The Klein-Gordon operator
with Atiyah-Patodi-Singer-type boundary conditions
on asymptotically Minkowski spacetimes

joint work w. Christian Gérard

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INVERSES OF THE KLEIN-GORDON OPERATOR

- On **Minkowski space** $(\mathbb{R}^{1,d}, \eta)$, $P = \partial_t^2 - \Delta_x$, has *four* natural inverses P_I^{-1}, $I = \{+, -, \mathrm{F}, \overline{\mathrm{F}}\}$:
  - advanced/retarded $\widehat{P}_{\pm}^{-1}(\tau, k) = ((\tau \mp i0)^2 - k^2)^{-1}$,
  - Feynman/anti-Feynman $\widehat{P}_{\mathrm{F}/\overline{\mathrm{F}}}^{-1}(\tau, k) = (\tau^2 - k^2 \pm i0)^{-1}$
- On **globally hyperbolic spacetimes** (M, g)

$$P := -\Box_g + m^2, \quad m \geq 0$$

  has *four classes* of parametrices $P_I^{(-1)}$, described by singularities of $P_I^{(-1)}(x, x')$ [Duistermaat, Hörmander '72]
  - Let $\mathcal{N} := p^{-1}(\{0\}) = \mathcal{N}_+ \cup \mathcal{N}_-$. For instance Feynman: $\mathrm{WF}'(P_{\mathrm{F}}^{(-1)}) = (\mathrm{diag}_{T^*M}) \cup \bigcup_{t \leq 0} (\Phi_t(\mathrm{diag}_{T^*M}) \cap \pi^{-1}\mathcal{N})$ (here Φ_t — bicharacteristic flow, π projects to left leg)

⚠ In applications in QFT & non-linear PDEs one needs approximate inverses in a stronger sense!

FREDHOLM SETUP FOR FEYNMAN PROPAGATOR

Two recent results give **Fredholm property** of $P : \mathcal{X}_F \to \mathcal{Y}_F$ with $\mathcal{X}_F, \mathcal{Y}_F$ Hilbert spaces with built-in generalized Atiyah - Patodi - Singer-type boundary conditions:

- ✓ P — **Dirac operator** on $[t_1, t_2] \times \Sigma$, Σ closed manifold [Bär, Strohmaier '15]
  - ✓ index theorem available
  - ✓ index related to particle creation and anomalies
- ✓ P — **wave operator** on asymptotically Minkowski spacetime [Gell-Redman, Haber, Vasy '15]
  - ✓ positivity $\mathrm{i}^{-1}(P_F^{-1} - (P_F^{-1})^*) \geq 0$ [Vasy '15]
  - ✓ $\mathrm{Ker} P|_{\mathcal{X}_F} \subset C^\infty$, [Vasy, W. '15]

$\Rightarrow$ P_F^{-1} — closest possible analogue of $(-\Delta + m^2)^{-1}$!!

In this talk: Fredholm problem for **Klein-Gordon operator** $P = -\Box_g + m^2$, $m > 0$ on asymptotically Minkowski spacetimes

GEOMETRICAL SETUP

We assume (M, g) asymptotically Minkowski spacetime, i.e. $M = \mathbb{R}^{1+d}$ and g Lorentzian metric s.t.

$$\text{(aM)} \quad \begin{aligned} & g_{\mu\nu} - \eta_{\mu\nu} \in S^{-\delta}(\mathbb{R}^{1+d}) \text{ for } \delta > 1, \\ & \textit{where } f \in S^{-\delta} \textit{ means } \partial_x^\alpha f \in O\left((1 + |x|^2)^{\frac{-\delta-\alpha}{2}} \right), \\ & \exists \text{ time function } \tilde{t} \text{ s.t. } \tilde{t} - t \in S^{1-\epsilon}(\mathbb{R}^{1+d}),\ \epsilon > 0. \end{aligned}$$

