

# Lecture notes: Models for Quantum Measurements

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## Abstract

We discuss two physical models (see [1], [2], [3], and [4]) involving a small quantum system coupled to a macroscopic apparatus. These models are simple enough to allow for explicit calculations of the joint dynamics of the measured system and the macroscopic variable of the apparatus used for readout (pointer). We study the two fundamental dynamical processes: (i) the entanglement of the measured system with the apparatus and (ii) decoherence of distinct pointer readouts, in some situations where these two processes proceed simultaneously.

## 1 Introduction

Since the birth of quantum mechanics, physicists and mathematicians have devoted a lot of works to the theoretical description of measurement processes on quantum systems (see e.g. [5, 6, 7, 8, 9, 10, 11]). Quantum measurements play a major role in quantum theory since they give us access to the quantum word. The primary motivation of these works was to investigate the foundation of the quantum theory and its interpretation problems, a subject still under debate. A renewal of interest for measurement processes came in the last decades from experiments which achieved to store, manipulate and study single quantum systems (one atom in a magnetic trap, few photons in a single mode of an optical cavity, charge or phase qubits in a Josephson junction,..., see e.g. the lectures of S. Seidelin, L. Lévy and L. Faoro). It has been realized that measurements can be used to manipulate such systems, by means of the quantum Zeno effect or with quantum trajectories (see the lecture of P. Degiovanni). These recent wonderful experiments have open the route to (and are now being stimulated by) applications to quantum information. Quantum measurements play an important role in this rapidly growing field. For instance, a measurement has to be performed to *extract classical information out of* the transmitted *quantum information* in quantum cryptography, or to get the result at the end of a quantum computation.

The aim of these lectures is neither to give an overview of the various theories proposed in the literature in order to explain the reduction of the wavepacket nor to discuss what could be a consistent interpretation of the quantum theory. Our goal is to discuss some *specific concrete models* describing a quantum measurement (QM). These models will be studied within the framework of modern quantum theory and its so-called “Copenhagen” interpretation. A measurement is viewed here as a *quantum dynamical process* originating from a unitary evolution on the microscopic scale.

## 2 Lecture 1: what is a good model for a QM?

### 2.1 Macroscopic measuring apparatus

In order to measure the value of an observable  $S$  of a quantum or of a classical system  $\mathcal{S}$ , this system must interact (during a definite period of time) with a measuring apparatus  $\mathcal{A}$ , in such a way that some information on the state of  $\mathcal{S}$  be transferred to  $\mathcal{A}$ . If the object  $\mathcal{S}$  is a classical macroscopic system, the perturbation of its state resulting from this interaction can be neglected, at least for a good enough measuring apparatus (such an apparatus can always be obtained in principle via technical improvements). On the contrary, it is never possible to neglect the perturbation made on the state of a small quantum object  $\mathcal{S}$  during its interaction with  $\mathcal{A}$  (excepted if  $\mathcal{S}$  is initially in an eigenstate of  $S$ ). For instance, if one sheds light on a small particle to measure its position, the photons will give small momentum kicks to the particle in arbitrary directions; the resulting uncertainty  $\Delta p$  in the momentum of the particle satisfies  $\Delta p \Delta x \gtrsim \hbar$ , where  $\Delta x$  is the precision of the position measurement [12].

Let us specify the properties that an ideal measuring apparatus  $\mathcal{A}$  must necessarily have.

1.  $\mathcal{A}$  must be *macroscopic* and possesses a “pointer” variable  $X$  capable of a *quasi-classical behaviour*. This variable is used as readout of the measurement results (e.g.  $X$  can be the position of the centre-of-mass of an ammeter needle). The initial value  $x_0$  of  $X$  is precisely known and its fluctuations remain negligible on the macroscopic scale during the whole measurement.
2. At the end of the measurement, there must be a *one-to-one correspondence* between the eigenvalues  $s$  of the measured observable  $S$  and the values  $x_s$  of  $X$ . These values must moreover be macroscopically distinguishable for distinct  $s$  (e.g., the positions  $x_s$  of the ammeter needle associated to different  $s$  are separated by macroscopic distances).
3. There is *initially no correlations* between  $\mathcal{A}$  and  $\mathcal{S}$  when they are put in contact and start to interact.

Thanks to the first requirement, the classical pointer will not be perturbed noticeably by an observer looking at the result of the measurement (see above). This observer does not need to perform a new QM to obtain the result. The second requirement means that the interaction between  $\mathcal{S}$  and  $\mathcal{A}$  provokes a macroscopic change in the state of  $\mathcal{A}$ . Since  $\mathcal{S}$  is a small system, it can only perturb  $\mathcal{A}$  weakly and this small perturbation must be subsequently amplified, so as to lead to macroscopic changes in the pointer variable  $X$ . Such amplifications of small signals are used e.g. in photo-detectors. Many measurements actually involve a chain  $\{\mathcal{A}_n\}_{n=1,\dots,N}$  of apparatus (cascade): only the first apparatus  $\mathcal{A}_1$  in the chain (which is not necessarily macroscopic) is in contact with  $\mathcal{S}$ ; each apparatus  $\mathcal{A}_n$  measures one after the other the observable  $X_{n-1}$  of the previous apparatus; finally, the observer reads the result on the pointer variable  $X_N$  of the last apparatus  $\mathcal{A}_N$  (which satisfies the above requirements 1 and 2). In what follows, we will not deal with the complications included in such chains of apparatus, but will restrict ourselves to the case of a single apparatus  $\mathcal{A}$ . In view of the requirements 1-3, we assume that  $\mathcal{A}$  is *initially in a metastable state*. In the model discussed in Sec. 3, this state is a quasi-bound state of a one-dimensional scattering problem; in the model of Sec. 4, it is an unstable state which may relax into one among several equilibria of the apparatus, the latter being in the critical regime of a phase transition.

Because the apparatus is made of atoms, it is appropriate to assume that:

4.  $\mathcal{A}$  can be described quantum-mechanically.

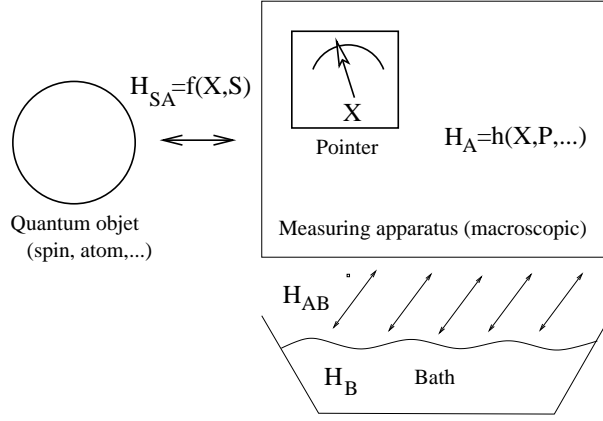


Figure 1: *Model for a QM: the quantum object  $\mathcal{S}$  is coupled to a macroscopic measuring apparatus  $\mathcal{A}$  having a macro-observable  $X$ ; the apparatus  $\mathcal{A}$  is coupled to a bath  $\mathcal{B}$  with infinitely many degrees of freedom. All relevant Hamiltonians are shown.*

Since  $\mathcal{A}$  is macroscopic, its precise microscopic state will be unknown;  $\mathcal{A}$  must then be described with the help of quantum statistical mechanics. The initial state of  $\mathcal{A}$  is given by a density matrix  $\rho_A^{(0)}$  (mixed state). To each eigenvalue  $s$  of the measured observable corresponds a specific apparatus state with density matrix  $\rho_A^{(s)}$  (which does not depend on the initial state of  $\mathcal{S}$ ):

Object $\mathcal{S}$	Apparatus $\mathcal{A}$
eigenprojector $P_s$ of the meas. observ. $S$	$\rho_A^{(s)}$
eigenvalue $s$ of $S$	pointer variable $x_s = \text{tr}(\rho_A^{(s)} X)$
	fluctuations $\Delta x_s = (\text{tr}(\rho_A^{(s)} X^2) - x_s^2)^{1/2}$ ,
	$\Delta x_s \ll \min_{s \neq s'}  x_s - x_{s'} $ .

## 2.2 Coupling the apparatus with a bath

It is well known that macroscopic bodies cannot be considered as isolated from their environment (the typical energy difference between their nearest levels being extremely small, any small interaction with the environment may induce transitions between these levels). Hence statistical physics does not only enter in a QM because the initial state of  $\mathcal{A}$  is a mixed state, but also because one must *take into account the coupling of  $\mathcal{A}$  with its environment* or with some uncontrollable microscopic degrees of freedom of the apparatus itself (which we separate from  $\mathcal{A}$  and call altogether the “bath  $\mathcal{B}$ ” in what follows). As a result of the coupling between  $\mathcal{A}$  and  $\mathcal{B}$ , the combined system  $\mathcal{S} + \mathcal{A}$  undergoes an *irreversible evolution* (see the lectures of C.-A. Pillet and S. Attal). A good model for a QM therefore necessarily includes all three subsystems  $\mathcal{S}$ ,  $\mathcal{A}$  and  $\mathcal{B}$  (see Figure 1); both the  $\mathcal{S}$ - $\mathcal{A}$  and the  $\mathcal{A}$ - $\mathcal{B}$  couplings play an important role. The total system  $\mathcal{S} + \mathcal{A} + \mathcal{B}$  can be assumed to be isolated and its dynamics is given by Schrödinger’s equation. The density matrix of  $\mathcal{S} + \mathcal{A} + \mathcal{B}$  at time  $t$  is given in terms of its value at  $t = 0$  by

$$\rho_{SAB}(t) = e^{-it(H_S + H_A + H_B + H_{SA} + H_{AB})} \rho_{SAB}^{(0)} e^{it(H_S + H_A + H_B + H_{SA} + H_{AB})} \quad (1)$$

where  $H_S$ ,  $H_A$  and  $H_B$  are the Hamiltonians of the object, apparatus and bath and  $H_{SA}$  and  $H_{AB}$  are the object-apparatus and apparatus-bath interaction Hamiltonians. The direct coupling

between  $\mathcal{S}$  and  $\mathcal{B}$  does not play an important role in the measurement<sup>1</sup> and has been neglected. We shall assume that we have no access to bath observables and define the object-apparatus density matrix by tracing out the bath degrees of freedom in the density matrix of  $\mathcal{S} + \mathcal{A} + \mathcal{B}$ ,

$$\rho_{SA}(t) = \text{tr}_B \rho_{SAB}(t) \quad (2)$$

(here  $\text{tr}_B$  denotes the partial trace over the bath  $\mathcal{B}$ ).

## 2.3 The von Neumann measurement postulate

As one learns during the first lectures in quantum mechanics, only probabilities of the possible measurement results can be predicted, even if the initial wavefunction  $|\psi_0\rangle$  of the quantum object  $\mathcal{S}$  is perfectly known (i.e.,  $\mathcal{S}$  is in a pure state). Probabilities are introduced as a fundamental aspect of the theory, unlike in classical statistical physics where they result from the impossibility to know in practise the positions and velocities of all particles. According to the requirement 3 of Sec. 2.1, the initial object-apparatus state is a product state  $\rho_{SA}^{(0)} = |\psi_0\rangle\langle\psi_0| \otimes \rho_A^{(0)}$ . Let  $c_s = \langle s|\psi_0\rangle$  be the component of  $|\psi_0\rangle$  in the eigenbasis  $\{|s\rangle\}$  of  $S$ . We shall assume here for simplicity that  $S$  has a discrete and non-degenerate spectrum. The state  $\rho_{SA}^{(0)}$  is transformed during the measurement as follows:

$$\rho_{SA}^{(0)} = \sum_{s,s'} c_s c_{s'}^* |s\rangle\langle s'| \otimes \rho_A^{(0)} \longrightarrow \rho_{SA}^{\text{post meas.}} = \sum_s |c_s|^2 |s\rangle\langle s| \otimes \rho_A^{(s)}. \quad (3)$$

To the expense that the density matrix  $\rho_{SA}^{\text{post meas.}}$  has *the same interpretation as an ensemble* as one gives to density matrices in statistical physics, (3) is the mathematical formulation of:

- **Born rules:** the value of the pointer variable  $X$  after the measurement is  $x_s = \text{tr}(\rho_A^{(s)} X)$  with probability  $|c_s|^2$ .
- **von Neumann postulate:** given that  $X$  has value  $x_{s_0}$ , the state of the object  $\mathcal{S}$  immediately after the measurement is  $|s_0\rangle$ .

The meaning of Born rules is that, if one repeats many times the measurement on identical objects initially in the same state  $|\psi_0\rangle$ , the fraction of results “ $X = x_s$ ” will be  $|c_s|^2$ . The von Neumann postulate (= reduction of the wavepacket), on the contrary, concerns a single run of the measurement. It can be interpreted in various ways: is the collapse

$$\sum_s |c_s|^2 |s\rangle\langle s| \otimes \rho_A^{(s)} \longrightarrow |s_0\rangle\langle s_0| \otimes \rho_A^{(s_0)} \quad \text{given that the result is “} X = x_{s_0} \text{”} \quad (4)$$

a *real* or an *apparent collapse*? We prefer the second terminology and share the point of view of D. Bohm in his 1951 book (from which the following citation is taken, see [7], Sec. 22.10): “*The sudden replacement of the statistical ensemble of wavefunctions by a single wavefunction represents absolutely no change in the state [of the object], but is analogous to the sudden changes in classical probability functions which accompany an improvement of the observer’s information*”. In other words, transformation (4) is simply due to the gain of information

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<sup>1</sup>Actually, it turns out that for a small quantum object  $\mathcal{S}$  strongly coupled to the pointer  $\mathcal{P}$ , decoherence processes caused by a direct  $\mathcal{S}$ - $\mathcal{B}$  coupling have a much smaller effect than the decoherence resulting from the quantum correlations between  $\mathcal{S}$  and  $\mathcal{B}$  which develop in time thanks to the  $\mathcal{S}$ - $\mathcal{P}$  and  $\mathcal{P}$ - $\mathcal{B}$  couplings, i.e., thanks to the *indirect coupling* of  $\mathcal{S}$  with  $\mathcal{B}$  via the pointer  $\mathcal{P}$ .

obtained from the knowledge of what the actual value of  $X$  is (in mathematical terms one should speak of conditional probabilities); the measurement problem is not to understand this “collapse” but to explain what kind of physical process can lead to the state transformation (3). The main problem is that it is not always obvious to give to the density matrix in the r.h.s. of (3) the necessary interpretation as an ensemble, in particular if this density matrix is obtained as a result of a partial tracing over the bath degrees of freedom of an object-apparatus-bath state as in (2). In Sec. 3 and 4, we will show on concrete models that the reduced object-apparatus density matrix  $\rho_{SA}(t)$  is very close to the r.h.s. of (3) at times  $t$  larger than the measurement time  $t_{\text{meas}}$ , but we will not address the above-mentioned delicate problem of the interpretation of  $\rho_{SA}(t)$  as an ensemble<sup>2</sup>.

It is worth noting that (3) implies in particular that *an object initially in an eigenstate  $|s\rangle$  of the measured observable remains unchanged* during the measurement,

$$|s\rangle\langle s| \otimes \rho_A^{(0)} \longrightarrow |s\rangle\langle s| \otimes \rho_A^{(s)}. \quad (5)$$

Let us assume that one can find a unitary operator  $U$  acting on the Hilbert space of  $\mathcal{S} + \mathcal{A}$  which implements the transformation (3), i.e., such that  $\rho_{SA}^{\text{post meas.}} = U \rho_{SA}^{(0)} U^\dagger$ . Writing  $\rho_A^{(0)} = \sum_i p_i |\chi_i^{(0)}\rangle\langle\chi_i^{(0)}|$  with  $p_i > 0$ , it follows from (5) that  $U|s\rangle \otimes |\chi_i^{(0)}\rangle = |s\rangle \otimes |\chi_i^{(s)}\rangle$  and  $\rho_A^{(s)} = \sum_i p_i |\chi_i^{(s)}\rangle\langle\chi_i^{(s)}|$ . By virtue of the linearity  $U$ , one gets for the initial state considered in (3):

$$\rho_{SA}^{\text{ent}} = U \rho_{SA}^{(0)} U^\dagger = \sum_{s,s'} c_s c_{s'}^* |s\rangle\langle s'| \otimes \rho_A^{(s,s')} \quad (6)$$

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<sup>2</sup>The lack of information in the reduced density matrix  $\rho_{SA}(t)$  defined in (2) concerns the entanglement with the bath. To simplify the discussion, let us assume that  $\mathcal{A}$  and  $\mathcal{B}$  are initially uncorrelated and in pure states  $|\chi^{(0)}\rangle$  and  $|\Phi^{(0)}\rangle$ , respectively. At time  $t$ , the total system  $\mathcal{S} + \mathcal{A} + \mathcal{B}$  has wavefunction  $|\Psi_{SAB}(t)\rangle = \sum_s c_s |s\rangle \otimes |\chi^{(s)}(t)\rangle \otimes |\Phi^{(s)}(t)\rangle$ . Let us imagine that the bath wavefunctions  $|\Phi^{(s)}(t)\rangle$  are nearly orthogonal for different  $s$ . (This orthogonality is indeed produced by the dynamics after a short *decoherence time*  $t_{\text{dec}}$  when a macroscopic system like  $\mathcal{A}$  is coupled to a bath, with e.g. a coupling proportional to  $X$ ). In this situation, the reduced object-apparatus density matrix (2) is indeed almost equal to the density matrix  $\rho_{SA}^{\text{p. meas.}}$  in the r.h.s. of (3), with  $\rho_A^{(s)} = |\chi^{(s)}(t)\rangle\langle\chi^{(s)}(t)|$ . If we increase the amount of information at our disposal, we will not find that  $\mathcal{S} + \mathcal{A}$  is in state  $|s_0\rangle\langle s_0| \otimes \rho_A^{(s_0)}$  for a given  $s_0$ , but that it is *entangled with the bath  $\mathcal{B}$*  and that  $\mathcal{S} + \mathcal{A} + \mathcal{B}$  is in a *linear superposition* of the states  $|\Psi_{SAB}^{(s)}(t)\rangle = |s\rangle \otimes |\chi^{(s)}(t)\rangle \otimes |\Phi^{(s)}(t)\rangle$ . It may be impossible in practise to distinguish (via an appropriate measurement) the entangled state  $|\Psi_{SAB}(t)\rangle\langle\Psi_{SAB}(t)|$  from (a member of) a statistical ensemble of systems  $\mathcal{S} + \mathcal{A} + \mathcal{B}$  prepared in states  $|\Psi_{SAB}^{(s)}(t)\rangle$  with probability  $p_s = |c_s|^2$ . Actually, one can show in many models of system-bath interaction that  $\langle\Phi^{(s')}(t)|O|\Phi^{(s)}(t)\rangle \simeq 0$  for  $s \neq s'$  and times  $t \gg t_{\text{dec}}$  and for any local observable  $O$  of  $\mathcal{S} + \mathcal{A} + \mathcal{B}$ . It follows that measurements on such local observables give no information about the coherences  $c_s c_{s'}^* |\Psi_{SAB}^{(s)}(t)\rangle\langle\Psi_{SAB}^{(s')}(t)|$  for  $s \neq s'$ , which are present in the entangled state  $|\Psi_{SAB}(t)\rangle\langle\Psi_{SAB}(t)|$  but absent in the statistical mixture. Therefore one cannot make a distinction between the two states. Nevertheless, even if one realizes that measuring other (nonlocal) observables is an impossible task, so that the aforementioned coherences pertain to some set of “unavailable information”, it seems difficult to say that  $\mathcal{S} + \mathcal{A}$  is *really* in one of the states  $|s\rangle\langle s| \otimes \rho_A^{(s)}$ . One should keep in mind that the identification of the object-apparatus state with the reduced density matrix  $\rho_{AS}(t)$  amounts to identify a linear superposition with a statistical ensemble, i.e., to ignore some quantum correlations which cannot be measured but exist nevertheless. We have to face a much more subtle situation than in classical statistical physics, where one usually ignores some microscopic degree of freedom without being obliged to ignore at the same time some *fundamental* correlations. Let us quote D. Zeh ([9], chapter 2): “*Identifying the [system-apparatus] superposition with an ensemble of states (represented by a statistical operator  $\rho$ ) which merely leads to the same expectation values  $\langle O \rangle = \text{tr}(O\rho)$  for an axiomatically limited set of observables  $O$  (such as local ones) would obviously beg the question. This insufficient argument is nonetheless found widely in the literature (cf Haag 1992). It would be equivalent to a quantum mechanical state space smaller than required by a general superposition principle.*”

with  $\rho_A^{(s,s')} = \sum_i p_i |\chi_i^{(s)}\rangle \langle \chi_i^{(s')}|$ . The superposition principle (= THE postulate of quantum mechanics) thus makes it impossible that  $\rho_{SA}^{\text{post meas.}} = U \rho_{SA}^{(0)} U^\dagger$  for all object initial wavefunction  $|\psi\rangle$ . In other words, (3) *cannot be a unitary transformation*. If that bothers you, remember that irreversibility (and thus non-unitary dynamics) was to be expected from the very fact that  $\mathcal{A}$  is macroscopic and its coupling with its environment cannot be neglected (subsection 2.2). The dynamics of  $\mathcal{A}$  cannot be governed by the Schrödinger equation as for closed systems. It is also worth noting that the two states in the r.h.s. of (3) and (6) do not have the same von Neumann entropy: for instance, if the apparatus states  $\rho_A^{(0)}$  and  $\rho_A^{(s)}$  have all the same entropy<sup>3</sup>,  $\rho_{SA}^{\text{post meas.}}$  has a higher entropy than  $\rho_{SA}^{(0)}$  by the amount  $-\sum_s |c_s|^2 \ln |c_s|^2$ . Hence the dynamical process leading to (3) produces entropy if  $|\psi_0\rangle$  is not an eigenstate of  $S$ .

