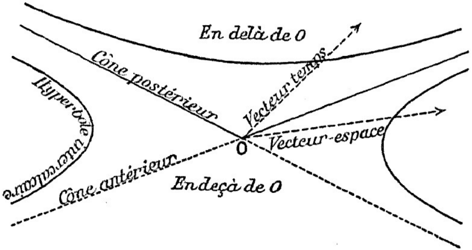
Quantum fields on curved spacetime:

the microlocal point of view

(part I)



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Let (M, g) Lorentzian. Three different levels for QFT:

- **Non-interacting quantum fields** (w. fixed $V \in C^\infty(M, \mathbb{R})$)

$$(-\Box_g + V)\psi(x) = 0, \qquad \psi \in \mathscr{D}'(M, \mathscr{C}(\mathfrak{H}))$$

  - ⚠ Hilbert space $\mathfrak{H}$ as unknown as ψ
  - ⚠ ψ subject to list of 'axioms'.
    - If $(M, g) = \mathbb{R}^{1,d}$, ψ determined by spectrum condition
    - If (M, g) globally hyperbolic, ψ subject to microlocal spectrum condition [Radzikowski '96]
    - ? What if (M, g) not globally hyperbolic / w. boundary?

- **Interacting quantum fields**

$$(-\Box_g + V)\psi(x) = \lambda :\psi^k(x):,$$

  - ⚠ $\psi^k(x)$ does not exist, renormalization required $(:\psi^k(x):)$
  - ⚠ $\psi(x)$ exists merely as ambiguous formal power series in λ

- **Back-reaction**: (M, g) (and/or V) and ψ both dynamical

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi (\Omega | :T_{\mu\nu}: \Omega)$$

  - ⚠ some progress in consistent formulation but no initial value problem so far [Moretti '03, Hollands & Wald '04]

PLAN

I. Introduction
- Globally hyperbolic spacetimes
- Quantum fields and states
- Radzikowski's theorem

II. Parametrix for $(-\Box_g + V)$ by ΨDOs

III. Symmetries and thermal effects
- Pure vs. mixed states
- Ground and thermal states
- Spacetimes with bifurcate Killing horizons
- Gauge theories

IV. Interacting theory

Globally hyperbolic spacetimes

A **Lorentzian spacetime** is

- (M, g) — connected Lorentzian C^∞-manifold
  of signature $(-, +, \ldots, +)$, dimension $n = 1 + d \geq 2$
- equipped with continuous time-like vector field
  - i.e., $v \in \Gamma(TM)$ space-like/time-like (null) if $\pm g(v, v) > 0$ $(g(v, v)=0)$
  - gives the notion of future/past-directed time-like (or null) $v \in \Gamma(TM)$

(M, g) — **globally hyperbolic** if (pick one!):

- no closed time-like/null curve, $J_+(x) \cap J_-(y)$ compact $\forall\, x, y \in M$
  - where $J_+(x) := \{y \in M :\ \exists\gamma$ future-directed s.t. $\gamma(0) = x, \gamma(1) = y\}$
- $\exists$ Cauchy surface S
  - i.e., each inextendible time-like curve intersects S *once*
- $M \overset{\text{iso}}{=} \mathbb{R}_t \times S$, $g = -\vartheta dt^2 + h_t$ with $\vartheta > 0$, (S, h_t) Riemannian, and $\{t\} \times S$ — Cauchy surface [Bernal, Sánchez '03]

Examples:

- Minkowski space $\mathbb{R}^{1,d}$ ($g = -dt^2 + dx^2$, Cauchy surface $\{t_0\} \times \mathbb{R}^d$), de Sitter
- Interior/exterior Schwarzschild

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\mathbb{S}^2$$

  - ✓ In $0 < r < 2m$, Cauchy surface $\{r_0\} \times \mathbb{R} \times \mathbb{S}^2$
  - ✓ In $r > 2m$, Cauchy surface $(2m, \infty) \times \{t_0\} \times \mathbb{S}^2$

Non-examples:

- Anti-de Sitter: $M = \mathbb{R} \times \mathbb{S}^{n-1}_+$

$$g = \frac{1}{x_n^2}(-dt^2 + d\mathbb{S}^{n-1}_+)$$

- Kerr (we'll see that laterr)

WAVE EQUATION

For $K \subset M$, $J_\pm(K) := \bigcup_{x \in K} J_\pm(x)$. Consider

$$P := (-\Box_g + V) \in \mathrm{Diff}^2(M)$$

with $V \in C^\infty(M, \mathbb{R})$. In particular $P = P^*$.