In practice: reduction to model case

$$P = \partial_t^2 + r(t)\partial_t + a(t, \mathrm{x}, D_\mathrm{x}),$$

and $\exists\, a_{\mathrm{out/in}}(\mathrm{x}, D_\mathrm{x})$ elliptic, $a_{\mathrm{out/in}}(\mathrm{x}, D_\mathrm{x}) \geq C > 0$ s.t.

$$\begin{aligned} & a(t, \mathrm{x}, D_\mathrm{x}) = a_{\mathrm{out/in}}(D_\mathrm{x}) + \Psi^{2,-\delta}_{\mathrm{std}},\ \delta > 1, \\ & r(t) \in \Psi^{0,-1-\delta}_{\mathrm{std}}. \end{aligned}$$

$\Rightarrow$ *This means decay of* $P - P_{\mathrm{out/in}}$*, where* $P_{\mathrm{out/in}} = \partial_t^2 + a_{\mathrm{out/in}}$.

IN/OUT STATES

- Let $\mathcal{U}(t,s), \mathcal{U}_{\text{free}}(t,s)$ — Cauchy evolution for P, $P_{\text{free}} = -\partial_t^2 - \Delta_x + m^2$
  i.e. maps Cauchy data $\varrho_s u := (u, \mathrm{i}^{-1}\partial_t u)\restriction_{t=s}$ to $\varrho_t u$
- **In/out covariances**: t-dependent projections

$$c_{\text{out}}^{\pm}(t) := \lim_{t_+\to\infty} \mathcal{U}(t,t_+)c_{\text{free}}^{\pm}\mathcal{U}(t_+,t),$$
$$c_{\text{in}}^{\pm}(t) := \lim_{t_-\to\infty} \mathcal{U}(t,t_-)c_{\text{free}}^{\pm}\mathcal{U}(t_-,t),$$

where

$$c_{\text{free}}^{\pm} = \begin{pmatrix} \mathbf{1} & \pm\sqrt{-\Delta_x + m^2} \\ \pm\sqrt{-\Delta_x + m^2} & \mathbf{1} \end{pmatrix} = \begin{matrix}\mathbb{R}_{\pm} - \text{spectral projection} \\ \text{of generator of } \mathcal{U}_{\text{free}}(t,s)\end{matrix}$$

Theorem ([Gérard, W.])

For all Cauchy data h, $\mathrm{WF}(\mathcal{U}(t,t_0)c^{\pm}(t_0)h) \subset \mathcal{N}^{\pm}$

$\Rightarrow$: In QFT terms, "the in/out states are Hadamard".

MAIN RESULT

Fix $t_0 \in \mathbb{R}$ and define Feynman/anti-Feynman scattering data:

$$\varrho_{\mathrm{F}} u := \lim_{t_\pm \to \pm\infty} \left(c^+_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_+) \varrho_{t_+} u + c^-_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_-) \varrho_{t_-} u \right),$$

$$\varrho_{\overline{\mathrm{F}}} u := \lim_{t_\pm \to \pm\infty} \left(c^+_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_-) \varrho_{t_-} u + c^-_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_+) \varrho_{t_+} u \right).$$

Theorem ([Gérard, W.])

Let $\mathcal{Y}^m := \langle t \rangle^{-\gamma} L^2(\mathbb{R}; H^m(\mathbb{R}^d))$, $\frac{1}{2}\gamma < \frac{1}{2} + \delta$ *and*

$$\mathcal{X}^m_{\mathrm{F}} := \{ u \in C^0(\mathbb{R}; H^m(\mathbb{R}^d)) \cap C^1(\mathbb{R}; H^{m+1}(\mathbb{R}^d)) : \ Pu \in \mathcal{Y}^m, \ \varrho_{\overline{\mathrm{F}}} u = 0 \}.$$