Another important difference between the two states in the r.h.s. of (3) and (6) is that the former is a separable state, i.e., they are no quantum correlations between  $\mathcal{S}$  and  $\mathcal{A}$ , whereas the latter is an object-apparatus entangled state. Let us recall that the apparatus wavefunctions  $|\chi_i^{(s)}\rangle$  correspond to quasi-classical states with expectation values  $x_{s,i} = \langle \chi_i^{(s)} | X | \chi_i^{(s)} \rangle \approx x_s$  differing from each other on a macroscopic scale for distinct  $s$  (requirements 1 and 2 in Sec. 2.1). Hence the state in the r.h.s. of (6) is a superposition of macroscopically distinguishable states (a so-called “Schödinger cat state”).

A second implication of (5) concerns the object-apparatus interaction Hamiltonian  $H_{SA}$ . In order that all eigenstates  $|s\rangle$  of  $S$  be left invariant by the object-apparatus interaction,  $H_{SA}$  must *commute with the measured observable*  $S$ . In all models studied below, we will consider Hamiltonians of the form  $H_{SA} = S \otimes P$  where  $P$  is a macro-observable of the apparatus (e.g.  $P = X$ ). Finally, the object Hamiltonian  $H_S$  does not play a significant role in an (ideal) QM: if  $S$  is not a constant of motion for the free dynamics of the object, i.e., if  $[S, H_S] \neq 0$ , in order to fulfil (5) one must assume that the typical time  $T_S$  of evolution of  $S$  under the dynamics implemented by  $H_S$  is much larger than the time duration  $t_{\text{meas}}$  of the measurement.

## 2.4 A simple example of apparatus: the Stern-Gerlach experiment

### 2.4.1 The three-partite system:

- *Quantum object*: spin 1/2 of an atom; the  $z$ -component of the spin  $S = \sigma_z$  is measured.
- *Pointer variable*: position  $X$  of the atom.
- *Environment*: screen (or fluctuations of the magnetic field in the magnet, or molecules in the air between the magnet and the screen scattering the atomic beam).

The magnetic field inside the magnet can be approximated at the vicinity of the line  $y = z = 0$  (along which the atoms move, see Fig. 2) by  $\vec{B}(x, y, z) \simeq (B_z(0) + \partial_z B_z(0)z)\vec{e}_z$  (we ignore the  $y$ -component of the inhomogeneous field). The object-pointer interaction Hamiltonian reads

$$H_{SA} = \mu_B \partial_z B_z(0) Z \sigma_z$$

where  $Z$  is the position operator of the atom along the  $z$ -direction and  $\mu_B$  the Bohr magneton. The constant part in  $\vec{B}$  contributes to the object Hamiltonian  $H_S = \mu_B B_z(0) \sigma_z$ . Since we ignore the component of the magnetic field in the  $x$  and  $y$  directions,  $\sigma_z$  is a constant of

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<sup>3</sup>It seems necessary that all the possible final states  $\rho_A^{(s)}$  have the same entropy in order to avoid any bias in the measurement produced by the apparatus.

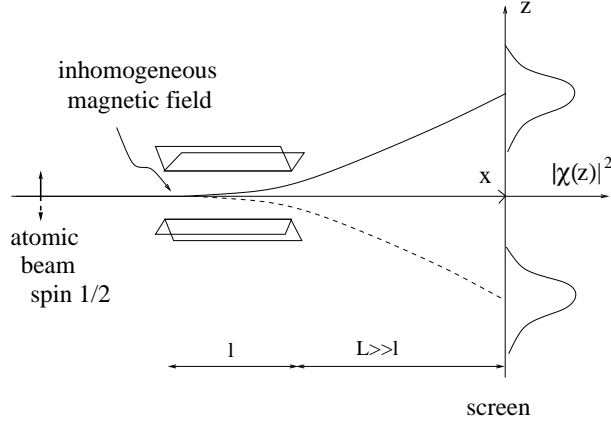


Figure 2: *Stern-Gerlach apparatus for measuring the spin of an atom.*

motion,  $[H_S, \sigma_z] = 0$ . The atom moves freely in a perfect vacuum, so that  $H_A = 0$ . The time spent by the atom in the magnet is  $t_{\text{int}} = lm_{\text{at}}/k_x$  ( $l$  is the length of the magnet,  $m_{\text{at}}$  the atomic mass, and  $\hbar = 1$ ).

### 2.4.2 Initial state

We assume that spin-orbit coupling is negligible, so that spin and position are uncorrelated before the atom enters the magnet (requirement 3 of Sec. 2.1). Before it enters the magnet, the spin of the atom is in an arbitrary linear superposition  $|\psi_0\rangle = c_+|\uparrow\rangle + c_-|\downarrow\rangle$  of eigenstates of  $\sigma_z$  and the atom is in a wavepacket  $|\chi^{(0)}\rangle$  with a sharply defined momentum  $\vec{k}^{(0)} = k_x^{(0)}\vec{e}_x$  along  $x$  and momentum uncertainties  $\Delta k_x^{(0)}$ ,  $\Delta k_y^{(0)}$  and  $\Delta k_z^{(0)}$ .

### 2.4.3 Dynamics

The crossing of the magnet entangles the spin and position degrees of freedom of the atom,

$$(c_+|\uparrow\rangle + c_-|\downarrow\rangle) \otimes |\chi^{(0)}\rangle \rightarrow c_+|\uparrow\rangle \otimes |\chi^{(+)}(t_{\text{int}})\rangle + c_-|\downarrow\rangle \otimes |\chi^{(-)}(t_{\text{int}})\rangle$$

where  $|\chi^{(\pm)}(t_{\text{int}})\rangle = e^{\mp i\mu_B \partial_z B_z(0)t_{\text{int}}Z}|\chi^{(0)}\rangle$  is a wavepacket with a sharply defined momentum  $\vec{k}_{\pm} = k_x^{(0)}\vec{e}_x \pm \mu_B \partial_z B_z(0)t_{\text{int}}\vec{e}_z$  (recall that  $Z$  is the generator of translations in the  $k_z$ -momentum space). After the exit of the magnet, the two wavepacket  $|\chi^{(\pm)}\rangle$  separate from each other as they propagate freely. The object-apparatus density matrix  $\rho_{SA}^{\text{ent}}$  just before the atom hits the screen (at time  $t_{\text{meas}}$ ) is given by

$$(i) \text{ Premeasurement: } \sum_{s,s'=\pm} c_s c_{s'}^* |s\rangle\langle s'| \otimes |\chi^{(0)}\rangle\langle\chi^{(0)}| \rightarrow \rho_{SA}^{\text{ent}} = \sum_{s,s'=\pm} c_s c_{s'}^* |s\rangle\langle s'| \otimes |\chi^{(s)}\rangle\langle\chi^{(s')}|$$

where  $|\chi^{(\pm)}\rangle$  are two wavepackets centred in position at  $z_{\pm} \simeq \pm \mu_B \partial_z B_z(0)t_{\text{int}}t_{\text{meas}}/m_{\text{at}}$ . By requirements 1 and 2 in Sec. 2.2, the distance  $d \approx 2\mu_B \partial_z B_z(0)t_{\text{int}}t_{\text{meas}}/m_{\text{at}}$  between the two centres of these wavepackets should be macroscopic, i.e., the distance  $L \simeq k_x^{(0)}t_{\text{meas}}/m_{\text{at}}$  between the magnet and the screen must be large. Moreover, the position uncertainty  $\Delta z$  of the two wavepackets should be much smaller than  $d$ . Taking into account the spreading of each wavepacket, it is easy to show that this is the case if [7]

$$\Delta k_z^{(0)} \ll \mu_B \partial_z B_z(0)t_{\text{int}}.$$

This condition means that the peaks in momentum of the two wavepackets can be resolved at the exit the magnet. The motion of the centres of the two wavepackets after the exit of the magnet is well described by the laws of classical mechanics.

If the atom was in state  $\rho_{SA}^{\text{ent}}$  at the end of the measurement this measurement would not give a definite answer. The last stage of the measurement is the decoherence process. The interaction of the atom with the molecules of the screen (or with any device measuring the position of the atom) will destroy the coherence between the two wavepackets. All information about the coherences in  $\rho_{SA}^{\text{ent}}$  is “transferred” during this interaction to the environment degrees of freedom and is irretrievably lost:

$$\begin{aligned} \text{(ii) Decoherence:} \quad \rho_{SA}^{\text{ent}} &= \sum_{s,s'=\pm} c_s c_{s'}^* |s\rangle\langle s'| \otimes |\chi^{(s)}\rangle\langle\chi^{(s')}|. \\ &\rightarrow \rho_{SA}^{\text{post meas.}} = \sum_{s=\pm} |c_s|^2 |s\rangle\langle s| \otimes |\chi^{(s)}\rangle\langle\chi^{(s)}|. \end{aligned} \quad (7)$$

Again, let us stress that the two atomic states  $\rho_{SA}^{\text{ent}}$  and  $\rho_{SA}^{\text{post meas.}}$  are quite different! For an atom in state  $\rho_{SA}^{\text{ent}}$ , one could in principle reconstruct a localised wavepacket and the spin state  $c_+|\uparrow\rangle + c_-|\downarrow\rangle$  by recombining the two beams. For instance if  $c_+ = c_- = 1/\sqrt{2}$ , this would lead to an eigenstate of  $\sigma_x$  with eigenvalue 1. However, for an atom in state  $\rho_{SA}^{\text{post meas.}}$  one would obtain the same mixed spin state  $|c_+|^2|\uparrow\rangle\langle\uparrow| + |c_-|^2|\downarrow\rangle\langle\downarrow|$  after having recombined the two beam as when they were separated. For any value of  $c_{\pm}$ , this state has a vanishing mean value of  $\sigma_x$ .



### 3 Lecture 2: Quantum object coupled to a pointer position

We discuss in this lecture the model studied in Ref [3, 4]. The quantum object is coupled to a single macroscopic variable of the measuring apparatus, e.g. its centre-of-mass position. A distinct pointer position is tied to each eigenvalue of the measured observable  $S$  of the object. The pointer is coupled to an infinite bath. The main novel features of the model are:

- (i) initial correlations between pointer and bath are taken into account by considering a pointer and a bath initially in a metastable local thermal equilibrium;
- (ii) unlike in the Stern-Gerlach apparatus of Sec. 2.4, object-pointer entanglement and decoherence of distinct pointer readouts proceed simultaneously; mixtures of macroscopically distinct object-pointer states may then arise without intervening macroscopic superpositions.

Our main goal is to determine the object-pointer dynamics; in particular, we shall give a quantitative treatment of decoherence which goes beyond the Markovian approximation.

#### 3.1 The model

##### 3.1.1 The three-partite system

- *Quantum object  $\mathcal{S}$* : any microscopic system  $\mathcal{S}$ ; the measured observable  $S$  has a discrete spectrum. We denote by  $H_S$  its Hamiltonian.
- *Pointer  $\mathcal{P}$* : it has one degree of freedom; the pointer variable is the position  $X$ . The Hilbert space is  $\mathcal{H}_P = L^2(\mathbb{R})$ . The pointer Hamiltonian is

$$H_P = \frac{P^2}{2M} + V(X) , \quad (8)$$

where  $P$  is the momentum conjugate to  $X$  and  $M$  the mass. We assume that the potential  $V(x)$  is even and has a local minimum at  $x = 0$ , i.e.,  $V'(0) = 0$  and  $V''(0) > 0$ . The height of the two potential barriers surrounding this minimum is supposed to be much larger than the thermal energy  $k_B T$  (see Fig. 3). This is necessary in order that the pointer has a well-defined rest state at  $x = 0$  (even if this may not be a global minimum of  $V$ ), see below. The object-pointer coupling Hamiltonian,

$$H_{SP} = \epsilon S \otimes P , \quad (9)$$

is chosen so as to (i) not change the measured observable  $S$  (i.e.,  $[H_{SP}, S] = 0$ , see Lecture 1, Sec. 2.3); (ii) be capable of shifting the pointer position by an amount proportional to  $S$ , such that each eigenvalue  $s$  of  $S$  becomes tied up with a specific pointer reading; (iii) involve a large coupling constant  $\epsilon$ , so that different eigenvalues  $s \neq s'$  end up associated with pointer readings separated by large distances.

- *Environment (bath  $\mathcal{B}$ )*: it includes all the other degrees of freedom  $\nu = 1, \dots, \mathcal{N}$  of the apparatus. Its Hilbert space is  $\mathcal{H}_B = \otimes_{\nu=1}^{\mathcal{N}} \mathcal{H}_\nu$ , where  $\mathcal{H}_\nu$  is the Hilbert space of the  $\nu$ th degree of freedom. We assume  $\mathcal{N} \gg 1$ ; all formulae below are meant to be valid in this limit<sup>4</sup>. The pointer-bath coupling Hamiltonian is

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<sup>4</sup>This means that we take  $\mathcal{N} \rightarrow \infty$  before all other limits, in particular, before the large time limit in Sec 3.5.2.

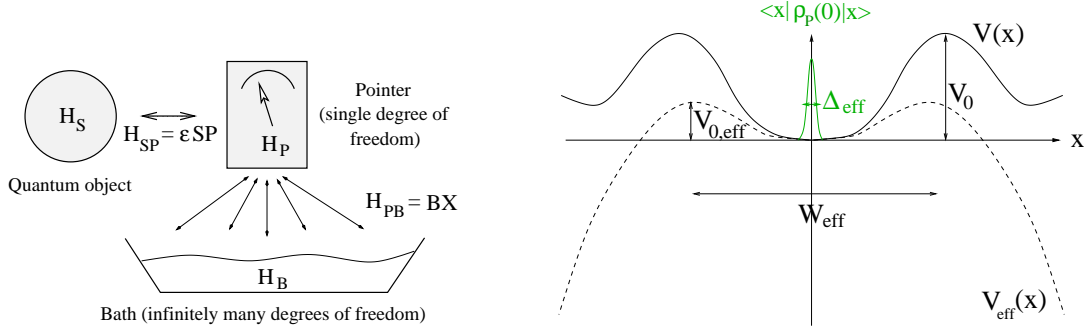


Figure 3: **Left:** the QM model of Ref [3, 4]. **Right:** the pointer potential  $V(x)$  (plain line) and effective potential  $V_{\text{eff}}(x)$  (broken lines) in arbitrary units. Given (16), the respective heights  $V_0$  and  $V_{0,\text{eff}}$  of the potential barriers of  $V(x)$  and  $V_{\text{eff}}(x)$  are roughly of the same order of magnitude, and so are the widths  $W \approx (V_0/V''(0))^{1/2}$  and  $W_{\text{eff}} \approx (V_{0,\text{eff}}/V''_{\text{eff}}(0))^{1/2}$  of the potential walls. We assume that  $V_0 \gg k_B T$ , i.e.,  $W \gg \Delta_{\text{th}} = (\beta V''(0))^{-1/2}$ . The density  $\langle x | \rho_P(0) | x \rangle$  of pointer position is represented in green; it has a width  $\Delta_{\text{eff}} \approx \Delta_{\text{th}} \ll W_{\text{eff}}$ .

$$H_{PB} = B \otimes X, \quad B = \mathcal{N}^{-1/2} \sum_{\nu=1}^{\mathcal{N}} B_{\nu}, \quad (10)$$

where the operators  $B_{\nu}$  act on the Hilbert space  $\mathcal{H}_{\nu}$ . More general Hamiltonians  $H_{PB} = B \otimes f(X)$  with  $f$  a smooth function can be considered<sup>5</sup>. The additivity of  $B$  in contributions  $B_{\nu}$  acting on single bath degree of freedom will allow us to invoke the quantum central limit theorem (QCLT) [16, 17]. We make no specific assumption on the bath Hamiltonian  $H_B$  excepted that there should be no long-range correlations in the free bath Gibbs state  $\rho_B^{\text{eq}} = Z_B^{-1} e^{-\beta H_B}$  at the inverse temperature  $\beta = (k_B T)^{-1}$  (more precisely,  $\text{tr}(B_{\mu} B_{\nu} \rho_B^{\text{eq}})$  should decay to zero faster than  $1/|\mu - \nu|$  for  $|\mu - \nu| \gg 1$  [17]). Such strong correlations would invalidate the QCLT. The QCLT implies Gaussian statistics (Wick theorem) for the time-correlation functions associated to  $B$  w.r.t. the free bath Gibbs state.

### 3.1.2 Separation of time scales

What are the different time scales in the model ?

