrm{i}r}(t,s) \end{pmatrix} S^{-1}(s):$$

- well-defined if $b+b^*$ invertible
- $\mathcal{U}_A^+(s,t)+\mathcal{U}_A^-(s,t)=\mathcal{U}_A(s,t)$
- $\mathcal{U}_A^\pm(s,t')\mathcal{U}_A^\pm(t',s)=\mathcal{U}_A^\pm(s,t)$, $\mathcal{U}_A^\mp(s,t')\mathcal{U}_A^\pm(t',s)=0$
- $\mathrm{WF}'(\mathcal{U}_A^+(s,t)) = \{(x,\xi,x',\xi')\in T^*S\times T^*S : (x,\xi) = \Phi^+(t,s)(x',\xi')\}$ (Φ^+ — flow generated by $\sigma_1(b(t))$)
  - OK if $\sigma_1(b(t)) = (\sigma_2(-\Delta_{h_t}))^{1/2}$
- $\sigma\mathcal{U}_A^\pm(t,t)\geq 0$ iff $b+b^*\geq 0$

$$\text{indeed, } \mathcal{U}_A^+(t,t) = (S^{-1}(t))^*\begin{pmatrix} 0 & b(t)+b^*(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t)$$

RICCATI EQUATION

$$(\partial_t + \mathrm{i}b(t) + r(t)) \circ (\partial_t - \mathrm{i}b(t)) = \partial_t^2 + a(t) + r(t)\partial_t$$

…is actually a Riccati equation $\mathrm{i}\partial_t b - b^2 + a + \mathrm{i}rb = 0$.

- pick $a_0 \in C^\infty(\mathbb{R})$ s.t. $\tilde{a}(t) := a(t) + a_0(t)\mathbf{1} \geq \mathbf{1}$

Theorem

Assume $S = \mathbb{R}^d$ *or compact, then* $\exists\, b(t) \in C^\infty(\mathbb{R}, \Psi^1(S))$ *s.t.*

$$\begin{aligned} i)&\quad b = \tilde{a}^{\frac{1}{2}} + C^\infty(\mathbb{R}, \Psi^0(S)),\\ ii)&\quad (b + b^*)^{-1} = \tilde{a}^{-\frac{1}{4}}(\mathbf{1} + r_{-1})\tilde{a}^{-\frac{1}{4}},\ r_{-1} \in C^\infty(\mathbb{R}, \Psi^{-1}(S)),\\ iii)&\quad (b + b^*)^{-1} \geq C\,\tilde{a}^{-\frac{1}{2}},\ \textit{for some } C(t) > 0\\ iv)&\quad \mathrm{i}\partial_t b - b^2 + a + \mathrm{i}rb \in C^\infty(\mathbb{R}, \Psi^{-\infty}(S)). \end{aligned}$$

Ansatz $b = \tilde{a}^{\frac{1}{2}} + b_0$ gives *transport equations*

$$\begin{aligned} b_0 &= \tfrac{\mathrm{i}}{2}(\tilde{a}^{-\frac{1}{2}}\partial_t\tilde{a}^{\frac{1}{2}} + \tilde{a}^{-\frac{1}{2}}r\tilde{a}^{\frac{1}{2}}) + F(b_0),\\ F(b_0) &= \tfrac{1}{2}\tilde{a}^{-\frac{1}{2}}(\partial_t b_0 + [\tilde{a}^{\frac{1}{2}}, b_0] + \mathrm{i}rb_0 - b_0^2). \end{aligned}$$

- Invertibility:
  - Write $b(t) = \tilde{a}^{-\frac{1}{4}}(\mathbf{1} + r_{-1}(t))\tilde{a}(t)^{-\frac{1}{4}}$, $r_{-1} \in \Psi^{-1}$
  - Replace r_{-1} by
$$r_{-1,R}(t) = \chi\left(\frac{\tilde{a}(t)}{R\lambda(t)}\right) r_{-1}(t)\chi\left(\frac{\tilde{a}(t)}{R\lambda(t)}\right)$$

Now **to get actual solutions** replace $\mathcal{U}_A^{\pm}(t,s)$ by $\mathcal{U}_A(t,0)\mathcal{U}_A^{\pm}(0,0)\mathcal{U}_A(0,s)$, where $\mathcal{U}_A(s,t)$ — 'true propagator'!

- WF unchanged by 'interaction picture argument'
  - $B(t)$ replaced by $\tilde{B}(t)$, $B(t) - \tilde{B}(t) \in C^\infty(\mathbb{R}, \Psi^{-\infty})$

We get:

- Two-point functions $\Lambda_+(s,t) = \pi_0\,\mathcal{U}_A^{+}(t,s)\,\pi_1^*$
- Feynman propagator
$G_{\mathrm{F}}(s,t) = \mathrm{i}^{-1}\pi_0\left(\mathcal{U}_A^{+}(t,s)\theta(t-s) - \mathcal{U}_A^{-}(t,s)\theta(s-t)\right)\pi_1^*$

REGULAR TWO-POINT FUNCTIONS

So far: $C^\infty(\mathbb{R}, \Psi^1(s)) \ni b(t) \mapsto$ two-point functions $\Lambda^b_\pm$.

Theorem
If $\Lambda_\pm$ two-point functions s.t. $\Lambda_\pm(t,t) \in \Psi^\infty(S)^{\oplus 2}$ then $\exists\, b(t)$ s.t. $\Lambda_\pm = \Lambda^b_\pm$.

- Bogoliubov transformations

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \mapsto U^* \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} U, \quad U = \underbrace{\begin{pmatrix} (\mathbf{1}+aa^*)^{\frac{1}{2}} & a \\ a^* & (\mathbf{1}+a^*a)^{\frac{1}{2}} \end{pmatrix}}_{\in \mathbf{1}+\Psi^{-\infty}}$$

- $a \in \Psi^{-\infty} \Rightarrow ((\mathbf{1}+aa^*)^{\frac{1}{2}} - a)^{\pm 1} \in \mathbf{1} + \Psi^{-\infty}$

On the other hand: $b(t)$ determines $-\Box_g + V$ by Riccati eqn.
Consequence: $b(t)$ determines **both quantum fields and the background geometry** !!

ASYMPTOTIC DATA

Piece of cake if $(M,g) = \mathbb{R}^{1,d}$, $\operatorname{supp} V \subset [-T/2, T/2] \times \mathbb{R}^d$:

$$\mathcal{U}_V^+(0,0) := \mathcal{U}_V(0,T)\mathcal{U}_0(T,0)\mathcal{U}_0^+(0,0)\mathcal{U}_0(0,T)\mathcal{U}_V(T,0).$$

More general V:

- If $\lim_{t\to\infty} V(t) = 0$ and Møller operator $S^+ := \lim_{T\to+\infty} \mathcal{U}_V(0,T)\mathcal{U}_0(T,0)$ exists

$$\mathcal{U}_V^+(0,0) := S^+ \circ \mathcal{U}_0^+(0,0) \circ (S^+)^\dagger$$

  ? is WF′ still OK? [in progress w. Gérard]

- Fix $\tilde{V}$ s.t. $\operatorname{supp} V \subset [-T/2, T/2] \times \mathbb{R}^d$, $\tilde{V} = V$ on $[-T/4, T/4]$

$$\mathcal{U}_V^+(0,0) := \mathcal{U}_{\tilde{V}}^+(0,0)$$

  This is the deformation argument [Fulling, Narcowich, Wald '79]

More challenging: black hole horizons…