Then $P : \mathcal{X}^m_{\mathrm{F}} \to \mathcal{Y}^m$ *is* ***Fredholm*** *of index*

$$\operatorname{ind} P = \operatorname{ind} \left(c^-_{\mathrm{free}} W^{-1}_{\mathrm{out}} + c^+_{\mathrm{free}} W^{-1}_{\mathrm{in}} \right),$$

$$\text{where } W^{-1}_{\mathrm{out/in}} = \lim_{t_\pm \to \pm\infty} \mathcal{U}_{\mathrm{free}}(0, t_\pm) \mathcal{U}(t_\pm, 0).$$

Furthermore, there exists $P^{-1}_{\mathrm{F}} : \mathcal{Y} \to \mathcal{X}_{\mathrm{F}}$ *with Feynman wave front set and s.t.* $\mathbf{1} - PP^{-1}_{\mathrm{F}}$ *and* $\mathbf{1} - P^{-1}_{\mathrm{F}} P$ *are compact with smooth kernel.*

1st STEP — APPROXIMATE DIAGONALIZATION

- *Suppose* we have $b(t) \in C^\infty(\mathbb{R}, \Psi^1(\mathbb{R}^d))$ s.t.

$$(\partial_t + \mathrm{i}b(t) + r(t)) \circ (\partial_t - \mathrm{i}b(t)) = \partial_t^2 + a(t) + r(t)\partial_t$$

- Hence also $(\partial_t - \mathrm{i}b^* + r) \circ (\partial_t + \mathrm{i}b^*) = \partial_t^2 + a + r\partial_t$

Set $\tilde{\psi}(t) := \begin{pmatrix} \partial_t + \mathrm{i}b^*(t) \\ \partial_t - \mathrm{i}b(t) \end{pmatrix} \phi(t)$. Now $(\partial_t^2 + a + r\partial_t)\phi(t) = 0 \Leftrightarrow$

$$\mathrm{i}^{-1}\partial_t\tilde{\psi}(t) = B(t)\tilde{\psi}(t), \quad B(t) = \begin{pmatrix} b^*(t) + \mathrm{i}r(t) & 0 \\ 0 & -b(t) + \mathrm{i}r(t) \end{pmatrix}.$$

This way, $\mathcal{U}_B(t,s) = \begin{pmatrix} \mathcal{U}_{-b^*+\mathrm{i}r}(t,s) & 0 \\ 0 & \mathcal{U}_{b+\mathrm{i}r}(t,s) \end{pmatrix}$ and
$\mathcal{U}(t,s) = T(t)\mathcal{U}_B(t,s)T^{-1}(s)$ where $\psi(t) = T(t)\tilde{\psi}(t)$.

- In practice we get the factorization modulo smooth errors, but we can correct $\mathcal{U}_B$ afterwards.

1^{st} STEP — APPROXIMATE DIAGONALIZATION

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- We construct $b(t)$ precisely enough to control decay of $b(t) - a^{1/2}(t)$, $b(t) - a_\infty^{1/2}$ (using $\Psi^{m,\delta}_{\text{std}}$-calculus)

2nd STEP — MØLLER OPERATORS

$$\varrho_{\mathrm{F}} u = \lim_{t_\pm \to \pm\infty} \left(c^+_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_+) \varrho_{t_+} u + c^-_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_-) \varrho_{t_-} u \right),$$

$$\varrho_{\overline{\mathrm{F}}} u = \lim_{t_\pm \to \pm\infty} \left(c^+_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_-) \varrho_{t_+} u + c^-_{\mathrm{free}} \mathcal{U}_{\mathrm{free}}(t_0, t_+) \varrho_{t_-} u \right),$$

$$\varrho_{\mathrm{out}} u := \lim_{t_+ \to +\infty} \mathcal{U}_{\mathrm{free}}(t_0, t_+) \varrho_{t_+} u, \quad \varrho_{\mathrm{in}} u := \lim_{t_- \to -\infty} \mathcal{U}_{\mathrm{free}}(t_0, t_-) \varrho_{t_-} u.$$