- The characteristic time  $T_P$  for the motion of the pointer under its Hamiltonian  $H_P$  is defined as the period  $T_P = 2\pi(M/V''(0))^{1/2}$  of oscillations around the minimum of the potential  $V(x)$ .
- The characteristic time  $T_S$  for the evolution of the measured observable  $S$  under the object Hamiltonian  $H_S$ : by definition, this is the largest time  $T_S$  such that  $\langle \psi | \tilde{S}^0(t_1) \cdots \tilde{S}^0(t_n) | \psi \rangle$

<sup>5</sup>For a pointer in a homogeneous medium it would be more physical to choose a translation-invariant pointer-bath Hamiltonian, e.g.  $H_{PB} = \mathcal{N}^{-1/2} \sum_q (B_q + B_{-q}^{\dagger}) \otimes e^{iqX}$  (with  $q$  the momentum of the  $q$ -th bath mode and  $B_q \rightarrow B_q e^{iqa}$  under a space translation by a distance  $a$ ). For small enough separations between the pointer positions, such an Hamiltonian can be approximated by the Hamiltonian (10), which is not translation invariant but has the advantage of simplifying the calculation. The generalisation of the foregoing results to the case  $f(x) = x^{\alpha}$ ,  $\alpha \in \mathbb{N}^*$ , does not present any major difficulty, see [4].

$\simeq \langle \psi | S^n | \psi \rangle$  for  $|t_1|, \dots, |t_n| \ll T_S$ , where  $\tilde{S}^0(t) = e^{iH_S t} S e^{-iH_S t}$  and  $|\psi\rangle$  is the initial wavefunction of  $\mathcal{S}$ .

- The object-pointer interaction time  $t_{\text{int}}$ .
- The decoherence time  $t_{\text{dec}}$  associated with the decay of the coherences of the object-pointer density matrix, i.e., the characteristic time of the dynamical process (7) in Lecture 1.

Finally, there are 3 (not necessarily independent) times scales are associated with the bath.

- The time  $t_B$  is defined in analogy with  $T_S$  in terms of the correlation function  $K(t)$  of the bath coupling agent  $B$  w.r.t. the free bath thermal state,

$$K(t) = \text{tr}(\tilde{B}(t) B \rho_B^{(\text{eq})}) = K(-t)^*, \quad \tilde{B}(t) = e^{iH_B t} B e^{-iH_B t}, \quad \rho_B^{(\text{eq})} = Z_B^{-1} e^{-\beta H_B}. \quad (11)$$

Here and in what follows we assume  $\text{tr}(B \rho_B^{(\text{eq})}) = 0$ . Then  $t_B$  is the largest time such that  $K(t) \simeq K(0)$  for  $|t| \ll t_B$ . By Wick theorem (see Sec. 3.3), higher-order correlation functions  $\text{tr}(\tilde{B}(t_1) \dots \tilde{B}(t_n) \rho_B^{(\text{eq})})$  can also be approximated by their values at times  $t_1, \dots, t_n = 0$  for  $|t_1|, \dots, |t_n| \ll t_B$ .

- The thermal time  $\beta = (k_B T)^{-1}$ .
- The bath correlation time  $T_B$  is the smallest time such that  $K(t) \simeq 0$  for  $t \gg T_B$ . Note that for an infinite bath ( $\mathcal{N} \rightarrow \infty$ ),  $K(t)$  indeed decays to zero at large times and thus  $T_B < \infty$ . For a bath in thermal equilibrium, the thermal time figures among the decay rates of  $K(t)$  and thus  $t_B \leq \beta \leq T_B$ . The equality  $T_B = \beta$  holds at low enough temperature.

As we already mentioned in Sec. 2.3, in an ideal measurement one must have  $t_{\text{int}}, t_{\text{dec}} \ll T_S$ , i.e., the free dynamics of  $\mathcal{S}$  remains ineffective on  $S$  during the measurement. Moreover, the pointer time scale  $T_P$  is a classical time which is much larger than all the other (quantum) time scales in the model. In particular,  $T_P$  is much larger than the thermal time  $\beta$ . In an ideal measurement we thus have the following important separation of time scales

$$t_{\text{dec}}, t_{\text{int}} \ll T_S, \quad t_{\text{dec}}, t_{\text{int}}, \beta \ll T_P. \quad (12)$$

### 3.2 Initial state

In view of the requirement 3 of Sec. 2.1, we consider an initial state having no correlation between  $\mathcal{S}$  and  $\mathcal{P} + \mathcal{B}$ . Assuming that  $\mathcal{S}$  is in a pure state  $|\psi\rangle$  with components  $c_s$  in the eigenbasis  $\{|s\rangle\}$  of  $S$ , the object-pointer-bath initial state is

$$\rho_{SPB}(0) = |\psi\rangle\langle\psi| \otimes \rho_{PB}(0), \quad |\psi\rangle = \sum_s c_s |s\rangle. \quad (13)$$

Here,  $\rho_{PB}(0)$  is a *metastable local pointer-bath thermal equilibrium* at inverse temperature  $\beta$ . In this local equilibrium the pointer is localised near  $x = 0$ . With an abuse of notation,  $\rho_{PB}(0) = Z_{PB}^{-1} e^{-\beta(H_P + H_B + H_{PB})}$ . Note that in such a state  $\mathcal{P}$  and  $\mathcal{B}$  are correlated. By invoking the high-temperature limit  $\beta \ll T_P$  and the Gaussian statistics of  $B$  (as implied by the QCLT) and by

tracing out the bath we find (see below) the reduced initial density matrix  $\rho_P(0) = \text{tr}_B(\rho_{PB}(0))$  of the pointer in the position representation,

$$\langle x | \rho_P(0) | x' \rangle \propto e^{-\beta(V_{\text{eff}}(x) + V_{\text{eff}}(x'))/2} e^{-2\pi^2(x-x')^2/\lambda_{\text{th}}^2} \quad (14)$$

where  $\lambda_{\text{th}} = 2\pi(\beta/M)^{1/2}$  is the thermal de Broglie wavelength. The pointer potential appears renormalised by the pointer-bath interaction as

$$V_{\text{eff}}(x) = V(x) - \gamma_0 x^2, \quad \gamma_0 = \int_{-\infty}^0 dt \Im K(t) \geq 0. \quad (15)$$

For local stability of the whole apparatus, the pointer-bath coupling must be weak enough so that  $V_{\text{eff}}''(0) > 0$ ; we even bound the latter curvature finitely away from zero by, say,  $V_{\text{eff}}''(0) > V''(0)/2$ , i.e.,

$$\gamma_0 < V''(0)/4. \quad (16)$$

This makes sure that the initial density of pointer positions has a single peak at  $x = 0$  with a renormalised width  $\Delta_{\text{eff}} = [\beta(V''(0) - 2\gamma_0)]^{-1/2}$  of the order of the bare thermal fluctuation  $\Delta_{\text{th}} = (\beta V''(0))^{-1/2}$ ,

$$\langle x | \rho_P(0) | x \rangle \propto e^{-\beta V_{\text{eff}}(x)} \simeq \exp\left(-\frac{x^2}{2\Delta_{\text{eff}}^2}\right) \quad \text{for} \quad |x| \lesssim \Delta_{\text{eff}}. \quad (17)$$

Condition (16) is fulfilled for a pointer-bath coupling satisfying

$$\Delta_{\text{th}}^2 \beta^2 \langle B^2 \rangle < 1/2, \quad \langle B^2 \rangle = \text{tr}(B^2 \rho_B^{\text{(eq)}}) = K(0). \quad (18)$$

This is a consequence of the inequality  $\gamma_0 \leq \beta \langle B^2 \rangle / 2$  which follows from general properties of bath correlation functions [4].

If  $V(x) = o(x^2)$  at large distances  $|x| \gtrsim W$ , the effective potential  $V_{\text{eff}}(x)$  is unstable. The matrix elements (14) then correspond to (the reduced pointer state of) a metastable local thermal equilibrium. That local equilibrium for the apparatus can be achieved by preparing  $\mathcal{P}$  in some state localised near  $x = 0$  at time  $t = -t_i$  and then letting it interact with  $\mathcal{B}$  between  $t = -t_i$  and  $t = 0$ . If the thermalization time is small compared with the tunnelling escape time, one may choose  $t_i$  larger than the former but much smaller than the latter time, so that  $\mathcal{P}$  is still within the effective potential well when the measurement starts at  $t = 0$ . In order to be able to prepare the apparatus in such a local equilibrium, the height  $V_{0,\text{eff}}$  of the two potential barriers surrounding the local minimum of the effective potential at  $x = 0$  must be large compared with the thermal energy  $\beta^{-1}$ . Thanks to (16), this is the case provided that the bare potential  $V(x)$  satisfies the same requirement, i.e.,  $V_0 \gg \beta^{-1}$  (see Fig. 3)<sup>6</sup>. Interestingly,  $V(x)$  can be chosen such that the two potential barriers of  $V_{\text{eff}}(x)$  are separated by a mesoscopic distance  $W_{\text{eff}} \approx (V_{0,\text{eff}}/V_{\text{eff}}''(0))^{1/2} \gg \Delta_{\text{eff}}$  (so that  $V_{0,\text{eff}} \gg \beta^{-1}$ ) which is small compared with the macroscopic read-out scale  $\Delta_{\text{class}}$ . The object-pointer interaction then just has to get the pointer out of the well, leaving the subsequent displacement growth to the action

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<sup>6</sup>For a pointer-bath coupling Hamiltonian of the form  $H_{PB} = B \otimes X^\alpha$  with  $\alpha > 1$ , the instability can be entirely due to the coupling. Actually, one can show [4] that  $V_{\text{eff}}(x) = V(x) - \gamma_0 x^{2\alpha}$ . Thus  $V_{\text{eff}}(x)$  is unstable even in the case of a bare potential  $V(x) \propto x^2$  for all  $x$ . In order to have a local equilibrium, one must then replace condition (18) by the stronger condition  $\eta^2 = \Delta_{\text{th}}^{2\alpha} \beta^2 \langle B^2 \rangle \ll 1$  (this insures that  $V_{0,\text{eff}} \approx (V''(0)^\alpha / \gamma_0)^{1/(\alpha-1)} \gg \beta^{-1}$ ). Moreover,  $W_{\text{eff}} \approx (V''(0)/(2\alpha\gamma_0))^{1/(2\alpha-2)}$  met the criterion (19) for sufficiently large pointer-bath coupling. If  $(\beta \langle B^2 \rangle / \gamma_0)(V_0 \beta)^{1-\alpha} \ll \eta^2 \ll 1$ , then the width  $W_{\text{eff}}$  of the effective potential is much smaller than that of the bare potential,  $W_{\text{eff}} \ll W$ .

of the effective potential. The instability of the effective potential (15) hence provides the amplification mechanism necessary to fulfil the requirement 1 of Sec. 2.1. For a macroscopic pointer at high temperature ( $\beta \ll T_P$ ), the different length scales are ordered as

$$\lambda_{\text{th}} \ll \Delta_{\text{th}} \approx \Delta_{\text{eff}} \ll W_{\text{eff}} \ll \Delta_{\text{class}} . \quad (19)$$

The first limit is equivalent to  $\beta \ll T_P$ . To fix ideas, for  $T_P = 1$  s,  $M = 1$  g and a temperature of 1 K, one has  $\lambda_{\text{th}} \approx 10^{-21}$  m and  $\Delta_{\text{th}} \approx 10^{-10}$  m.

### 3.3 Dynamics

We outline in this section the main steps in the calculation of the object-pointer density matrix. We skip some technical details related to the Quantum Central Limit Theorem (QCLT) and to general properties of the bath correlation function (11). A more complete derivation can be found in Ref.[4].

The object-pointer density matrix at time  $t$  is obtained by tracing out the bath degrees of freedom in the time-evolved density matrix of  $\mathcal{S} + \mathcal{P} + \mathcal{B}$  (see Lecture 1, Sec. 2.2), i.e.,

$$\rho_{PS}(t) = \text{tr}_B(e^{-iHt} \rho_{SPB}(0) e^{iHt}), \quad H = H_S + H_P + H_B + H_{SP} + H_{PB} . \quad (20)$$

We first simplify the time evolution operator at time  $t \ll T_P, T_S$  as  $e^{-iHt} \simeq U(t) e^{-i(H_S + H_P)t}$ , with

$$U(t) = e^{-i(H_B + H_{SP} + H_{PB})t} = e^{-iH_B t} e^{-i\epsilon S \otimes P t} \mathcal{T} e^{-i \int_0^t d\tau (X + \epsilon S \tau) \tilde{B}(\tau)} . \quad (21)$$

In the last identity in (21),  $\mathcal{T}$  denotes time ordering; this identity is a simple consequence of the relation  $e^{i\epsilon S \otimes P t} X e^{-i\epsilon S \otimes P t} = X + \epsilon S t$  (the momentum  $P$  being the generator of pointer displacements in position). Similarly, we invoke  $\langle s, x | e^{-i\epsilon S \otimes P t} = \langle s, x - \epsilon s t |$ , the cyclicity of the trace and the product initial state (13) to get the matrix elements of  $\rho_{PS}(t)$  in the joint eigenbasis  $\{|s, x\rangle\}$  of  $S$  and  $X$ ,

$$\langle s, x | \rho_{PS}(t) | s', x' \rangle \simeq \langle s | \psi^0(t) \rangle \langle \psi^0(t) | s' \rangle \langle x | \rho_P^{(ss')}(t) | x' \rangle, \quad t \ll T_S, T_P, \quad (22)$$

with

$$|\psi^0(t)\rangle = e^{-iH_S t} |\psi\rangle \quad (23)$$

evolving as for the free object, while the pointer matrix elements

$$\langle x | \rho_P^{(ss')}(t) | x' \rangle = \langle x_s(t) | \text{tr}_B \left( \tilde{U}_{sx}(t) e^{-iH_P t} \rho_{PB}(0) e^{iH_P t} \tilde{U}_{s'x'}^\dagger(t) | x'_{s'}(t) \rangle \right) \quad (24)$$

involve the bath evolution operator and shifted positions

$$\tilde{U}_{sx}(t) = \mathcal{T} \exp \left\{ -i \int_0^t d\tau x_s(t - \tau) \tilde{B}(\tau) \right\}, \quad x_s(t) = x - \epsilon s t, \quad x'_{s'}(t) = x' - \epsilon s' t. \quad (25)$$

To evaluate the matrix elements (24) we use the high-temperature approximation

$$\rho_{PB}(0) \simeq Z_{PB}^{-1} e^{-\beta H_P / 2} e^{-\beta(H_B + H_{PB})} e^{-\beta H_P / 2}, \quad \beta \ll T_P \quad (26)$$

for the pointer-bath Gibbs state. Given the weak-coupling condition, the error incurred is  $\mathcal{O}(\beta^2/T_P^2)$ , as easily seen from the Baker-Campbell-Hausdorff formula. One can also show [4] that  $e^{-iH_P t} \rho_{PB}(0) e^{iH_P t} \simeq \rho_{PB}(0)$  for  $t \ll T_P$ , i.e., the pointer Hamiltonian  $H_P$  can be neglected

in (24). The situation is different for the object Hamiltonian  $H_S$ . This Hamiltonian cannot be neglected in (23) even for  $t \ll T_S$ . However, one has<sup>7</sup> for such times  $|\langle s|\psi^0(t)\rangle| \simeq |c_s|$  for all  $s$ . Thus

$$\langle x|\rho_P^{(ss')}(t)|x'\rangle = Z_{\mathcal{PB}}^{-1} \int_{-\infty}^{\infty} dy \langle x_s(t)|e^{-\beta H_P/2}|y\rangle \langle y|e^{-\beta H_P/2}|x'_{s'}(t)\rangle Z_{B,y} \langle \tilde{U}_{s'x'}(t)^\dagger \tilde{U}_{sx}(t) \rangle_y \quad (27)$$

where  $\langle \cdot \rangle_y$  is the bath average w.r.t. a  $y$ -dependent renormalised bath state,

$$\langle \cdot \rangle_y = \text{tr}(\cdot \rho_{B,y}) \quad , \quad \rho_{B,y} = Z_{B,y}^{-1} e^{-\beta(H_B+yB)} . \quad (28)$$

Let us determine the normalisation factor

$$Z_{B,y} = \text{tr}_B(e^{-\beta(H_B+yB)}) = Z_{B,0} \left\langle \mathcal{T} \exp \left\{ -y \int_0^\beta dz \tilde{B}(-iz) \right\} \right\rangle_0 , \quad \tilde{B}(-iz) = e^{zH_B} B e^{-zH_B} . \quad (29)$$

By virtue of the QCLT and the additivity of the bath coupling agent  $B$  in contributions  $B_\nu$  acting on single degree of freedom, see (10), the  $n$ -point correlation functions associated to  $B$  satisfy the bosonic Wick theorem in the limit  $\mathcal{N} \rightarrow \infty$  [16]. That is,  $\langle \tilde{B}(t_1) \dots \tilde{B}(t_n) \rangle_0$  vanishes for odd  $n$  (since we have assumed  $\langle B \rangle_0 = 0$ ) and is given for even  $n$  by the sum over all pairings  $\{(i_1, j_1), \dots, (i_{n/2}, j_{n/2})\}$  of  $\{1, \dots, n\}$  of products  $K(t_{i_1} - t_{j_1}) \dots K(t_{i_{n/2}} - t_{j_{n/2}})$  of 2-point correlators. One can show [4] that Wick theorem is equivalent to the identity

$$\begin{aligned} & \left\langle \left[ \mathcal{T} \exp \left\{ -i \int_0^t d\tau k(\tau) \tilde{B}(\tau) \right\} \right]^\dagger \mathcal{T} \exp \left\{ -i \int_0^t d\tau l(\tau) \tilde{B}(\tau) \right\} \right\rangle_0 \\ &= \exp \left\{ - \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 (k(\tau_1) - l(\tau_1)) (k(\tau_2) K(\tau_1 - \tau_2)^* - l(\tau_2) K(\tau_1 - \tau_2)) \right\} , \end{aligned} \quad (30)$$

where  $k(\tau)$  and  $l(\tau)$  are two arbitrary real-valued functions. By applying (30) with  $t = -i\beta$ ,  $k(\tau) = 0$  and  $l(\tau) = y$ , one gets

$$Z_{B,y} = e^{\beta \gamma_0 y^2} Z_{B,0} , \quad \gamma_0 = \frac{1}{\beta} \int_0^\beta dz_1 \int_0^{z_1} dz_2 K(-iz_2) . \quad (31)$$

The fact that  $\gamma_0$  is also given by the r.h.s. of the second formula in (15) can be established by using the analyticity and KMS properties of the bath correlator  $K(\tau)$  [4]. We shall, however, only need later on that  $\gamma_0 \geq 0$ .

At this point we momentarily pause with dynamics and show that at  $t = 0$ , when  $\tilde{U}_{s'x'} = \tilde{U}_{sx} = 1$ ,  $x_s = x$  and  $x'_{s'} = x'$ , (27) yields the initial pointer state announced in (14). To that end we invoke the high temperature limit  $\beta \ll T_P$  again to approximate the matrix element  $\langle x|e^{-\beta H_P/2}|y\rangle$  by

$$\langle x|e^{-\beta H_P/2}|y\rangle \simeq e^{-\beta(V(x)+V(y))/4} e^{-4\pi^2(x-y)^2/\lambda_{\text{th}}^2} . \quad (32)$$

The reader may recognise in this expression the short-time behaviour of the quantum propagator  $\langle x|e^{-itH_P}|x'\rangle$  for  $t = -i\beta/2$  (see e.g. [19]). Replacing  $V(y)$  by  $V''(0)y^2/2$  in (27), using (31) and doing the Gaussian  $y$ -integral, we arrive at the initial state (14) by neglecting terms  $\mathcal{O}(\lambda_{\text{th}}^2/\Delta_{\text{th}}^2, \lambda_{\text{th}}^2/\Delta_{\text{eff}}^2)$ .