Theorem

Assume (M, g) globally hyperbolic. Then $\exists!\ G_\pm : C^\infty_{\mathrm{c}}(M) \to C^\infty(M)$ (the ***retarded/advanced*** *propagator) s.t.*

(1) $PG_\pm = G_\pm P = \mathbf{1}$ *on* $C^\infty_{\mathrm{c}}(M)$

(2) $\mathrm{supp}(G_\pm f) \subset J_\pm(\mathrm{supp} f)$, $f \in C^\infty_{\mathrm{c}}(M)$

Moreover, $(G_+)^* = G_-$

Indeed $\langle f, G_+ g\rangle = \langle PG_- f, G_+ g\rangle = \langle G_- f, PG_+ g\rangle = \langle G_- f, g\rangle$.

SYMPLECTIC SPACE OF SOLUTIONS

- $C^\infty_{\pm\pm} := \{f \in C^\infty(M) : \exists K \subset\subset M \text{ s.t. } \operatorname{supp} f \subset J^\pm(K)\}$
- $G_\pm P = \mathbf{1}$ on $\{f \in C^\infty_{\pm\pm}(M) : Pf \in C^\infty_c(M)\}$, because $\langle h, G_+ Pf\rangle = \langle G_- h, Pf\rangle = \langle PG_- h, f\rangle = \langle h, f\rangle, h \in C^\infty_c(M)$.

$$\mathrm{Sol}(P) := \{u \in C^\infty_{++} + C^\infty_{--} : Pu = 0\}, \quad G := G_+ - G_-$$

Fix $\eta_\pm$ smooth approx. of $\mathbb{1}_{\pm t>0}$. For $u \in \mathrm{Sol}(P)$,

$$u = G_+ P\eta_+ u + G_- P\eta_- u = G_+ P\eta_+ u - G_- P\eta_+ u = G \underbrace{P\eta_+ u}_{\in C^\infty_c(M)}$$

Proposition

Isomorphism of ***symplectic spaces****:*

$$\frac{C^\infty_c(M)}{PC^\infty_c(M)} \xrightarrow{[G]} \mathrm{Sol}(P)$$

In QFT: relevant properties are: $G^* = -G$ and $G(x, y) = 0$ for space-like related $x, y \in M$.

NON-INTERACTING QUANTUM FIELDS

Given (M, g), we want $\mathfrak{H}$ and field operators $\hat{\psi}$ s.t. 'axioms' hold

- $C_c^\infty(M) \ni f \to \hat{\psi}(f) \in \mathscr{C}(\mathfrak{H})$ anti-linear

  (so $C_c^\infty \ni f \to \hat{\psi}^*(f)$ linear)
- $(-\Box_g + V)\hat{\psi}(x) = 0$
- $[\hat{\psi}(x), \hat{\psi}(y)] = 0, \quad [\hat{\psi}(x), \hat{\psi}^*(y)] = \mathrm{i}^{-1} G(x,y)\mathbf{1}$
- $\Lambda^+(x,y) := (\Omega|\hat{\psi}^*(x)\hat{\psi}(y)\Omega), \ \ \Lambda^-(x,y) := (\Omega|\hat{\psi}(y)\hat{\psi}^*(x)\Omega)$ satisfy **microlocal spectrum condition**

$$\mathrm{WF}'(\Lambda^\pm) = \{(X, X') \in \mathcal{N}_\pm \times \mathcal{N}_\pm : \ X \sim X'\}$$

where $\mathcal{N}_\pm$ — connected components of $p^{-1}(\{0\}) \subset \dot{T}^*M$

- **algebraic quantization** → construct everything from $\Lambda^\pm$
- two ways of proving existence
- open questions concern existence of $\hat{\psi}$ with *concrete* properties on *concrete* space-times, and also generalizations *beyond globally hyperbolic/smooth* setup

A SHORTCUT THROUGH ALGEBRAIC QUANTIZATION

Theorem ('Araki-Woods representation')

Suppose $\Lambda^{\pm} : C_c^{\infty}(M) \to C^{\infty}(M)$ *s.t.*

- $P\Lambda^{\pm} = \Lambda^{\pm}P = 0$
- $\Lambda^{\pm} \geq 0, \quad \Lambda_{+} - \Lambda_{-} = \mathrm{i}(G_{+} - G_{-}),$

Then there exists Hilbert space $\mathfrak{H}$, $\Omega \in \mathfrak{H}$ *and* $\hat{\psi} \in \mathscr{D}'(M, \mathscr{C}(\mathfrak{H}))$ *verifying 'axioms'*, $\left(\mathrm{Span} \bigcup_k \hat{\psi}(f_1) \dots \hat{\psi}(f_k)\Omega\right)^{\mathrm{cl}} = \mathfrak{H}$, *and*

$$\Lambda_{+}(x,y) = (\Omega | \hat{\psi}^{*}(x)\hat{\psi}(y)\Omega), \quad \Lambda_{-}(x,y) = (\Omega | \hat{\psi}(y)\hat{\psi}^{*}(x)\Omega)$$