For solutions, $\varrho_{t_+} u = \mathcal{U}(t_+, t_0) \varrho_{t_0} u$, so $\varrho_I u = W_I^\dagger u(t_0)$, where

$$W_{\mathrm{out/in}} = \mathrm{s-}\lim_{t_\pm \to \pm\infty} \mathcal{U}(t_+, t_0) \mathcal{U}_{\mathrm{free}}(t_0, t_\pm), \quad W^\dagger_{\mathrm{out/in}} W_{\mathrm{out/in}} = \mathbf{1},$$

$$W_{\mathrm{F}} = W_{\mathrm{out}} c^+_{\mathrm{free}} + W_{\mathrm{in}} c^-_{\mathrm{free}}, \quad W_{\overline{\mathrm{F}}} = W_{\mathrm{in}} c^+_{\mathrm{free}} + W_{\mathrm{out}} c^-_{\mathrm{free}},$$

So Cauchy data of $\mathrm{Ker} D \cap \mathrm{Ker} \varrho_{\overline{\mathrm{F}}}$ is simply $\mathrm{Ker} W^\dagger_{\overline{\mathrm{F}}}$.

Lemma ([Gérard, W.])

For $\alpha < \delta/2$, $\sharp = \mathrm{in}, \mathrm{out}$,

$$c^\pm_\sharp(t_0) = W_\sharp c^+_{\mathrm{free}} W_\sharp^{-1} = c^\pm_{\mathrm{ref}}(t_0) + T(t_0) \langle \mathrm{x} \rangle^{-\alpha} \Psi^{-\infty}(\mathbb{R}^d) \langle \mathrm{x} \rangle^{-\alpha} T^{-1}(t_0).$$

… so $W^\dagger_{\overline{\mathrm{F}}} W_{\overline{\mathrm{F}}} = \mathbf{1} +$ compact, smoothing term, same for $W_{\overline{\mathrm{F}}} W^\dagger_{\overline{\mathrm{F}}}$!

3rd step — inhomogeneous Cauchy problem

Need Hilbert space $\mathcal{Y}$ s.t.

$$\begin{cases} Pu = f, & f \in \mathcal{Y} \\ \varrho_{t_0} u = v, & v \in H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \end{cases}$$

has unique solution, eg. $\mathcal{Y} := \langle t \rangle^{-\gamma} L^2(\mathbb{R}; H^1(\mathbb{R}^d))$, $\gamma > \frac{1}{2}$.

This gives

$u \in \mathcal{X} := \{u \in C^0(\mathbb{R}; H^1(\mathbb{R}^d)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^d)) : Pu \in \mathcal{Y}\}$.

Fredholm analysis:

- $\varrho_{t_0} \oplus P : \mathcal{X} \to (H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)) \oplus \mathcal{Y}$ isomorphism
- $\varrho_{\overline{\mathrm{F}}} \oplus P : \mathcal{X} \to (H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)) \oplus \mathcal{Y}$ Fredholm
  - reduction to $\varrho_{\overline{\mathrm{F}}} : \mathrm{Ker} P|_{\mathcal{X}} \to (H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$ Fredholm
- $P : \underbrace{\mathrm{Ker}\, \varrho_{\overline{\mathrm{F}}}|_{\mathcal{X}}}_{\mathcal{X}_{\mathrm{F}}} \to \mathcal{Y}$ Fredholm! The index is $\mathrm{ind}\, W_{\overline{\mathrm{F}}}$.

FREDHOLM INVERSE

First guess: 'Feynman propagator associated to in (or out) state'

$$(P_{\mathrm{F,in}}^{-1} f)(t) := -\mathrm{i}\pi_0 \int_{-\infty}^{\infty} \mathcal{U}(t, t_0)(c_{\mathrm{in}}^{+}(t_0) - \mathbf{1}_{t<s})\mathcal{U}(t_0, s)\varrho_s f \, ds$$

with π_0 — proj. to first component (of Cauchy data).

⚠ $P_{\mathrm{F,in}}^{-1}$ does not map $\mathcal{Y} \to \mathcal{X}_{\mathrm{F}}$ (recall $\mathcal{X}_{\mathrm{F}} = \mathcal{X} \cap \mathrm{Ker}\,\varrho_{\overline{\mathrm{F}}}$)

Theorem ([Gérard,W.])