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<sup>7</sup>For indeed, for all positive integer  $n$ ,  $\langle \psi|\tilde{S}^0(t)^n|\psi\rangle = \sum_s |\langle s|\psi^0(t)\rangle|^2 s^n$  is almost equal at time  $t \ll T_S$  to its value  $\sum_s |c_s|^2 s^n$  at time  $t = 0$ . This follows from the definition of  $T_S$ , see Sec. 3.1.2.

Let us return to the time-evolved pointer matrix (27). The QCLT implies that the bath coupling agent  $B$  has also a Gaussian statistics for the modified average (28), with a variance independent of  $y$  [4],

$$\langle \tilde{B}(t)\tilde{B}(t') \rangle_y - \langle \tilde{B}(t) \rangle_y \langle \tilde{B}(t') \rangle_y = K(t-t'). \quad (33)$$

If one replaces the average  $\langle \cdot \rangle_0$  by  $\langle \cdot \rangle_y$  and  $\tilde{B}(\tau)$  by  $\tilde{B}(\tau) - \langle \tilde{B}(t) \rangle_y$  in (30), the l.h.s. then remains unchanged: it is equal to the r.h.s. with an unmodified bath correlator  $K(\tau)$ . Setting  $k(\tau) = x'_{s'}(t-\tau)$  and  $l(\tau) = x_s(t-\tau)$  in this r.h.s., this yields

$$\begin{aligned} \langle \tilde{U}_{s'x'}(t)^\dagger \tilde{U}_{sx}(t) \rangle_y &= \exp \left\{ -D_t(x_s(t), x'_{s'}(t); s, s') - i\phi_t(x, x'; s, s') \right\} \\ &\times \exp \left\{ i \int_0^t d\tau \langle \tilde{B}(t-\tau) \rangle_y (x'_{s'}(\tau) - x_s(\tau)) \right\} \end{aligned} \quad (34)$$

with a *non negative decoherence exponent*  $D_t$  given by

$$D_t(x, x'; s, s') = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \Re K(\tau_1 - \tau_2) (x'_{s'}(-\tau_1) - x_s(-\tau_1)) (x'_{s'}(-\tau_2) - x_s(-\tau_2)) \quad (35)$$

and an ( $y$ -independent) real phase  $\phi_t$  irrelevant for decoherence. The non-negativity of  $D_t$  is a consequence of the fact that  $K(\tau)$  and its real part  $\Re K(\tau)$  are of positive type, i.e., they have nonnegative Fourier transforms  $\widehat{K}(\omega)$  and  $\widehat{(\Re K)}(\omega)$ . Actually, by using the parity property  $\Re K(\tau) = \Re K(-\tau)$  of the bath correlator, one may rewrite (35) as

$$D_t(x, x'; s, s') = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \widehat{(\Re K)}(\omega) \left| \int_0^t d\tau (x'_{s'}(-\tau) - x_s(-\tau)) e^{-i\omega\tau} \right|^2 \geq 0. \quad (36)$$

It turns out that the phase factor  $\langle \tilde{U}_{s'x'}(t)^\dagger \tilde{U}_{sx}(t) \rangle_y / \langle \tilde{U}_{s'x'}(t)^\dagger \tilde{U}_{sx}(t) \rangle_0$  (factor in the second line of (34)) entails nothing but a correction of relative order  $(\lambda_{\text{th}}/\Delta_{\text{eff}})^2$  to the decoherence exponent  $D_t$  under the stability condition (16) [4]. Dropping that correction, the  $y$ -integral in (27) reduces to the initial pointer density matrix (14), albeit with additional decoherence and phase factors  $e^{-D_t} e^{-i\phi_t}$  reflecting the action of the pointer-bath coupling. The action of the object-pointer coupling manifests itself in the shifted pointer positions  $x \rightarrow x_s(t) = x - \epsilon s t$  and  $x' \rightarrow x'_{s'}(t) = x' - \epsilon s' t$ . Thanks to (22), our final result for the object-pointer state at time  $t \ll T_S, T_P$  is

$$\langle s, x | \rho_{PS}(t) | s', x' \rangle = \langle s | \psi^0(t) \rangle \langle \psi^0(t) | s' \rangle \langle x_s(t) | \rho_P(0) | x'_{s'}(t) \rangle e^{-D_t(x_s(t), x'_{s'}(t); s, s')} e^{-i\phi_t} \quad (37)$$

with the notations specified in (14), (23), (25) and (35). Entanglement and decoherence contribute separately in that remarkably simple “final state”; they lead respectively to the third and fourth factors in (37).

Let us stress that the aforementioned results (in particular (35)) are exact (not lowest order in the pointer-bath coupling). They are consequences of Wick theorem as implied by the QCLT, the additivity (10) of the bath coupling agent  $B$  and the infinite bath limit  $\mathcal{N} \gg 1$ . Direct proofs of (31), (33) and (35) are easy in the particular case of a bath composed of harmonic oscillators linearly coupled to  $\mathcal{P}$ .

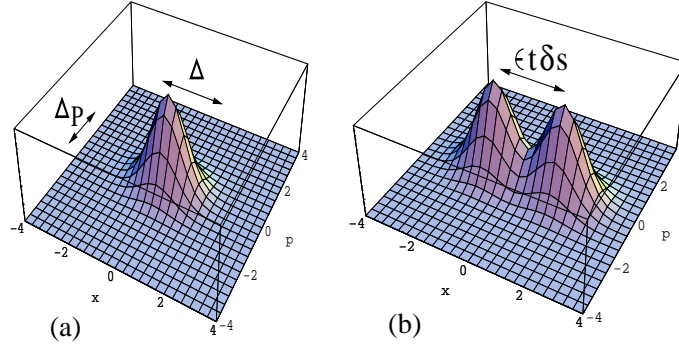


Figure 4: Wigner function  $W_P(x, p; t)$  of the pointer reduced state (38) at times (a)  $t = 0$  and (b)  $t \approx t_{\text{ent}}$ ;  $S$  is a spin one-half with two eigenvalues  $\pm \delta s/2$ . We have set the pointer-bath coupling to zero, so that  $D_t = 0$  in (39). In the horizontal axis, position and momentum are measured in units of  $\Delta$  and  $2\Delta p$ ; in the vertical axis, units are such that  $W_P(x, p; t)$  has maximum value 1.

### 3.4 Entanglement and decoherence times

Let us first look at the reduced pointer state. In view of (37), it is given at time  $t \ll T_S, T_P$  by (recall that  $|\langle s | \psi^0(t) \rangle| \simeq |c_s|$  for  $t \ll T_S$ )

$$\rho_P(t) = \text{tr}_S(\rho_{SP}(t)) \simeq \sum_s |c_s|^2 \rho_P^{(s)}(t) \quad (38)$$

where the pointer matrix elements

$$\langle x | \rho_P^{(s)}(t) | x' \rangle = \langle x - \epsilon s t | \rho_P(0) | x' - \epsilon s t \rangle e^{-D_t(x_s(t), x'_s(t); s, s) - i\phi_t} \quad (39)$$

correspond to the initial pointer state  $\rho_P(0)$  shifted by  $\epsilon s t$  in position (i.e.,  $e^{-i\epsilon s t P} \rho_P(0) e^{i\epsilon s t P}$ ), up to an additional phase factor and a decoherence factor

$$e^{-D_t(x_s(t), x'_s(t); s, s)} = \exp \left\{ -\frac{(x - x')^2}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 \Re K(\tau_1 - \tau_2) \right\}. \quad (40)$$

For a fixed  $s$ ,  $\langle x | \rho_P^{(s)}(t) | x \rangle = \langle x_s(t) | \rho_P(0) | x_s(t) \rangle$  has a narrow peak of width  $\Delta_{\text{eff}}$  centred around  $x = \epsilon t s$ . The probability density  $\langle x | \rho_P(t) | x \rangle$  in pointer position has thus a peak centred at  $x = \epsilon t s$  for each eigenvalue  $s$  present in the (decomposition in the  $|s\rangle$ -basis of the) object initial state  $|\psi\rangle$ . In the case of an observable  $S$  with two eigenvalues  $s = \pm \delta s/2$ , the pointer state  $\rho_P(t)$  has the shape represented in Fig. 4. Two peaks in the pointer density  $\langle x | \rho_P(t) | x \rangle$  associated to distinct eigenvalues  $s$  and  $s'$  begin to be resolved at the *entanglement time*

$$t_{\text{ent}}(s, s') = \frac{\Delta_{\text{eff}}}{\epsilon |s - s'|}. \quad (41)$$

Recalling that  $S$  has a discrete spectrum, we denote by  $\delta s$  the minimum of  $|s - s'|$  over all pairs  $(s, s')$  of eigenvalues present in the object initial state (i.e., such that  $c_s = \langle s | \psi \rangle \neq 0$  and  $c_{s'} \neq 0$ )<sup>8</sup>. At time  $t > t_{\text{ent}} = \Delta_{\text{eff}}(\epsilon \delta s)^{-1}$ , neighbouring peaks of the pointer densities can be

<sup>8</sup>We suppose that  $c_s = 0$  if  $s$  belongs to a part of the spectrum containing arbitrarily close eigenvalues, near an accumulation point, so that  $\delta s > 0$ .



resolved. Each eigenvalue  $s$  of the measured observable  $S$  is then uniquely tied up with “its” pointer position  $\epsilon st$ .

The object-pointer entanglement in the state  $\rho_{PS}(t)$  comes from the off-diagonal ( $s \neq s'$ ) contributions in (37). Given the narrow peaks at  $x = x' = 0$  (of width  $\Delta_{\text{eff}}$ ) of the initial pointer density matrix (14),  $\langle x_s(t) | \rho_P(0) | x_{s'}(t) \rangle$  and thus  $\langle s, x | \rho_{PS}(t) | s', x' \rangle$  almost vanish when  $|x_s(t)| \geq \Delta_{\text{eff}}$  or  $|x_{s'}(t)| \geq \Delta_{\text{eff}}$ . One can thus appreciate the fate of the  $s \neq s'$  coherences in the final state (37) by setting  $x_s(t) = x_{s'}(t) = 0$  there. The decoherence factor then reads

$$e^{-D_t^{\text{peak}}(s,s')} = \exp \left\{ -\frac{\epsilon^2(s-s')^2}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 \tau_1 \tau_2 \Re K(\tau_1 - \tau_2) \right\} \quad (42)$$

and reveals irreversible decay as soon as the time  $t$  much exceeds the *decoherence time*  $t_{\text{dec}}(s, s')$ . We may define that time implicitly as  $D_{t_{\text{dec}}}^{\text{peak}}(s, s') = 1$ . We will show in the next paragraph that  $D_t^{\text{peak}}(s, s')$  is a *positive increasing convex function of time* if  $s \neq s'$  (see the inset in Fig. 5). The decoherence and entanglement times are related by

$$\left( \frac{t_{\text{ent}}(s, s')}{\eta} \right)^2 = \frac{1}{\beta^2} \int_0^{t_{\text{dec}}(s,s')} d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_1 \tau_2 \frac{\Re K(\tau_1 - \tau_2)}{\langle B^2 \rangle} \quad (43)$$

where  $\langle B^2 \rangle = \text{tr}(B^2 \rho_B^{\text{(eq)}})$  and

$$\eta = \langle B^2 \rangle^{1/2} \Delta_{\text{eff}} \beta \quad (44)$$

is a dimensionless measure of the strength of the pointer-bath coupling<sup>9</sup>. Note that  $\eta \lesssim 1$  by condition (18). It follows from the increasing property of  $D_t^{\text{peak}}$  that the largest decoherence time  $t_{\text{dec}}(s, s')$  for all pairs  $(s, s')$  of eigenvalues present in the object initial state  $|\psi\rangle$  is  $t_{\text{dec}} = t_{\text{dec}}(s, s + \delta s)$ , i.e.,  $t_{\text{dec}}$  is given by replacing  $t_{\text{ent}}(s, s')$  by  $t_{\text{ent}} = t_{\text{ent}}(s, s + \delta s) = \Delta_{\text{eff}}(\epsilon \delta s)^{-1}$  in (43). We also set  $D_t^{\text{peak}} = D_t^{\text{peak}}(s, s + \delta s)$  and write the subsequent formulae for  $t_{\text{dec}}$  and  $t_{\text{ent}}$ ; all these formulae remain valid upon substitution of  $t_{\text{dec}}$  and  $t_{\text{ent}}$  by  $t_{\text{dec}}(s, s')$  and  $t_{\text{ent}}(s, s')$ , for arbitrary  $s \neq s'$ .

We now prove that  $D_t^{\text{peak}}$  is an increasing convex function of  $t$ . We take  $x = x' = 0$  in (36), differentiate both sides with respect to  $t$ , and do the time integral by parts to get

$$\frac{\partial}{\partial t} D_t^{\text{peak}} = \epsilon^2 \delta s^2 t \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{(\widehat{\Re K})(\omega)}{\omega^2} (1 - \cos(\omega t)) . \quad (45)$$

Bearing in mind that  $(\widehat{\Re h})(\omega) \geq 0$ , this establishes that  $\partial D_t^{\text{peak}} / \partial t > 0$  for  $t > 0$ . Hence  $D_t^{\text{peak}}$  is an increasing function of  $t$ . By a similar argument,  $\partial^2 D_t^{\text{peak}} / \partial t^2 > 0$  and thus  $D_t^{\text{peak}}$  is convex.

Let us recall that the results (35)-(43) are valid provided that  $t$  and  $t_{\text{dec}}$  remain much smaller than both  $T_S$  and  $T_P$ , a condition that must be checked *a posteriori*. The left part of Fig. 5 shows  $t_{\text{dec}}/\beta$  as a function of  $t_{\text{ent}}/(\beta\eta)$ . We choose here a specific bath correlator  $K(t)/\langle B^2 \rangle$  given by  $\widehat{\Re K}(\omega) = i \coth(\beta\omega/2) \widehat{\Im K}(\omega)$  (KMS relation) and  $\widehat{\Im K}(\omega) = -i \hat{\gamma} \omega^m \exp\{-(\omega/\omega_D)^2\}$  with  $\omega_D = 5/\beta$ . The larger decay time of  $K(t)$  is then the thermal time  $T_B = \beta > \omega_D^{-1}$ . This choice corresponds to a bath of harmonic oscillators linearly coupled to the pointer position  $X$  (i.e.,  $H_B = \sum_{\nu} \omega_{\nu} b_{\nu}^{\dagger} b_{\nu}$  and  $B = \sum_{\nu} (\kappa_{\nu} b_{\nu}^{\dagger} + \kappa_{\nu}^* b_{\nu}) / \sqrt{N}$ , where  $\omega_{\nu}$  is the frequency and  $\kappa_{\nu}$  the coupling constant of the  $\nu$ th oscillator) with a power spectrum function  $J(\omega) = \hat{\gamma} \omega^m \exp\{-(\omega/\omega_D)^2\}$

<sup>9</sup>More precisely,  $\eta$  is the fluctuation of the pointer-bath coupling energy in state  $\rho_P(0) \otimes \rho_B^{\text{(eq)}}$ , in units of  $\beta^{-1} = k_B T$ .

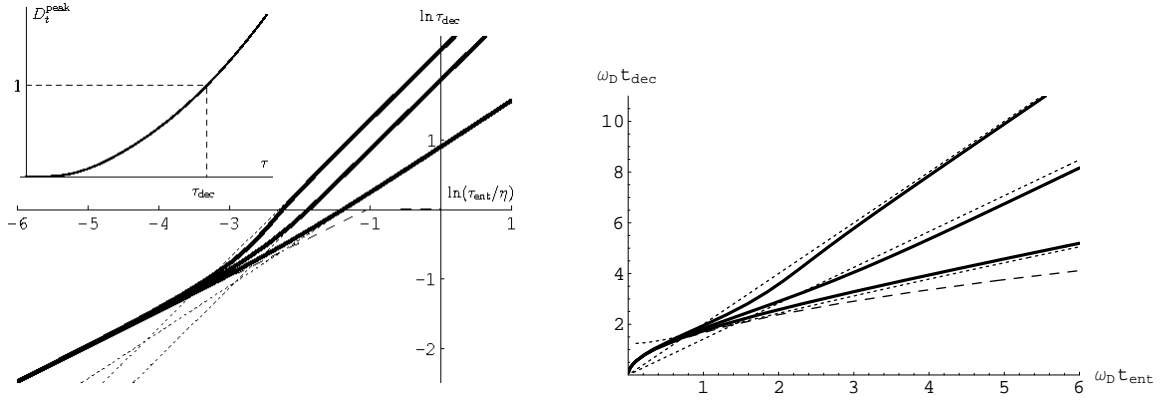


Figure 5: **Left:** Solid curves: Decoherence time  $\tau_{\text{dec}} = t_{\text{dec}}/T_B$  in units of  $T_B$  against  $t_{\text{ent}}/(\eta T_B)$  in a log-log scale. The bath correlator is chosen as indicated in the text with  $\omega_D = 5/\beta$  and  $m = 5, 3, 1$  (from left to right). Broken curves: approximate expressions (47) and (51) for  $\tau_{\text{dec}} \ll 1$  (dashed lines) and  $\tau_{\text{dec}} \gg 1$  (dotted lines). **Inset (left):** Decoherence exponent  $D_t^{\text{peak}}$  against  $\tau = t/T_B$  ( $m = 3$ ). **Right:** Decoherence against entanglement times in units of  $t_B = \omega_D^{-1}$  at low temperature  $\beta \gg t_B$ ; the bath correlator is chosen as on the left part and  $\eta_D = 1$  (see text). Solid curves:  $m = 5, 3, 1$  (from top to bottom). The approximations (47) and (53) for  $t_{\text{dec}} \ll t_B$  and  $t_B \ll t_{\text{dec}} \ll \beta$  are shown in dashed and dotted lines, respectively.

and a fixed cut-off frequency  $\omega_D = 5/\beta$  (see e.g. [13]). The 3 plain curves are obtained by solving numerically the implicit equation (43).

It is worthwhile to stress that without pointer-bath coupling (i.e, for  $\eta = 0$ ), the object-pointer coherences do not decay with time for  $x = \epsilon s t$  and  $x' = \epsilon s' t$ . Actually, by (37),

$$\langle s, x = \epsilon s t | \rho_{PS}(t) | s', x' = \epsilon s' t \rangle = \langle s | \psi^0(t) \rangle \langle \psi^0(t) | s' \rangle \langle 0 | \rho_P | 0 \rangle \quad \text{if } \eta = 0. \quad (46)$$

In the model under study, when  $\eta = 0$  all coherences between different eigenstates of  $S$  present in the initial state of  $\mathcal{S}$  are still alive, no matter how large the time  $t$  is. At times  $t \gtrsim t_{\text{class}} = \Delta_{\text{clas}}(\epsilon \delta s)^{-1}$ , object and pointer are then in a “Schrödinger cat state” characterised by nonzero matrix elements between macroscopically distinguishable pointer position eigenstates. For  $\mathcal{S}$  and  $\mathcal{P}$  in such a Schrödinger cat state at the end of the measurement, no classical probabilistic interpretation of the QM is possible<sup>10</sup>. We see here again (see Lecture 1) that the pointer-bath coupling plays a central role in the QM. As we shall see in Sec. 3.6.1, the decoherence process due to this coupling suppresses all  $(s \neq s')$ -coherences (37) at time  $t \gg t_{\text{dec}}$ .