- $\mathfrak{H} := \oplus_{k=0}^{\infty} \bigotimes_{\mathrm{s}}^{k} \mathfrak{h}$[1], $\mathfrak{h} :=$ completion of C_c^{∞} w.r.t. $\Lambda_{+} + \Lambda_{-}$
- $\hat{\psi}(f)$ made of $\hat{a}^{*}(\pi^{\pm}f)\Psi := (\pi^{\pm}f) \otimes_{\mathrm{s}} \Psi$, $\pi^{\pm} := \pm \mathrm{i}G^{-1}\Lambda_{\pm}$
  $\Omega := 1 \oplus (0 \otimes 0) \oplus \dots,$ number operator $\hat{N} := \oplus_{k=0}^{\infty} \bigotimes_{\mathrm{s}}^{k} \mathbf{1}$
- ✓ uniqueness *up to unitary equivalence* upon fixing m-point functions $(\Omega | \hat{\psi}^{(*)}(x_1) \dots \hat{\psi}^{(*)}(x_m)\Omega)$

[1] Actually a doubling of degrees of freedom is additionally needed if $\Lambda_{\pm}$ do not have the *purity property*, cf. part III.

TWO-POINT FUNCTIONS VS. PARAMETRICES

Let us focus on:

(1) $P\Lambda_{\pm} = \Lambda_{\pm}P = 0$

(2) $\Lambda_{\pm} \geq 0$, $\Lambda_{+} - \Lambda_{-} = \mathrm{i}(G_{+} - G_{-})$,

(3) $\mathrm{WF}'(\Lambda_{\pm}) = \{(X, X') \in \mathcal{N}_{\pm} \times \mathcal{N}_{\pm} : \; X \sim X'\}$

formulated in [Radzikowski '96]

Theorem ([Duistermaat, Hörmander '72])

Existence and uniqueness of parametrices $G_{+}, G_{-}, G_{\mathrm{F}}, G_{\overline{\mathrm{F}}}$ for P s.t.

$$\mathrm{WF}'(G_{\mathrm{F}}) = \Delta^{*} \cup \left(\mathcal{C} \cap \mathcal{N}_{\pm} \times \mathcal{N}_{\pm} \textit{ above } \{x_1 \in J^{\pm}(x_2)\}\right),$$

$$\mathrm{WF}'(G_{\overline{\mathrm{F}}}) = \Delta^{*} \cup \left(\mathcal{C} \cap \mathcal{N}_{\mp} \times \mathcal{N}_{\mp} \textit{ above } \{x_1 \in J^{\pm}(x_2)\}\right),$$

$$\mathrm{WF}'(G_{\pm}) = \Delta^{*} \cup \left(\mathcal{C} \cap \{x_1 \in J^{\mp}(x_2)\right)$$

$\Delta^{*} :=$ *diag. of* $\dot{T}^{*}M \times \dot{T}^{*}M$, $\mathcal{C} := \{(X_1, X_2) \in \mathcal{N} \times \mathcal{N} : \; X_1 \sim X_2\}$

Theorem ([Radzikowski '96])

(1)-(3) *determines* $\Lambda_{\pm}$ ***uniquely*** mod C^{∞}

Consider *weaker* conditions:

(1) $P\Lambda_\pm = \Lambda_\pm P = 0$

(2) $\Lambda_\pm \geq 0, \ \Lambda_+ - \Lambda_- = \mathrm{i}(G_+ - G_-)$

(3′) $\mathrm{WF}'(\Lambda_\pm) \subset \mathcal{N}_\pm \times \mathcal{N}_\pm$

If $\Lambda_\pm, \widetilde{\Lambda}_\pm$ subject to (2) mod C^∞ and (3′):

$$\underbrace{(\Lambda_+ - \widetilde{\Lambda}_+)}_{\mathrm{WF}' \text{ in } \mathcal{N}_+ \times \mathcal{N}_+} + \underbrace{(\Lambda_- - \widetilde{\Lambda}_-)}_{\mathrm{WF}' \text{ in } \mathcal{N}_- \times \mathcal{N}_-} = 0 \quad \text{mod } C^\infty$$

$\Rightarrow \Lambda_+ - \widetilde{\Lambda}_+ \in C^\infty$. □

Actually, $\widetilde{\Lambda}_\pm := \mathrm{i}(G_\mathrm{F} - G_\mp)$ satisfies (2) mod C^∞ and (3).
We can replace (3) by (3′)!

AVAILABLE CONSTRUCTIONS

- **Asymptotic constructions**
  - ✓ Deformation argument [Fulling, Narcowich, Wald '81]
  - ✓ Conformal compactification [Dappiaggi, Morretti, Pinamonti 06-09'; Gérard, W. '14]
- **Global constructions**
  - ✓ $\Lambda_{\pm} := \mathbb{1}_{\pm}(\mathrm{i}G)$ works on special spacetimes [Fewster, Verch '13; Brum, Fredenhagen '14]
  - ? Feynman propagator from Fredholm setup [Gell-Redman, Haber, Vasy '14]
- **Cauchy evolution**
  - ✓ Diagonalize P by elliptic ΨDOs [Junker, Schrohe; Gérard, W. '14]

t.b.c.