One can find $P_{\mathrm{F}}^{-1} : \mathcal{Y} \to \mathcal{X}_{\mathrm{F}}$ that has Feynman wave front set, satisfies $\mathrm{i}^{-1}(P_{\mathrm{F}}^{-1} - (P_{\mathrm{F}}^{-1})^*) \geq 0$, and $\mathbf{1} - PP_{\mathrm{F}}^{-1}$ and $\mathbf{1} - P_{\mathrm{F}}^{-1}P$ are compact with smooth kernel.

In practice: $P_{\mathrm{F}}^{-1} :=$ analogue of $P_{\mathrm{F,in}}^{-1}$ constructed using diagonalizable part of the dynamics.

OUTLOOK

The new:

- ✓ **Fredholm property** of Klein-Gordon operator on **asymptotically Minkowski spacetimes**
- ✓ Hadamard property of in/out states
  - works also for asymptotically static spacetimes with Cauchy surface of bounded geometry

Questions:

- Index formula as in [Bär, Strohmaier '15]? Dirac case? Applications of the latter to anomalies?
- Relation to [Gell-Redman,Haber,Vasy '15]? Complex powers??
- More general spacetimes: non-static free dynamics
- **Dereziński's conjecture**: P_F^{-1}= *boundary value of resolvent of a self-adjoint extension of* P?

Thank you for your attention!

APPENDIX — $\Psi^{m,\delta}_{\mathrm{std}}$ CLASSES

For $m, \delta \in \mathbb{R}$, $S^{m,\delta}_{\mathrm{std}}(\mathbb{R}; T^*\mathbb{R}^d) :=$ functions $a(t, \mathrm{x}, k)$ s.t.

$$\partial_t^\gamma \partial_{\mathrm{x}}^\alpha \partial_k^\beta a(t, \mathrm{x}, k) \in O((\langle t\rangle + \langle \mathrm{x}\rangle)^{\delta-\gamma-|\alpha|} \langle k\rangle^{m-|\beta|}),\ \gamma \in \mathbb{N},\ \alpha, \beta \in \mathbb{N}^d.$$

Let $\mathcal{W}^{-\infty}_{\mathrm{std}}(\mathbb{R}; \mathbb{R}^d)$ — space of operator-valued functions $a(t)$ s.t.

$$\|(D_{\mathrm{x}}^2 + \mathrm{x}^2)^m \partial_t^\gamma a(t) (D_{\mathrm{x}}^2 + \mathrm{x}^2)^m\|_{B(L^2(\mathbb{R}^d))} \in O(\langle t\rangle^{-n}),\ \forall\, m, n \in \mathbb{N}.$$

Finally (Op^{w} – Weyl quantization, ph – k-polyhomogeneity):

$$\Psi^{m,\delta}_{\mathrm{std}}(\mathbb{R}; \mathbb{R}^d) := \mathrm{Op}^{\mathrm{w}}(S^{m,\delta}_{\mathrm{std,ph}}(\mathbb{R}; T^*\mathbb{R}^d)) + \mathcal{W}^{-\infty}_{\mathrm{std}}(\mathbb{R}; \mathbb{R}^d).$$

Theorem ([Gérard,W.])

Let $a \in \Psi^{m,0}_{\mathrm{std}}(\mathbb{R}; \mathbb{R}^d)$ be elliptic, selfadjoint with $a(t) \geq c_0\mathbf{1}$ with $c_0 > 0$. Then $a^\alpha \in C^\infty_{(\mathrm{b})}(\mathbb{R}; \Psi^{m\alpha}(\Sigma))$ for any $\alpha \in \mathbb{R}$ and $\sigma_{\mathrm{pr}}(a^\alpha)(t) = \sigma_{\mathrm{pr}}(a(t))^\alpha$.

- proof by reduction to [Ammann, Lauter, Nistor, Vasy '05]