### 3.5 Limiting regimes

Formula (43) explicitly yields the decoherence time in several interesting limits.

#### 3.5.1 Interaction-dominated regime

In the time regime  $t \ll t_B$ , the dynamics is dominated by the interactions  $H_{SP}$  and  $H_{PB}$ . One may approximate the bath correlator  $K(\tau)$  by  $K(0) = \langle B^2 \rangle$  in (42) and (43). We conclude

<sup>10</sup>In fact, one cannot assign a probability to the pointer being located in the vicinity of  $x = \epsilon s t$ , henceforth implying that  $S$  has the value  $s$ . This would invalidate the Born rules.

$$e^{-D_t^{\text{peak}}} = e^{-(t/t_{\text{dec}})^4}, \quad \frac{t_{\text{dec}}}{\beta} = 2^{3/4} \left( \frac{t_{\text{ent}}}{\beta \eta} \right)^{1/2} \quad (47)$$

for  $t, t_{\text{dec}} \ll t_{\mathcal{B}}, T_S, T_P$ . The decoherence time depends on the bath through the pointer-bath coupling strength  $\eta$  only. One has  $t_{\text{ent}} \ll t_{\text{dec}}$ . This follows from the consistency condition  $t_{\text{dec}} \ll t_{\mathcal{B}}$  and the inequality  $t_{\mathcal{B}} \leq \beta$ , which imply  $t_{\text{dec}} \ll \beta$  and thus  $t_{\text{ent}} \ll \beta$  by (47), and from the stability condition  $\eta \lesssim 1$ .

Invoking (42) and  $(\widehat{\Re h})(\omega) \geq 0$ , it is easy to show that  $D_t^{\text{peak}} \leq (t/t_{\text{dec}}^{\text{int}})^4$  for all times  $t \geq 0$ , where  $t_{\text{dec}}^{\text{int}}/\beta$  is given by the r.h.s. of the second formula in (47). Since  $t_{\text{dec}}$  is defined by  $D_{t_{\text{dec}}}^{\text{peak}} = 1$ , this implies that  $t_{\text{dec}}^{\text{int}}$  gives a lower bound on the decoherence time, even when  $t_{\text{dec}}$  is larger than  $t_{\mathcal{B}}$ . This is indeed what is observed on Fig. 5.

### 3.5.2 Markovian regime

The opposite regime  $t_{\text{dec}} \gg T_{\mathcal{B}}$  defines the so-called singular coupling limit [18] (Markovian regime)<sup>11</sup>. Decoherence is governed in this regime by the small-frequency behaviours of the Fourier transforms  $(\widehat{\Re K})(\omega)$  and  $(\widehat{\Im K})(\omega)$  of the real and imaginary parts of the bath correlator (11). We shall make use of a few properties of these Fourier transforms. We assume that  $(\widehat{\Im K})(\omega) \sim -i\hat{\gamma}\omega^m$  for  $\omega \ll T_{\mathcal{B}}^{-1}$  ( $\hat{\gamma}$  is a positive constant). Bearing in mind that  $(\widehat{\Im K})(\omega)$  is an odd function of  $\omega$  and must admit differentials of sufficiently high orders (in such a way that  $\Im K(t)$  decays rapidly to zero as  $t \rightarrow \pm\infty$ ), we take  $m$  to be a positive odd integer. By analogy with the case of a bath of harmonic oscillators linearly coupled to  $\mathcal{P}$ , we speak of *Ohmic damping* when  $m = 1$  and of *super-Ohmic damping* when  $m > 1$  [13]. The behaviour of  $(\widehat{\Re K})(\omega)$  at small frequencies can be deduced from that of  $(\widehat{\Im K})(\omega)$  thanks to the KMS relation

$$(\widehat{\Re K})(\omega) = i \frac{(\widehat{\Im K})(\omega)}{\tanh(\beta\omega/2)}. \quad (48)$$

The KMS relation holds for all frequencies; it relies on the fact that the average in the bath correlator (11) is taken w.r.t. a Gibbs state  $\rho_B^{(\text{eq})}$  (see [14]). It gives  $(\widehat{\Re K})(\omega) \sim 2\hat{\gamma}\omega^{m-1}/\beta$  for  $\omega \ll T_{\mathcal{B}}^{-1}$ .

Let us first discuss the super-Ohmic case  $m \geq 3$ . In the limit  $t \gg T_{\mathcal{B}}$  we can neglect in (45) the oscillatory integral. This yields

$$\frac{\partial}{\partial t} D_t^{\text{peak}} \simeq \epsilon^2 \delta s^2 t \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{(\widehat{\Re K})(\omega)}{\omega^2} = -\frac{\Delta_{\text{eff}}^2 t}{t_{\text{ent}}^2} \int_0^{\infty} d\tau \tau \Re K(\tau). \quad (49)$$

For an Ohmic bath ( $m = 1$ ) the frequency integral in (49) diverges. We shall show that one can in this case replace  $(\widehat{\Re K})(\omega)$  by  $(\widehat{\Re K})(0) = 2\hat{\gamma}/\beta$  in the r.h.s. of (45), which becomes in the limit  $t \gg T_{\mathcal{B}}$

$$\frac{\partial}{\partial t} D_t^{\text{peak}} \simeq 2\epsilon^2 \delta s^2 t (\widehat{\Re K})(0) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sin^2(\omega t/2)}{\omega^2} = \frac{\Delta_{\text{eff}}^2 t^2}{t_{\text{ent}}^2} \int_0^{\infty} d\tau \Re K(\tau). \quad (50)$$

Note that this amounts to replace  $\Re K(t)$  by a white-noise correlator  $(2\hat{\gamma}/\beta)\delta(t)$  in (42). To estimate the error, let us determine the difference between the l.h.s. and the r.h.s. of (50). Since

<sup>11</sup>Note that a rotating-wave approximation is inappropriate here due to our restriction (12)! The Markovian regime studied here is different from the one obtained in the van Hove (or weak-coupling) limit.

the integral  $\int_{-\infty}^{\infty} d\omega ((\widehat{\Re K})(\omega) - (\widehat{\Re K})(0)) \omega^{-2}$  converges (and is equals to  $-2\pi \int_0^{\infty} d\tau \tau \Re K(\tau)$ ), this difference is given in the limit  $t \gg T_B$  by (49) modulo the replacement of  $(\widehat{\Re K})(\omega)$  by  $(\Re K)(\omega) - (\Re K)(0)$ . One thus finds that the relative error made by approximating the l.h.s. of (50) by its r.h.s. is small, of the order of  $T_B/t$ . We conclude, by integrating (49) and (50) w.r.t. time, that in the limit  $T_B \ll t_{\text{dec}} \ll T_S, T_P$  it holds

$$e^{-D_t^{\text{peak}}} = \begin{cases} \exp\left\{-\left(\frac{t}{t_{\text{dec}}}\right)^3\right\} \\ \exp\left\{-\left(\frac{t}{t_{\text{dec}}}\right)^2\right\} \end{cases}, \quad \frac{t_{\text{dec}}}{\beta} = \begin{cases} c_1^{1/3} \left(\frac{t_{\text{ent}}}{\beta \eta}\right)^{2/3} & \text{if } m = 1 \\ c_m^{1/2} \frac{t_{\text{ent}}}{\beta \eta} & \text{if } m \geq 3. \end{cases} \quad (51)$$

The constant  $c_m$  is independent of the strengths of the couplings,

$$c_1 = \frac{3\beta \langle B^2 \rangle}{\int_0^{\infty} d\tau \Re K(\tau)}, \quad c_{m \geq 3} = \frac{2\beta^2 \langle B^2 \rangle}{\left| \int_0^{\infty} d\tau \tau \Re K(\tau) \right|}. \quad (52)$$

Qualitatively different results are obtained for Ohmic and super-Ohmic baths. Ohmic baths win in efficiency for decoherence over super-Ohmic baths. Actually,  $(t_{\text{dec}}^{m=1}/t_{\text{dec}}^{m \geq 3})^3$  is equal to the product of  $c_1\beta/(T_B c_{m \geq 3})$  by  $(T_B/t_{\text{dec}}^{m \geq 3})$ . Since the last factor must be small compared with 1 for consistency and the first one is  $\lesssim 1$ , it follows that  $t_{\text{dec}}^{m=1} \ll t_{\text{dec}}^{m \geq 3}$ . This result should be compared with the known saturation of the decoherence factor of a single system coupled to a super-Ohmic bath in the singular coupling limit (see Sec. 3.6.2 and [20]). We see here that, as a result of the indirect coupling *via the pointer* of the object  $\mathcal{S}$  to the super-Ohmic bath, the decoherence factor does not saturate to a positive value but decays to zero, although more slowly than for a Ohmic bath.

One sees on the left part of Fig. 5 a remarkably good agreement between the exact and asymptotic behaviours of  $t_{\text{dec}}$  as soon as  $t_{\text{dec}} \geq T_B = \beta$ . The plain curves representing  $t_{\text{dec}}$  split by increasing  $t_{\text{ent}}$  into distinct branches corresponding to distinct  $m$ 's, as predicted by (51). This splitting occurs when  $t_{\text{dec}}$  is in the transition region  $t_B \lesssim t_{\text{dec}} \lesssim T_B = \beta$ . After this splitting  $t_{\text{dec}}$  is larger for larger  $m$ . In particular, a Ohmic bath ( $m = 1$ ) has a smaller decoherence time than a super-Ohmic bath ( $m = 3, 5 \dots$ ), as stated above.

### 3.5.3 Bath at low temperature

Let us briefly discuss the case of a bath initially in thermal equilibrium at low temperature. Strictly speaking, extremely low temperatures have to be proscribed because of our hypothesis  $\hbar\beta \ll T_P$ . However, taking e.g.  $T_P = 1$  s, this separation of time scales holds even for  $T$  of the order of  $10^{-8}$ - $10^{-9}$  degree Kelvin! Furthermore, the stability condition (18) has a better chance to be met at low temperature  $T$  since  $\Delta_{\text{th}}$  decreases with  $T$ . By “low temperature” we mean here  $\beta = (k_B T)^{-1} \gg t_B$ ,  $t_B$  being the inverse of a cut-off frequency  $\omega_D$ . In the interaction-dominated regime  $t_{\text{dec}} \ll t_B$ , the decoherence time is still given by (47) (note that  $\beta$  can be eliminated from both sides of (47), so that temperature only enters in this eq. through the variance  $\langle B^2 \rangle$  of  $B$ , which has a finite limit when  $\beta \gg t_B$ ). The opposite limit  $t_B \ll t_{\text{dec}} \ll \beta$  can be treated in a similar way as in Sec. 3.5.2 if one assumes that the correlator  $K(\tau)$  has only 2 timescales  $t_B$  and  $T_B$ , the latter being equal to the thermal time  $T_B = \beta \gg t_B = \omega_D^{-1}$ . For instance, if one takes a temperature-independent  $(\widehat{\Im K})(\omega) = -iJ(\omega) = -i\hat{\gamma}\omega^m \exp\{-(\omega/\omega_D)^2\}$ ,  $\omega \geq 0$  (see Sec. 3.4), one finds thanks to the KMS relation (48) that  $(\widehat{\Re K})(\omega)$  can be replaced by

$J(|\omega|)$  in the second integral in (49) when  $t_B \ll t \ll \beta$ . For a super-Ohmic bath ( $m \geq 3$ ), the approximation (49) is thus valid in this regime modulo this replacement. Similarly, one has  $\langle B^2 \rangle = \int_{-\infty}^{\infty} d\omega (\widehat{\Re K})(\omega)/(2\pi) \sim \int_{-\infty}^{\infty} d\omega J(|\omega|)/(2\pi)$ . For an Ohmic bath ( $m = 1$ ) one can clearly not use (50). One finds [4]

$$t_{\text{ent}} \simeq \begin{cases} \eta_D t_{\text{dec}} \left\{ \ln\left(\frac{t_{\text{dec}}}{t_B}\right) - 0.2114 \right\}^{\frac{1}{2}} & \text{if } m = 1 \text{ (Ohmic)} \\ \eta_D t_{\text{dec}} / \sqrt{\tilde{c}_m} & \text{if } m \geq 3 \text{ (super-Ohmic),} \end{cases} \quad (53)$$

where  $\eta_D = \langle B^2 \rangle^{1/2} \Delta_{\text{eff}} t_B$  is the pointer-bath coupling strength in units of  $1/t_B$  and  $\tilde{c}_{m \geq 3} = 2t_B^2 \int_0^{\infty} d\omega J(\omega) / \int_0^{\infty} d\omega J(\omega) \omega^{-2} = m - 1$ . For a given  $t_{\text{ent}}$ , the ratio between the decoherence times for Ohmic and super-Ohmic baths is logarithmically small in the dimensionless time  $t_{\text{dec}}/t_B$ . Hence a Ohmic bath is not dramatically more efficient than a super-Ohmic bath at very low temperature, in contrast with our previous findings at “high” temperatures. The right part of Fig. 5 shows  $t_{\text{dec}}$  as a function of  $t_{\text{ent}}$  in the low temperature limit  $t_B \ll \beta$  (obtained by solving numerically (43) in this limit), for the aforementioned choice of  $(\widehat{\Im K})(\omega)$ . A good agreement with (53) is found in the regimes  $t_B \ll t_{\text{dec}} \ll \beta$  (dotted lines) and  $t_{\text{dec}} \ll t_B$  (dashed line).

### 3.6 Decay of the object-pointer coherences

In this section we will show (at least in the two limiting regimes studied in Secs. 3.5.1 and 3.5.2) that the 2 following statements are true:

1. For  $s \neq s'$ , the object-pointer matrix element  $\langle s, x | \rho_{PS}(t) | s', x' \rangle$  is vanishingly small at time  $t \gg t_{\text{dec}}$  for all values of  $(x, x')$ .
2. The decoherence factor  $e^{-D_t(x_s(t), x'_s(t); s, s')}$  for the pointer matrix elements  $\langle x | \rho_P^{(s)} | x' \rangle$  in (39) remains close to unity at time  $t \approx t_{\text{dec}}$  if  $|x_s(t)|$  and  $|x'_s(t)|$  are smaller than  $\Delta_{\text{eff}}$ .

We may conclude from the first statement that the reduction of the wavepacket (i.e., the disappearance of all the object-pointer coherences (37) for  $s \neq s'$ ) is entirely governed in our model by the two timescales  $t_{\text{ent}}$  and  $t_{\text{dec}}$ . Let us recall that  $t_{\text{dec}}$  has been defined in Sec. 3.4 as the decay time of  $\langle s, x | \rho_{PS}(t) | s', x' \rangle$  for a *specific value* of  $(x, x')$  (namely,  $(x, x') = (\epsilon t s, \epsilon t s')$ ). The second statement means that decoherence does away with the “off-diagonal” ( $s \neq s'$ ) object-pointer matrix elements before the “diagonal” ones change noticeably.

The justifications of the 2 statements are somehow technical; the reader is advised to skip them in a first reading and proceed directly to Sec. 3.7.

#### 3.6.1 Justification of statement 1

We first show that statement 1 holds true if one assumes  $t_{\text{dec}} \geq t_{\text{ent}}$ . We have seen above that the  $(s \neq s')$ -coherences (37) almost vanish when  $|x_s(t)| \geq \Delta_{\text{eff}}$  or  $|x'_{s'}(t)| \geq \Delta_{\text{eff}}$ . We may therefore restrict our attention to values of  $(x, x')$  satisfying  $|x_s(t)|, |x'_{s'}(t)| < \Delta_{\text{eff}}$ . By using (35), (41) and (42), one obtains

$$D_t(x_s(t), x'_{s'}(t); s, s' = s + \delta s) = D_t^{\text{peak}} \left( 1 + 4t_{\text{ent}} \frac{I_{0,1,t}}{I_{1,1,t}} + 4t_{\text{ent}}^2 \frac{I_{0,0,t}}{I_{1,1,t}} \right) \quad (54)$$

where  $I_{a,b,t} = \int_0^t d\tau_1 \int_0^t d\tau_2 \tau_1^a \tau_2^b \Re K(\tau_1 - \tau_2) = I_{b,a,t}$  satisfies  $\sup_{t \geq 0} t^{2-a-b} I_{a,b,t} / I_{1,1,t} < \infty$  for any  $a, b = 0, 1$ <sup>12</sup>. As a result, the r.h.s. of (54) equals  $D_t^{\text{peak}}(1 + \mathcal{O}(t_{\text{ent}}/t))$ . If  $t$  is large compared with both  $t_{\text{dec}}$  and  $t_{\text{ent}}$  then  $D_t^{\text{peak}} \gg 1$ ,  $\mathcal{O}(t_{\text{ent}}/t) \ll 1$  and thus  $D_t(x_s(t), x'_{s'}(t); s, s' = s + \delta s) \gg 1$  for all  $(x, x')$  such that  $|x_s(t)|$  and  $|x'_{s'}(t)|$  are smaller than  $\Delta_{\text{eff}}$ . Therefore, the product of the two last factors in the r.h.s. of (37) is vanishingly small for all values of  $(x, x')$ .

We now argue that  $t_{\text{dec}} \geq t_{\text{ent}}$  for pointer-bath coupling strength  $\eta \lesssim \beta/T_{\mathcal{B}}$ , excepted for Ohmic baths at large values of  $t_{\text{ent}}$  (i.e., for weak object-pointer couplings). In the interaction-dominated regime, one has even  $t_{\text{dec}} \gg t_{\text{ent}}$ , see Sec. 3.5.1. Let us study the Markov regime  $t_{\text{dec}} \gg T_{\mathcal{B}}$ . By using  $|K(\tau)| \leq \langle B^2 \rangle$  (Cauchy-Schwarz inequality) and  $\Re K(\tau) \simeq 0$  for  $\tau \gg T_{\mathcal{B}}$  (Sec. 3.1.2), one finds that the integrals  $\int_0^\infty d\tau \Re K(\tau)$  and  $|\int_0^\infty d\tau \tau \Re K(\tau)|$  are at most of the order of  $\langle B^2 \rangle T_{\mathcal{B}}$  and  $\langle B^2 \rangle T_{\mathcal{B}}^2$ , respectively. Hence the coefficient  $c_{m \geq 3}$  in (52) is of the order of  $(\beta/T_{\mathcal{B}})^2$  or larger. By virtue of (51), the condition  $t_{\text{ent}} \leq t_{\text{dec}}$  holds for super-Ohmic baths provided that  $\eta \lesssim \beta/T_{\mathcal{B}}$ . Note that the latter condition is necessarily fulfilled by virtue of the stability condition (18) when the temperature is not too large (so that  $T_{\mathcal{B}} = \beta$ ). The situation is different for Ohmic baths: then  $t_{\text{dec}} \leq t_{\text{ent}}$  even for small  $\eta$  if

$$\frac{t_{\text{ent}}}{\beta} \gtrsim \frac{\langle B^2 \rangle \beta}{\eta^2 \int_0^\infty d\tau \Re K(\tau)}. \quad (55)$$

Assuming that  $\eta \lesssim \beta/T_{\mathcal{B}}$  and that  $t_{\text{dec}}$  can be “smoothly interpolated” between the regimes  $t_{\text{dec}} \ll t_{\mathcal{B}}$  and  $t_{\text{dec}} \gg T_{\mathcal{B}}$  (see Fig. 5), it seems reasonable to conclude that the condition  $t_{\text{dec}} \geq t_{\text{ent}}$  is fulfilled save for Ohmic baths when  $t_{\text{ent}}$  satisfies (55).

The existence of the above-mentioned exceptional situation for Ohmic bath leads us to ask us the following question. When the condition  $t_{\text{dec}} \geq t_{\text{ent}}$  is not fulfilled, can the object-pointer coherences (37) be still  $\approx 1$  at time  $t \approx t_{\text{dec}}$ ? To answer this question, we shall determine the decay of the maximum decoherence factor obtained by minimising  $D_t(x, x'; s, s')$  over all values of  $(x, x')$  in  $\mathbb{R}^2$ . We transform (36) with the help of the change of variable  $\tau \rightarrow \tau - t/2$  into

$$\begin{aligned} D_t(x, x'; s, s') &= \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} (\widehat{\Re K})(\omega) \left\{ \left( x' - x + t\epsilon(s' - s)/2 \right)^2 \left( \int_{-t/2}^{t/2} d\tau \cos(\omega\tau) \right)^2 \right. \\ &\quad \left. + \epsilon^2(s' - s)^2 \left( \int_{-t/2}^{t/2} d\tau \tau \sin(\omega\tau) \right)^2 \right\}. \end{aligned} \quad (56)$$

Fixing  $(s, s')$  and  $t$  and letting  $(x, x')$  vary,  $D_t(x, x', s, s')$  reaches its minimum value when  $x' - x = -t\epsilon(s' - s)/2$ . This minimum reads

$$D_t^{\text{min}}(s, s') = \min_{x, x'} \{ D_t(x, x', s, s') \} = \frac{\epsilon^2(s' - s)^2}{2} \int_{-t/2}^{t/2} d\tau_1 \int_{-t/2}^{t/2} d\tau_2 \tau_1 \tau_2 \Re K(\tau_1 - \tau_2). \quad (57)$$

Therefore, the decoherence factor  $\exp\{-D_t(x_s(t), x'_{s'}(t); s, s')\}$  in the object-pointer state (37) is maximum when  $x' - x = t\epsilon(s' - s)/2$ , i.e., when the distance  $|x' - x|$  is half the distance between the peaks of the shifted pointer densities  $\rho_p^{(s)}$ . We may now compare the decay of  $\exp\{-D_t^{\text{min}}(s, s')\}$  with that of the decoherence factor  $\exp\{-D_t^{\text{peak}}(s, s')\}$  associated with the object-pointer coherences for  $(x, x') = (\epsilon ts, \epsilon ts')$ . We first do that in the interaction-dominated regime  $t \ll t_{\mathcal{B}}$ . To obtain the expression of (57) in that regime, an expansion of  $K(\tau_1 - \tau_2)$  up

<sup>12</sup>It is clear that  $I_{a,b,t} \sim 4t^{a+b-2} I_{1,1,t} (a+1)^{-1} (b+1)^{-1}$  as  $t \ll t_{\mathcal{B}}$ . The large time limit  $t \gg T_{\mathcal{B}}$  can be handled by a similar approach as in Sec. 3.5.2.

to second order in  $\tau_1 - \tau_2$  is required (the zeroth and odd derivatives do not contribute due to the symmetry  $\tau_1 \leftrightarrow \tau_2$ ,  $\tau_1 \leftrightarrow -\tau_1$  and  $\tau_2 \leftrightarrow -\tau_2$  of the range of integration in (57)). One gets

$$D_t^{\min}(s, s' = s + \delta s) = \left( \frac{t}{t_{\text{dec}}^{\max}} \right)^6, \quad \frac{t_{\text{dec}}^{\max}}{\beta} = \left( \frac{288 \langle B^2 \rangle}{\beta^2 |K''(0)|} \right)^{1/6} \left( \frac{t_{\text{ent}}}{\eta \beta} \right)^{1/3} \quad (58)$$

As before, the conditions  $t_{\text{dec}}^{\max} \ll t_{\mathcal{B}}, T_S, T_P$  must be checked for consistency. The modulus of the second time derivative  $K''(0)$  of  $K(\tau)$  at  $\tau = 0$  is at most of the order of  $\langle B^2 \rangle (t_{\mathcal{B}})^{-2}$ . Comparing (47) and (58), one sees that the maximum decoherence factor  $\exp\{-D_t^{\min}\}$  decays more slowly than  $\exp\{-D_t^{\text{peak}}\}$  at small time  $t \ll t_{\mathcal{B}}$  (the ratio between  $t_{\text{dec}}^{\max}$  and  $t_{\text{dec}}$  is large, of the order of  $(t_{\mathcal{B}}/t_{\text{dec}})^{1/3}$ ). In contrast, in the opposite Markov regime  $t \gg T_{\mathcal{B}}$  the object-pointer coherences  $\langle s, x | \rho_{PS}(t) | s', x' \rangle$  for fixed  $s \neq s'$  decay in the same way for all values of  $(x, x')$ , apart from irrelevant numerical factors. Actually, a similar calculation as in Sec. 4.2.5 shows

$$D_t^{\min} \simeq \begin{cases} D_t^{\text{peak}}/4 & \text{if } m = 1 \text{ (Ohmic)} \\ D_t^{\text{peak}}/2 & \text{if } m \geq 3 \text{ (super-Ohmic)} \end{cases}, \quad t \gg T_{\mathcal{B}}. \quad (59)$$

In particular,  $D_t^{\min} \simeq 1/4$  when  $t = t_{\text{dec}} \gg T_{\mathcal{B}}$  for a Ohmic bath. More generally, in the Markov regime  $t_{\text{dec}} \gg T_{\mathcal{B}}$ , the decoherence factor  $e^{-D_t(x_s(t), x'_s(t); s, s')}$  in (37) is vanishingly small at time  $t \gg t_{\text{dec}}(s, s')$  for all values of  $(x, x')$ . Since we have seen above that in the regime  $t_{\text{dec}} \lesssim T_{\mathcal{B}}$  the object-pointer coherence are vanishingly small for all  $(x, x')$  thanks to the smallness of the second factor in (37), we conclude that the statement 1 at the beginning of this Section is correct in all cases.

### 3.6.2 Justification of statement 2

It is appropriate to demonstrate that the decay of the pointer matrix elements (39) remains negligible for times  $t$  until well after the disappearance of the off-diagonal ( $s \neq s'$ ) terms in  $\rho_{PS}(t)$ . Due to the factor  $\langle x_s(t) | \rho_P(0) | x'_s(t) \rangle$  in (39) and the form (14) of the pointer initial state, the relevant values of  $(x, x')$  satisfy  $|x - x'| \lesssim \lambda_{\text{th}}$ .

Let us first discuss the interaction-dominated regime  $t \ll t_{\mathcal{B}}$ . Replacing  $K(\tau_1 - \tau_2)$  by  $\langle B^2 \rangle$  in (40), one obtains

$$e^{-D_t(x_s(t), x'_s(t); s, s)} \simeq \exp\left\{-\frac{1}{2} \langle B^2 \rangle t^2 (x - x')^2\right\}, \quad t \ll t_{\mathcal{B}}. \quad (60)$$

Given  $|x - x'| \leq \lambda_{\text{th}}$ , it follows from  $t_{\text{ent}} \ll t_{\text{dec}}$  and  $\lambda_{\text{th}} \ll \Delta_{\text{eff}}$  that  $|x - x'|$  is much smaller than the inter-peak distance  $t_{\text{dec}} \epsilon \delta s$  relevant for the decay of the  $(s, s' = s + \delta s)$  matrix elements of  $\rho_{PS}(t)$ . Hence  $|x - x'| \leq \lambda_{\text{th}}$  entails  $D_t(x_s(t), x'_s(t); s, s) \ll \eta^2 t^2 \beta^{-2} \ll 1$  (recall that  $t \ll t_{\mathcal{B}} \leq \beta$  and  $\eta \lesssim 1$  by the stability condition (18)). The formula (47) can be deduced heuristically from (60) in the following way. Recalling that the distance between neighbouring peaks in the pointer density  $\rho_P(t)$  grows proportionally with time, one replaces  $x' - x$  in (60) by the inter-peak distance  $\epsilon t \delta s$  and recovers (47) apart from a numerical factor.

In the Markov regime  $t \gg T_{\mathcal{B}}$ , one can use a similar method as in Sec. 3.5.2 to deduce from (39) that

$$e^{-D_t(x_s(t), x'_s(t); s, s)} = \begin{cases} \exp\left\{-\frac{3t_{\text{ent}}^2}{t_{\text{dec}}^3} t \frac{(x' - x)^2}{\Delta_{\text{eff}}^2}\right\} & \text{if } m = 1 \text{ (Ohmic)} \\ \exp\left\{-\frac{2\eta^2}{c_m} \frac{(x - x')^2}{\Delta_{\text{eff}}^2}\right\} & \text{if } m \geq 3 \text{ (super-Ohmic)} \end{cases}, \quad t \gg T_{\mathcal{B}} \quad (61)$$

where  $t_{\text{dec}}$  and  $c_{m \geq 3}$  are given by the upper line in (51) and by (52), respectively. For an Ohmic bath, the  $(s = s')$  object-pointer matrix elements  $\langle s, x | \rho_{PS}(t) | s, x' \rangle$  decay exponentially with time when  $x \neq x'$ . Such an exponential decay is familiar to quantum systems coupled to an Ohmic bath in the Markov regime. Note the difference with the decay like  $\exp\{-(t/t_{\text{dec}})^3\}$  of the decoherence factor (51) for the  $(s \neq s')$  object-pointer matrix elements. There, the superposition of distinct eigenvectors  $|s\rangle$  and  $|s'\rangle$  decohere through the entanglement with the pointer, which is coupled to the bath (indirect decoherence). For a super-Ohmic bath, the decoherence factor (61) saturates to a finite value as  $t$  goes to infinity. This means that there is no complete damping of the  $(s = s')$  object-pointer matrix elements, even for large finite distances  $|x - x'|$ . Such a saturation was already mentioned in Sec. 3.5.2. It is in striking contrast with what happens for  $s \neq s'$  in (51). The formula (51) can be deduced heuristically from (61) (for both  $m = 1$  and  $m \geq 3$ ) by the same argument as used above in the interaction-dominated regime: replacing  $x' - x$  in (61) by the inter-peak distance  $\epsilon t \delta s$ , one recovers (51) apart from numerical factors. For an Ohmic bath, if  $t_{\text{dec}}$  is not in the time regime indicated in (55) so that  $t_{\text{ent}} \leq t_{\text{dec}}$ , then (19) entails  $D_t(x_s(t), x'_s(t); s, s) \ll 1$  for  $|x - x'| \leq \lambda_{\text{th}}$  and  $t \lesssim t_{\text{dec}}$ . For a super-Ohmic bath one has even  $D_t(x_s(t), x'_s(t); s, s) \ll 1$  for  $|x - x'| \leq \lambda_{\text{th}}$  at all times  $t$ .

In conclusion, the decoherence caused by the pointer-bath coupling has a small effect on the pointer states  $\rho_P^{(s)}(t)$  up to times  $t \lesssim t_{\text{dec}}$ . Actually, at those times (39) reduces to

$$\langle x | \rho_P^{(s)}(t) | x' \rangle \simeq \langle x_s(t) | \rho_P(0) | x'_s(t) \rangle e^{-i\phi t}, \quad t \lesssim t_{\text{dec}}. \quad (62)$$

### 3.7 Conclusion

If  $t$  is much larger than the maximum decoherence time  $t_{\text{dec}} = t_{\text{dec}}(s, s \pm \delta s)$  ( $t$  being still smaller than  $T_S$  and  $T_P$ ), the matrix elements (37) are vanishingly small for all values of  $s \neq s'$  and all values of  $(x, x')$  (see Sec. 3.6). Assuming moreover that the spectrum of  $S$  is non-degenerate, we conclude that object and pointer are in a *separable mixed state*

$$\rho_{PS}(t) \simeq \sum_s p_s |s\rangle \langle s| \otimes \rho_P^{(s)}(t) \quad , \quad t \gg t_{\text{dec}} , \quad (63)$$

with  $p_s = |c_s|^2$ . Hence, with probability  $p_s$  the object is in the eigenstate  $|s\rangle$  of the measured observable  $S$  and the pointer is in a state  $\rho_P^{(s)}(t)$  localised in position near  $x = \epsilon s t$ . The state in (63) looks like the post-measurement state (3) of the von Neumann postulate. The only difference between (3) and (63) is that the pointer states  $\rho_P^{(s)}(t)$  are not macroscopically distinguishable expected at very large times  $t \gtrsim t_{\text{class}}$ . One can, however, use the instability induced by the pointer-bath coupling as an amplification mechanism (see Sec. 3.2). Let the coupling  $H_{SP}$  be switched off at time

$$t_{\text{int}} \approx W_{\text{eff}}(\epsilon \delta s)^{-1} \gg t_{\text{ent}}. \quad (64)$$

Then all pointer states  $\rho_P^{(s)}(t)$  are outside the effective potential well save for possibly one eigenvalue  $s \simeq 0$ . The “mesoscopic” inter-peak distance  $W_{\text{eff}}$  at time  $t = t_{\text{int}}$  is amplified at time  $t > t_{\text{int}}$  by the effective pointer dynamics, till it reaches a macroscopically resolvable magnitude  $\Delta_{\text{class}}$ . Then a pointer reading, while still a physical process in principle perturbing  $\mathcal{P}$ , surely cannot blur the distinction of the peaks.

The result (63) shows that our apparatus with a single-degree-of-freedom macroscopic pointer coupled via the Hamiltonians (9) and (10) to the object and bath performs a measurement of the object observable  $S$ . The object-pointer state is transformed at time  $t \gg t_{\text{dec}}$



into a statistical mixture of eigenstates of  $S$ , in agreement with the von Neumann postulate (3). Let us stress that, in contrast to the description of measurement processes presented in von Neumann's and other old quantum mechanics textbooks [6], it has not been necessary to postulate the existence of a new kind of dynamics - different from that given by Schrödinger's equation and even non-unitary. We have only used "conventional" quantum theory, namely, the superposition principle and Schrödinger's equation. Unfortunately, the fundamental issue of interpreting the object-pointer reduced density matrix (20) as a *true statistical ensemble* of object-pointer states has not been elucidated at all! (see the discussion in Sec.2.3). Despite of this major difficulty, the "object+pointer+bath" approach used here (which is, after two decades of active work, by now widely spread in the physics community) is a step forward in the theory of QM. This approach gives us some hope that measurement processes could be understood without introducing a new dynamical equation generalising Schrödinger's equation.

However, the main interest of the model studied in this lecture is not that it explains (with the limitations mentioned above) the von Neumann's postulate. Many other models in the literature based on the "object+pointer+bath" approach do so, see e.g. [9]. The interesting point is that our model, like the model of Ref. [1] that will be studied in the next lecture, allows for explicit calculations of the measurement times. Two fundamental times characterising the measurement have been introduced. The *entanglement time*  $t_{\text{ent}}$  is the time after which pointer positions corresponding to distinct eigenvalues  $s$  of  $S$  begin to be resolved. It is given by  $t_{\text{ent}} = \Delta_{\text{eff}}(\epsilon\delta s)^{-1}$ , where  $\epsilon$  is the object-pointer coupling constant,  $\delta s$  the separation between neighbouring eigenvalues and  $\Delta_{\text{eff}} \approx \Delta_{\text{th}}$  the uncertainty in the initial pointer position. Object and apparatus must interact during a time  $t_{\text{int}}$  much larger than  $t_{\text{ent}}$ , see (64). Accordingly,  $t_{\text{ent}}$  provides a good measure of the efficiency of the object-pointer interaction (the smaller  $t_{\text{ent}}$ , the more efficient the coupling). The second fundamental time of the measurement is the time  $t_{\text{dec}}$  associated to the *decoherence process*. This is the time after which the object-pointer density matrix gets close to the statistical mixture (63). In an ideal measurement, both  $t_{\text{ent}}$  and  $t_{\text{dec}}$  must be much smaller than the times  $T_S$  and  $T_P$  associated with the dynamics of the isolated object and pointer. Under this hypothesis, we have shown that

$$\tau_{\text{dec}} = c_{\gamma,m}^{1/\gamma} \left( \frac{\tau_{\text{ent}}}{\eta} \right)^{2/\gamma}, \quad \gamma = \begin{cases} 4 & \text{if } t_{\text{dec}} \lesssim t_B \quad (\text{interaction-dominated regime}) \\ 3 & \text{if } t_{\text{dec}} \gtrsim T_B \quad \text{for an Ohmic bath (Markov)} \\ 2 & \text{if } t_{\text{dec}} \gtrsim T_B \quad \text{for a super-Ohmic bath (Markov)} \end{cases} \quad (65)$$

where  $\tau_{\text{dec}} = t_{\text{dec}}/\beta$  and  $\tau_{\text{ent}} = t_{\text{ent}}/\beta$  are the decoherence and entanglement times in units of the thermal time  $\beta$ ,  $\eta = \langle B^2 \rangle^{1/2} \Delta_{\text{eff}} \beta \lesssim 1$  is the pointer-bath coupling energy in units of  $k_B T$  and  $c_{\gamma,m}$  are constants independent of the strength of the couplings. Two distinct regimes ought to be identified in (65): in the *interaction-dominated regime*,  $t_{\text{dec}}$  is shorter than the characteristic time  $t_B$  after which the bath correlation function  $K(t)$  differs significantly from its value  $\langle B^2 \rangle$  at  $t = 0$ ; in the opposite *Markov regime*, one must wait more than the bath correlation time  $T_B$ , i.e., the largest decay time of  $K(t)$ , to obtain the required statistical mixture. While  $t_{\text{dec}}$  presents a universal behaviour in the interaction-dominated regime (it depends on the bath through the single parameter  $\eta$ , i.e.,  $c_{4,m} = 8$  is independent on the details of the pointer-bath coupling), in the Markov regime it is determined by the small-frequency behaviour of  $\Im K(t)$ ,  $(\widehat{\Im K})(\omega) \sim -i \hat{\gamma} \omega^m$ . Larger values of  $t_{\text{dec}}$  are found for larger  $m$ 's, with a significant change of behaviour between  $m = 1$  (Ohmic bath) and  $m > 1$  (super-Ohmic bath). For a bath at very low temperature,  $\beta \gg t_B$ , (65) still holds with  $\tau_{\text{dec}}$  and  $\eta$  replaced by  $t_{\text{dec}}/t_B$  and  $\eta_D = \langle B^2 \rangle^{1/2} \Delta_{\text{eff}} t_B$ , save for the Ohmic case where  $\eta_D t_{\text{dec}}/t_{\text{ent}}$  becomes logarithmically small in  $t_{\text{dec}}/t_B$ .

The entanglement and decoherence times are ordered as follows:

- (1)  $t_{\text{ent}} \ll t_{\text{dec}} \lesssim t_{\mathcal{B}}$  in the interaction-dominated regime;
- (2) in the Markov regime ( $t_{\text{dec}} \gtrsim T_{\mathcal{B}}$ ),  $t_{\text{ent}} \leq t_{\text{dec}}$  for a super-Ohmic bath with a small enough pointer-bath coupling ( $\eta \lesssim \beta/T_{\mathcal{B}}$ ), whereas for an Ohmic bath this inequality holds for strong object-pointer coupling only (more precisely, for  $t_{\text{ent}} \lesssim c_{3,1} \eta^{-2} \beta$ ).

Therefore, the only regime with a decoherence faster than resolution of pointer peaks is the Markov regime with  $m = 1$  (Ohmic bath).

It is worthwhile to note that for reasonably strong pointer-bath coupling and not too strong object-pointer coupling, the decoherence time can be so small that *the whole measurement is performed without producing a Schrödinger cat state* as an intermediate step. More precisely, one can deduce from (64) and (65) that  $t_{\text{dec}} \leq t_{\text{int}}$  if  $\eta \geq c_{\gamma,m}^{1/2} \Delta_{\text{eff}}/W_{\text{eff}}$  ( $\gamma = 2, 3, 4$ ) and  $\tau_{\text{ent}} \geq \Delta_{\text{eff}}/W_{\text{eff}}$ , with the object-pointer interaction time given by (64) and  $W_{\text{eff}} \gg \Delta_{\text{eff}}$  the width of the effective pointer potential (15). In such a case, due to the simultaneous action of the object-pointer and pointer-bath couplings, mesoscopic superpositions decay to mixtures faster than entanglement can create them.

## 4 Lecture 3: Curie-Weiss model

In this third lecture, we study a model for a QM due to Allahverdyan, Balian and Nieuwenhuizen [1, 2] and compare the results for the entanglement and decoherence times to those obtained in the second lecture. The measurement apparatus is an interacting spin chain. The object is a spin 1/2 coupled identically to all spin of the apparatus. The latter spins are coupled to independent thermal baths at the same temperature (Fig. 6). One assumes that there are initially no apparatus-bath correlations. The main advantages of this model are:

- (i) it displays interesting statistical physics phenomena such as phase transition and spin-spin long-range correlations in the apparatus;
- (ii) like in the model discussed in the previous lecture, object-apparatus entanglement and environment-induced decoherence proceed simultaneously and mixtures of macroscopically distinct object-pointer product states arise without intervening macroscopic superpositions;
- (iii) by relying on a mean-field approximation and treating the apparatus-bath coupling perturbatively, explicit results for the object-apparatus density matrix can be obtained in the non-Markovian as well as in the Markovian regime.

Let us mention that another model in the same spirit, which involves an ideal Bose gas and Bose-Einstein condensation, has been worked out by the aforementioned authors [21].

### 4.1 The model

#### 4.1.1 The three-partite system

- *Quantum object*: spin 1/2; the  $z$ -component  $\hat{S} = \hat{\sigma}_z^{(0)}$  of the spin is measured (here and in what follows the upper index 0 refers to the object).  
We ignore the Hamiltonian of the object,  $\hat{H}_S = 0$  (this is justified provided the time scale  $T_S$  for the dynamics of the isolated object is much larger than the decoherence and object-apparatus interaction times, i.e.  $t_{\text{dec}}, t_{\text{int}} \ll T_S$ , see Sec. 3.1.2).
- *Apparatus*:  $N$  spin 1/2, labelled by the index  $n$ ,  $n = 1, \dots, N$  (Hilbert space  $\mathcal{H}_A = \mathbb{C}^{2N}$ ). The pointer variable is the total magnetisation  $\hat{\mathcal{M}}_z = N\mu \hat{m}_z$  in the  $z$ -direction. Here  $\mu$  is the magnetic moment of one spin and

$$\hat{m}_z = \frac{1}{N} \sum_{n=1}^N \hat{\sigma}_z^{(n)}, \quad (66)$$

$\hat{\sigma}_z^{(n)}$  being the  $z$ -component of spin  $n$ . The apparatus and object-apparatus Hamiltonians are

$$\hat{H}_A = -\frac{1}{4} J N \hat{m}_z^4, \quad \hat{H}_{SA} = -g N \hat{\sigma}_z^{(0)} \otimes \hat{m}_z. \quad (67)$$

$\hat{H}_A$  describes e.g. super-exchange spin interactions. The exchange integral  $J > 0$  is identical for all spin pairs; it is positive and thus favours spin alignment. Unlike in the model discussed in Sec. 3, where the object was coupled to a single degree of freedom of the apparatus,  $\hat{H}_{SA}$  couples  $\mathcal{S}$  *identically to all spins of  $\mathcal{A}$* , with the same coupling energy  $g > 0$ .

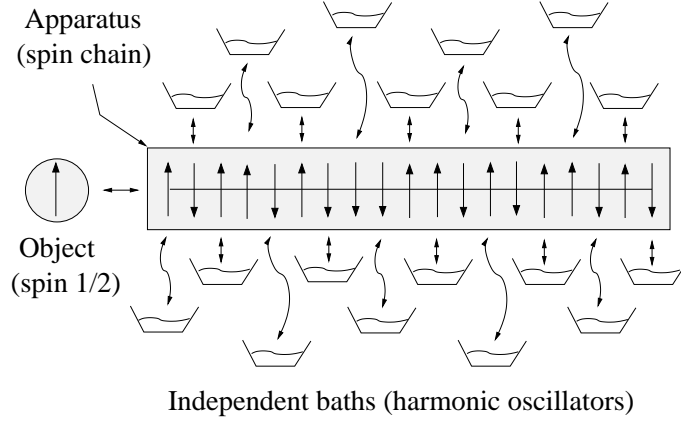


Figure 6: *The QM model of Allahverdyan et al. [1, 2].*

The  $N$  spins of the apparatus are initially in a paramagnetic state with  $m_z = 0$ . In order to fulfil the requirements 1 and 2 of Sec. 2.1, we assume  $N \gg 1$ . Moreover, we are interested in values of  $g$  large enough (or temperatures  $T$  small enough) so that the equilibrium state of  $\mathcal{A}$  coupled via  $\hat{H}_{SA}$  to the magnetic moment of  $\mathcal{S}$  is a *ferromagnetic state* with magnetisation  $\mathcal{M}_z \simeq \pm \mu N$  if the object is initially in an eigenstate  $|s = \pm 1\rangle$  of  $\hat{\sigma}_z^{(0)}$  (critical regime of the phase transition).

- *Environment:* Ohmic bath of harmonic oscillators coupled independently to the 3 spin components  $\hat{\sigma}_a^{(n)}$  ( $a = 1, 2, 3$ ) of the  $N$  spins ( $n = 1, \dots, N$ ).

The coupling with the apparatus is linear in the creation and annihilation operators  $b_{\nu,a}^{(n)\dagger}$  and  $b_{\nu,a}^{(n)}$  in the mode  $(\nu, a, n)$  (here  $\nu = 1, \dots, \mathcal{N}$  counts the distinct modes coupled to the same spin component  $\hat{\sigma}_a^{(n)}$ ),

$$\hat{H}_{AB} = \sum_{n=1}^N \sum_{a=1}^3 \hat{\sigma}_a^{(n)} \otimes \hat{B}_a^{(n)} \quad , \quad \hat{B}_a^{(n)} = \mathcal{N}^{-1/2} \sum_{\nu=1}^{\mathcal{N}} \left( \kappa_{\nu} b_{\nu,a}^{(n)\dagger} + \kappa_{\nu}^* b_{\nu,a}^{(n)} \right) . \quad (68)$$

The coupling constant  $\kappa_{\nu}$  between the  $(\nu, a, n)$ -mode and the  $a$ -component of spin  $n$  is independent of  $a$  and  $n$ . The same is true for the mode frequency  $\omega_{\nu}$ . Hence all spin of  $\mathcal{A}$  are coupled to *identical* independent baths. The Hamiltonian of the latter is

$$\hat{H}_B = \sum_{n=1}^N \sum_{a=1}^3 \sum_{\nu=1}^{\mathcal{N}} \omega_{\nu} b_{\nu,a}^{(n)\dagger} b_{\nu,a}^{(n)} . \quad (69)$$

In absence of spin-bath coupling, the object-apparatus comes back to its initial state at the recurrence times  $t_{\text{rec}}^{(j)} = \pi g^{-1} j$ ,  $j = 1, 2, \dots$  (see below). Between  $t = 0$  and  $t_{\text{rec}}^{(1)}$ , the information on the object coherences spreads out among the  $N$  spins of the apparatus. The role of the spin-bath coupling is twofold: firstly, it introduces some irreversibility in the dynamics, which prevents  $\mathcal{S}$  and  $\mathcal{A}$  to come back to the initial state at times  $t_{\text{rec}}^{(j)}$  and allows for an irretrievable loss of information on the initial coherences of  $\mathcal{S}$  (this information, after being stored in the  $N$ -spin state, can “escape” to the baths)<sup>13</sup>.

<sup>13</sup>Let us recall that in the model of Sec. 3, the information about the object coherences had first to be “transmitted” to the pointer before being spread out among the infinitely many degrees of freedom of the bath

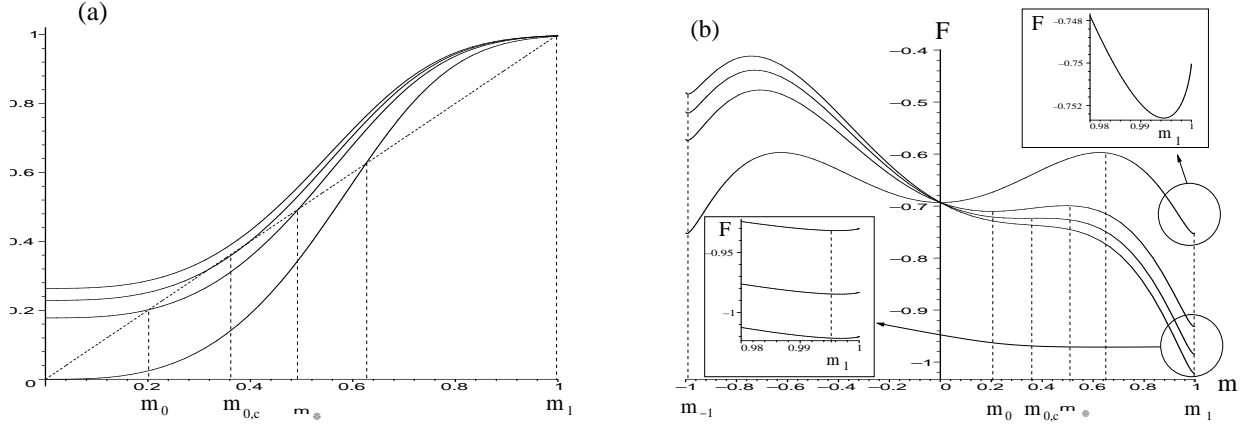


Figure 7: (a)  $\tanh(\kappa(m^3 + \alpha))$  as a function of  $m$  for  $\kappa = 3$  and  $\alpha = 0.09, 0.077, 0.06$  and  $0$  (solid curves, from top to bottom). The abscissa of the vertical dashed lines give the solutions of (73). (b) Free energy per spin of the apparatus,  $F/N$ , as a function of  $m$  for  $s = 1$  and for the same values of  $\kappa$  and  $\alpha$  as in (a). Energy is measured in unit of  $k_B T$ . The curve with the higher maximum near  $m = 1$  corresponds to  $\alpha = 0$ , the other ones having a lower maxima correspond to  $\alpha = 0.06, 0.077$  and  $0.09$ . (Insets: same for  $m$  varying between  $0.98$  and  $1$ ).

Secondly, it allows for a symmetry breaking, leading to a spontaneous magnetisation of  $\mathcal{A}$  depending on the initial value  $s = \pm 1$  of the spin  $\mathcal{S}$ .

#### 4.1.2 Initial state

Unlike in the model discussed in Sec. 3, we assume that  $\mathcal{A}$  and  $\mathcal{B}$  are *not initially correlated*. In view also of the requirement 1-3 of Sec. 2.1, we consider the object-apparatus-bath initial state

$$\hat{\rho}_{SAB}(0) = |\psi\rangle\langle\psi| \otimes \hat{\rho}_A(0) \otimes \hat{\rho}_B^{\text{eq}} \quad \text{with} \quad |\psi\rangle = c_+ |\uparrow\rangle + c_- |\downarrow\rangle \quad (70)$$

and

$$\hat{\rho}_A(0) = 2^{-N} \quad , \quad \hat{\rho}_B^{\text{eq}} = Z_B^{-1} e^{-\beta \hat{H}_B} . \quad (71)$$

The object is initially in a pure state given by a linear superposition of spin up and down. The apparatus is initially in a metastable state with a vanishing magnetisation  $m_z = \text{tr}(\hat{\rho}_A(0) \hat{m}_z) = 0$ . Its density matrix  $\hat{\rho}_A(0)$  is proportional to the identity operator on  $\mathbb{C}^{2N}$ . The bath is initially in thermal equilibrium at temperature  $T = (k_B \beta)^{-1}$ .

#### 4.1.3 Equilibrium states of the apparatus

As we will see below, the initial state of  $\mathcal{A}$  evolves under the couplings  $\hat{H}_{SA}$  and  $\hat{H}_{AB}$  to an equilibrium state. What are the apparatus equilibria in absence of coupling with the bath ?

and therefore irretrievably lost. In the present model, this information first spreads out among the  $N$  spins of  $\mathcal{A}$ , thanks to the direct coupling of  $\mathcal{S}$  with these  $N$  spins. For finite  $N$  this information could still be accessible if the spins were not coupled to  $\mathcal{B}$ : one could then reconstruct the object initial state at times  $t_{\text{rec}}^{(j)}$ . As far as decoherence is concerned,  $\mathcal{B}$  only plays the role of preventing this reconstruction by introducing irreversibility in the dynamics. The choice of a bath of harmonic oscillators with linear couplings should therefore not matter much on the decoherence process if the apparatus-bath coupling strength is small enough or  $N$  is large enough. This fact will be confirmed in Sec. 4.2.

Such equilibria minimise the apparatus free energy  $F = E - TS$ . Replacing the  $z$ -component  $\hat{\sigma}_z^{(0)}$  of the spin  $\mathcal{S}$  by its eigenvalue  $s = \pm 1$  in (67) and using the entropy of  $N$  independent spins (as will be justified in the next section by invoking the mean-field approximation), we obtain

$$F = -\frac{1}{4}JNm^4 - gNsm + Nk_B T \left\{ \frac{1+m}{2} \ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2} \ln\left(\frac{1-m}{2}\right) \right\}. \quad (72)$$

The extrema of  $F$  are given by the implicit equation

$$m = \tanh(\beta\mu h(m)) \quad , \quad h(m) = \mu^{-1}(Jm^3 + gs). \quad (73)$$

Let us first consider the case  $s = 1$ . We set  $\kappa = \beta J$  and  $\alpha = g/J$ . For fixed  $\kappa$  and  $\alpha$ , the approximate solutions of (73) can be easily obtained graphically, see Fig. 7(a). It not difficult to prove that, for a fixed  $\kappa$ , there exists some  $\alpha_c > 0$  with the following property:

- (1) for  $\alpha < \alpha_c$ , the function  $f_{\kappa,\alpha}(m) = \tanh(\kappa(m^3 + \alpha)) - m$  has three roots  $m_0$ ,  $m_*$  and  $m_1$  on  $[0, 1]$ ; for  $g = \alpha = 0$ , the first root is  $m_0 = 0$ .
- (2) for  $\alpha > \alpha_c$  it has only one root  $m_1$  on  $[0, 1]$ ;
- (3) for  $\alpha = \alpha_c$ , it has two roots  $m_{0,c}$  and  $m_{1,c}$  on  $[0, 1]$ .

The critical value  $\alpha_c$  can be obtained from the condition  $f_{\kappa,\alpha}'(m_{0,c}) = f_{\kappa,\alpha}(m_{0,c}) = 0$  for some  $m_{0,c} \in [0, 1]$ . This gives  $\cosh(\kappa(m_{0,c}^3 + \alpha)) = \sqrt{3\kappa} m_{0,c}$  and  $m_{0,c}^2 = \sqrt{1/4 + (3\kappa)^{-1}} - 1/2$ . One finds

$$\alpha_c = \frac{1}{\kappa} \operatorname{argth} \left( \left( \sqrt{1/4 + (3\kappa)^{-1}} - 1/2 \right)^{1/2} \right) - \left( \sqrt{1/4 + (3\kappa)^{-1}} - 1/2 \right)^{3/2}. \quad (74)$$

In Fig. 7(b),  $F$  is shown as a function of  $m$  for  $\kappa = 3$  and for three values of  $\alpha$  smaller, equal and larger than  $\alpha_c \simeq 0.0777$ .

- (1) For  $\alpha < \alpha_c$ , the smallest positive solution  $m_0$  of (73) corresponds to a local minimum of  $F$ . In fact,  $(\partial^2 F / \partial m^2)(m_0) = N\beta^{-1}(-3\kappa m_0^2 + (1 - m_0^2)^{-1})$  is positive since  $m_0^2 < m_{0,c}^2 = \sqrt{1/4 + (3\kappa)^{-1}} - 1/2 < 1/2 - \sqrt{1/4 - (3\kappa)^{-1}}$ . The largest solution  $m_1 \in [m_0, 1]$  of (73) has also  $(\partial^2 F / \partial m^2)(m_1) > 0$  and has a lower free energy than  $m_0$ . It corresponds to a ferromagnetic equilibrium state, with magnetisation  $\mathcal{M}_z = \mu N m_1 \simeq \mu N$  for large  $\kappa$ . The smallest solution  $m_{-1}$  of (73) in the interval  $[-1, 0]$  gives another local minimum of  $F$  which is higher than  $m_1$  when  $g > 0$ . Note that for  $g = 0$ , the paramagnetic state with  $m_z = m_0 = 0$  is a metastable equilibrium. The free energy is then an even function of  $m$  and one has two equilibria with magnetisation  $m_{\pm 1}$  (see Fig. 7(b)).
- (2) For  $\alpha > \alpha_c$ , the only states for which  $F$  has a local minimum are the ferromagnetic states  $m = m_{\pm 1} \simeq \pm 1$ . The equilibrium state corresponds to the largest solution  $m_1 \in [m_0, 1]$ . Hence, when  $g > g_c = J\alpha_c$  the object-apparatus interaction is sufficient to suppress the barrier in the free energy (Fig. 7(b)), in much the same way as for the pointer potential in the model of lecture 2.

The case  $s = -1$  can be deduced from  $s = 1$  by reverting the sign of  $m$  (the free energy (72) is then left unchanged).

In presence of a small amount of dissipation, the apparatus relaxes to the metastable or equilibrium state closest to its initial paramagnetic state. For  $s = \pm 1$ , this state has a magnetisation

$$\mathcal{M}_z = \begin{cases} \pm \mu N m_0 & \text{if } g < g_c(J, \beta) = J\alpha_c \\ \pm \mu N m_1 & \text{if } g > g_c(J, \beta). \end{cases} \quad (75)$$

The pointer variable  $\mathcal{M}_z$  ends up in macroscopically distinct values according to the spin  $\mathcal{S}$  eigenvalue  $s$ , up to small fluctuations. This is in agreement with the requirement 2 of Sec. 2.1.

Let the object-apparatus interaction be switched-off gradually after the aforementioned relaxation has taken place, say after time  $t_{\text{int}}$ . The magnetisation then evolves adiabatically at times  $t > t_{\text{int}}$  to the solution of (73) for  $g = 0$ .

- (1) If the object-apparatus coupling constant  $g$  in the time interval  $[0, t_{\text{int}}]$  is smaller than the critical value  $g_c(J, \beta)$ , the pointer variable  $\mathcal{M}_z$  thus returns to its initial value  $\mathcal{M}_z = 0$  after the  $\mathcal{S}\text{-}\mathcal{A}$  coupling has been switched off.
- (2) On the contrary, if  $g > g_c(J, \beta)$  the apparatus remains in a ferromagnetic state, with magnetisation  $\mathcal{M}_z \simeq \pm \mu N$  if  $\kappa \gg 1$ . In this case the pointer variable  $\mathcal{M}_z$  acts as a *permanent register* of the measurement result.

#### 4.1.4 Bath correlation function

Denoting by  $\langle \cdot \rangle_{\text{eq}}$  the average w.r.t. the bath equilibrium state  $\hat{\rho}_B^{\text{eq}}$ , we remark that  $\langle \hat{B}_a^{(n)} \rangle_{\text{eq}} = 0$  and, by virtue of the independence of the bath modes  $(\nu, a, n)$  and  $(\nu', a', n')$  for  $a \neq a'$  or  $n \neq n'$ ,

$$K_{a,a'}^{(n,n')}(t) = \langle e^{it\hat{H}_B} \hat{B}_a^{(n)} e^{-it\hat{H}_B} \hat{B}_{a'}^{(n')} \rangle_{\text{eq}} = \delta_{a,a'} \delta_{n,n'} K(t). \quad (76)$$

It is well known that for a bath of harmonic oscillators with a linear coupling as in (68), the imaginary part of  $K(t)$  is temperature-independent, its Fourier transform being given by  $\widehat{(\Im K)}(\omega) = -\widehat{(\Im K)}(-\omega) = -iJ(\omega)$ ,  $\omega \geq 0$ , where

$$J(\omega) = \frac{\pi}{\mathcal{N}} \sum_{\nu=1}^{\mathcal{N}} |\kappa_\nu|^2 \delta(\omega - \omega_\nu) \quad (77)$$

is the so-called power spectrum function [13]. Moreover, the real part of  $K(t)$  is temperature-dependent, its Fourier transform is given by the KMS relation (48). For a Ohmic bath,  $J(\omega)$  behaves linearly with  $\omega$  at low frequencies. We take here:

$$J(\omega) = \frac{1}{8} \hat{\gamma} \omega e^{-\omega/\omega_D}, \quad \omega \geq 0$$

where  $\omega_D$  a cutoff frequency (Debye frequency)<sup>14</sup>. We conclude that the bath correlation function  $K(t)$  appearing in (76) is given by

$$\Re K(t) = \hat{\gamma} \int_0^\infty \frac{d\omega}{8\pi} \cos(\omega t) \omega \coth\left(\frac{\beta\omega}{2}\right) e^{-\omega/\omega_D}, \quad \Im K(t) = -\hat{\gamma} \int_0^\infty \frac{d\omega}{8\pi} \sin(\omega t) \omega e^{-\omega/\omega_D}. \quad (78)$$

At low temperature  $k_B T \leq \omega_D$ , the bath correlation time after which  $K(t) \simeq 0$  is equal to  $T_B = \beta = (k_B T)^{-1}$  (see Sec.3.1.2).

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<sup>14</sup>Note that a slightly different power spectrum function was taken in lecture 2.

### 4.1.5 Separation of time scales

We assume in what follows the following separation of time scales

$$t_B = \omega_D^{-1} \ll T_B = \beta, J^{-1}, g^{-1}. \quad (79)$$

The first limit means that one considers baths at very low temperatures, see Sec.3.5.3. In addition,  $T_B$ ,  $J^{-1}$  and  $g^{-1}$  must be much smaller than the time scale  $T_S$  for the free evolution of the object (in order to justify that one can neglect  $H_S$ , see the discussion in Sec. 2.3). Unlike in the model of lecture 2, we may take the object-apparatus interaction time  $t_{\text{int}}$  to be of the same order as the time scales  $J^{-1}$  and  $g^{-1}$  for the evolution of the pointer variable  $\mathcal{M}_z$  under the apparatus Hamiltonian  $H_A$ . We shall see below that  $J^{-1}$  and  $g^{-1}$  are much larger than the decoherence time for  $N \gg 1$ , although it is not necessary to make such an assumption at this point.

## 4.2 Dynamics

### 4.2.1 Redfield equation

In the weak coupling limit  $K(0)^2 T_B^3 t \ll 1$ , the reduced object-apparatus density matrix (2) satisfies the time-dependent Bloch-Redfield equation [15]

$$\frac{d\hat{\rho}_{SA}}{dt} = -i[\hat{H}_A + \hat{H}_{SA}, \hat{\rho}_{SA}(t)] + \sum_{n=1}^N \sum_{a=1}^3 \int_0^t d\tau \left\{ K(\tau) [\tilde{\sigma}_a^{(n)}(-\tau) \hat{\rho}_{SA}(t), \hat{\sigma}_a^{(n)}] + \text{h.c.} \right\} \quad (80)$$

where  $\tilde{\sigma}_a^{(n)}(\tau)$  is the  $a$ -component of the  $n$ th spin in the interaction picture,

$$\tilde{\sigma}_a^{(n)}(\tau) = e^{i\tau(\hat{H}_A + \hat{H}_{SA})} \hat{\sigma}_a^{(n)} e^{-i\tau(\hat{H}_A + \hat{H}_{SA})}. \quad (81)$$

It is convenient to introduce for each pair  $(s, s')$  of spin  $\mathcal{S}$  values the corresponding ‘‘apparatus density matrix’’  $\hat{\rho}_{A,ss'}(t)$  defined by

$$\hat{\rho}_{A,ss'}(t) c_s c_{s'}^* = \langle s | \hat{\rho}_{SA}(t) | s' \rangle. \quad (82)$$

Note that  $\hat{\rho}_{A,\uparrow\downarrow}(t) = \hat{\rho}_{A,\downarrow\uparrow}(t)^\dagger$  is not necessarily hermitian. The ‘‘diagonal’’ contributions  $\hat{\rho}_{A,\uparrow\uparrow}(t)$  and  $\hat{\rho}_{A,\downarrow\downarrow}(t)$  are hermitian and satisfy

$$|c_\uparrow|^2 \text{tr} \hat{\rho}_{A,\uparrow\uparrow}(t) + |c_\downarrow|^2 \text{tr} \hat{\rho}_{A,\downarrow\downarrow}(t) = 1. \quad (83)$$

One may associate to each spin sector  $(s, s')$  its dimensionless magnetisation per spin at time  $t$ ,

$$m_{ss'}(t) = \langle \hat{m}_z \rangle_{ss'}(t) = \frac{\text{tr}(|\hat{\rho}_{A,ss'}(t)| \hat{m}_z)}{\text{tr}(|\hat{\rho}_{A,ss'}(t)|)}. \quad (84)$$

In general  $|\hat{\rho}_{A,ss'}(t)| = (\hat{\rho}_{A,ss'}(t)^\dagger \hat{\rho}_{A,ss'}(t))^{1/2}$  may differ from  $|\hat{\rho}_{A,s's}(t)|$  for  $s \neq s'$  and thus  $m_{ss'}(t) \neq m_{s's}(t)$ .

The Redfield equation (80) can be rewritten as four differential equations in each spin sector  $(s, s')$ ,

$$\begin{aligned} \frac{d\hat{\rho}_{A,ss'}}{dt} = & -i[\hat{H}_A, \hat{\rho}_{A,ss'}(t)] + i g N s \left( \delta_{s,s'} [\hat{m}_z, \hat{\rho}_{A,ss'}(t)] + \delta_{s,-s'} \{ \hat{m}_z, \hat{\rho}_{A,ss'}(t) \} \right) \\ & + \sum_{n=1}^N \sum_{a=1}^3 \int_0^t d\tau \left\{ K(\tau) [\tilde{\sigma}_a^{(n)}(s; -\tau) \hat{\rho}_{A,ss'}(t), \hat{\sigma}_a^{(n)}] + (\text{h.c.}, s \leftrightarrow s') \right\} \end{aligned} \quad (85)$$

with  $\tilde{\sigma}_a^{(n)}(s; \tau) = \langle s | \tilde{\sigma}_a^{(n)}(\tau) | s \rangle$ .



### 4.2.2 Dynamical Mean-Field Approximation

The interactions among the spins of  $\mathcal{A}$  will be described with the help of a Dynamical Mean-Field Approximation (DMFA). One neglects in this approximation the spin fluctuations  $\langle (\hat{m}_z - m_{ss'}(t))^k \rangle_{ss'}(t)$  with  $k = 2, 3, 4$ , where  $\langle \cdot \rangle_{ss'}(t)$  denotes the average w.r.t.  $|\rho_{ss'}(t)|$ , see (84). This amounts to substitute the apparatus Hamiltonian (67) by

$$\hat{H}_A \longrightarrow \hat{H}_A^{\text{M.F.}}(s, s'; t) = -JNm_{ss'}(t)^3 (\hat{m}_z - m_{ss'}(t)) \quad (86)$$

apart from an irrelevant constant  $-JNm_{ss'}(t)^4/4$ . Within the DMFA, we seek a solution of (85) in the product form

$$\rho_{A,ss'}(t) = \rho_{ss'}^{(1)}(t) \otimes \cdots \otimes \rho_{ss'}^{(N)}(t) . \quad (87)$$

In other terms, we disregard the entanglement between the spins of  $\mathcal{A}$ . It is important to notice that making a standard DMFA for the  $(N+1)$  spins of  $\mathcal{S} + \mathcal{A}$  would amount to disregard also the entanglement between  $\mathcal{S}$  and  $\mathcal{A}$ , a very bad idea ! For indeed, as was pointed out in the first lecture, this entanglement plays a major role in a QM. The justification of the DMFA used here should certainly merit more attention. We will, however, not elaborate on this point.

Replacing (86) and (87) into (85), one gets for all  $n = 1, \dots, N$ ,

$$\begin{aligned} \frac{d\hat{\rho}_{ss'}^{(n)}}{dt} = & \text{i}Jm_{ss'}(t)^3 [\hat{\sigma}_z^{(n)}, \hat{\rho}_{ss'}^{(n)}(t)] + \text{i}gs \left( \delta_{s,s'} [\hat{\sigma}_z^{(n)}, \hat{\rho}_{ss'}^{(n)}(t)] + \delta_{s,-s'} \{ \hat{\sigma}_z^{(n)}, \hat{\rho}_{ss'}^{(n)}(t) \} \right) \\ & + \sum_{a=1}^3 \int_0^t d\tau \left\{ K(\tau) [\tilde{\sigma}_{a,\text{M.F.}}^{(n)}(s, s'; -\tau, t) \hat{\rho}_{ss'}^{(n)}(t), \hat{\sigma}_a^{(n)}] + (\text{h.c.}, s \leftrightarrow s') \right\} \end{aligned} \quad (88)$$

where  $\tilde{\sigma}_{a,\text{M.F.}}^{(n)}(s, s'; \tau, t)$  is the spin operator  $\langle s | \tilde{\sigma}_a^{(n)}(\tau) | s \rangle$  in the interaction picture in the DMFA. This spin operator is obtained by replacing  $\hat{H}_A + \hat{H}_{SA}$  in (81) by the following mean-field Hamiltonian in the  $(s, s')$ -sector

$$\hat{H}_A^{\text{M.F.}}(s, s'; t) - gNs\hat{m}_z = -\mu Nh_{ss'}(t)\hat{m}_z + \text{const.} . \quad (89)$$

The  $z$ -component of each spin of  $\mathcal{A}$  is coupled to the magnetic field

$$h_{ss'}(t) = \mu^{-1} (Jm_{ss'}(t)^3 + gs) \quad (90)$$

created by all other spins of  $\mathcal{A}$  and by the spin  $\mathcal{S}$  in the eigenstate  $|s\rangle$  of  $\hat{\sigma}_z^{(0)}$ . This magnetic field depends on the spin sector  $(s, s')$  and on time  $t$  at which one looks at the apparatus density matrix  $\rho_{A,ss'}(t)$ . It is clear on the form of the Hamiltonian (89) that  $\tilde{\sigma}_{a,\text{M.F.}}^{(n)}(s, s'; \tau, t)$  acts as an identity operator on the Hilbert spaces of all spins save for spin  $n$ . To simplify the notation, we momentarily omit the indices  $n, s$  and  $s'$ . In view of (66), (81) and (89),

$$\frac{\partial \tilde{\sigma}_{a,\text{M.F.}}(\tau, t)}{\partial \tau} = -\text{i}\mu h(t) [\hat{\sigma}_z, \tilde{\sigma}_{a,\text{M.F.}}(\tau, t)] . \quad (91)$$

Expanding  $\tilde{\sigma}_{a,\text{M.F.}}(\tau, t)$  in terms of the Pauli matrices  $\hat{\sigma}_1, \hat{\sigma}_2$  and  $\hat{\sigma}_3 = \hat{\sigma}_z$  and using the commutation relation  $[\hat{\sigma}_a, \hat{\sigma}_b] = 2\text{i} \sum_{c=1}^3 \epsilon_{abc} \hat{\sigma}_c$  (where  $\epsilon_{abc} = 1$  if  $(a, b, c)$  is a circular permutation of  $(1, 2, 3)$ ,  $-1$  if it is a circular permutation of  $(2, 1, 3)$  and  $0$  otherwise), a short calculation gives

$$\begin{aligned} \tilde{\sigma}_{1,\text{M.F.}}(\tau, t) &= \cos(2\mu h(t)\tau) \hat{\sigma}_1 + \sin(2\mu h(t)\tau) \hat{\sigma}_2 \\ \tilde{\sigma}_{2,\text{M.F.}}(\tau, t) &= -\sin(2\mu h(t)\tau) \hat{\sigma}_1 + \cos(2\mu h(t)\tau) \hat{\sigma}_2 \\ \tilde{\sigma}_{3,\text{M.F.}}(\tau, t) &= \hat{\sigma}_3 . \end{aligned} \quad (92)$$

The last identity is a consequence of  $[\hat{\sigma}_3, \hat{H}_A + \hat{H}_{SA}] = 0$  and does in fact *not* rely on the DMFA.

Let us write

$$\hat{\rho}_{ss'}(t) = \frac{1}{2} \sum_{b=0}^3 \zeta_{b,ss'}(t) \hat{\sigma}_b \quad (93)$$

where  $\hat{\sigma}_0 = 1$  is the  $2 \times 2$  identity matrix and  $\zeta_{b,ss'}(t)$  are complex-valued coefficients. These coefficients fulfil  $\zeta_{b,ss'}(t)^* = \zeta_{b,s's}(t)$  because  $\hat{\rho}_{A,ss'}(t)^\dagger = \hat{\rho}_{A,s's}(t)$ . It is easy (but somehow tedious) to derive the system of coupled differential equations satisfied by the  $\zeta_{b,ss'}(t)$ 's. By replacing (92) and (93) into (88), using the aforementioned commutation relations for the Pauli matrices and  $\{\hat{\sigma}_a, \hat{\sigma}_b\} = 2(\delta_{a,b} + \delta_{a,0}\hat{\sigma}_b + \delta_{b,0}\hat{\sigma}_a - 2\delta_{a,0}\delta_{b,0})$ , one gets

$$\begin{cases} \frac{d\zeta_{0,ss'}}{dt} &= 2igs \delta_{s,-s'} \zeta_{3,ss'}(t) \\ \frac{d\zeta_{1,ss'}}{dt} &= -[\Lambda_{ss'}(t; \underline{h}(t)) + \Lambda_0(t)] \zeta_{1,ss'}(t) + [2Jm_{ss'}(t)^3 + 2gs\delta_{s,s'} + \Upsilon_{ss'}(t; \underline{h}(t))] \zeta_{2,ss'}(t) \\ \frac{d\zeta_{2,ss'}}{dt} &= -[2Jm_{ss'}(t)^3 + 2gs\delta_{s,s'} + \Upsilon_{ss'}(t; \underline{h}(t))] \zeta_{1,ss'}(t) - [\Lambda_{ss'}(t; \underline{h}(t)) + \Lambda_0(t)] \zeta_{2,ss'}(t) \\ \frac{d\zeta_{3,ss'}}{dt} &= 2ig[s\delta_{s,-s'} + \eta_{ss'}(t; \underline{h}(t))] \zeta_{0,ss'}(t) - 2\Lambda_{ss'}(t; \underline{h}(t)) \zeta_{3,ss'}(t) \end{cases} \quad (94)$$

where  $\underline{h}(t)$  is the  $2 \times 2$  matrix with coefficients given by (90),  $\Lambda_0(t) = 4 \int_0^t d\tau \Re K(\tau)$  and

$$\begin{aligned} \Lambda_{ss'}(t; \underline{h}(t)) &= 2 \int_0^t d\tau \left\{ K(\tau) \cos(2\mu h_{ss'}(t)\tau) + K(\tau)^* \cos(2\mu h_{s's}(t)\tau) \right\} \\ \Upsilon_{ss'}(t; \underline{h}(t)) &= 2 \int_0^t d\tau \left\{ K(\tau) \sin(2\mu h_{ss'}(t)\tau) + K(\tau)^* \sin(2\mu h_{s's}(t)\tau) \right\} \\ \eta_{ss'}(t; \underline{h}(t)) &= \frac{2}{g} \int_0^t d\tau \left\{ K(\tau) \sin(2\mu h_{ss'}(t)\tau) - K(\tau)^* \sin(2\mu h_{s's}(t)\tau) \right\}. \end{aligned} \quad (95)$$

#### 4.2.3 Solution of the Bloch-Redfield equation within the DMFA

The initial condition (71) can be rewritten thanks to (82), (87) and (93) as

$$\zeta_{a,ss'}(0) = \delta_{a,0} \quad , \quad a = 0, \dots, 3, \quad s, s' = \uparrow, \downarrow. \quad (96)$$

The solution of (94) with this initial condition satisfies

$$\begin{aligned} \zeta_{1,ss'}(t) = \zeta_{2,ss'}(t) &= 0 \quad \forall s, s' = \uparrow, \downarrow, \quad \forall t \\ \zeta_{0,\uparrow\uparrow}(t) = \zeta_{0,\downarrow\downarrow}(t) &= 1 \quad \forall t \end{aligned} \quad (97)$$

The last inequality implies  $\text{tr}(\hat{\rho}_{\uparrow\uparrow}(t)) = \text{tr}(\hat{\rho}_{\downarrow\downarrow}(t)) = 1$  at any time  $t \geq 0$  (this is consistent with (83)). A short calculation using (97) yields

$$|\rho_{\uparrow\downarrow}(t)| = |\rho_{\downarrow\uparrow}(t)| = \frac{1}{2} \begin{pmatrix} |\zeta_{0,\uparrow\downarrow}(t) + \zeta_{3,\uparrow\downarrow}(t)| & 0 \\ 0 & |\zeta_{0,\uparrow\downarrow}(t) - \zeta_{3,\uparrow\downarrow}(t)| \end{pmatrix}. \quad (98)$$

By (84) and (87),  $m_{ss'}(t) = \text{tr}(|\rho_{ss'}(t)|\sigma_z) / \text{tr} |\rho_{ss'}(t)|$ . Thus, in view of (93), (97) and (98),

$$m_{ss}(t) = \zeta_{3,ss}(t) \quad , \quad m_{\uparrow\downarrow}(t) = m_{\downarrow\uparrow}(t) = \frac{2\Re\{\zeta_{0,\uparrow\downarrow}(t)^* \zeta_{3,\uparrow\downarrow}(t)\}}{|\zeta_{0,\uparrow\downarrow}(t)|^2 + |\zeta_{3,\uparrow\downarrow}(t)|^2}. \quad (99)$$

For  $s = s' = 1$ , the last equation in (94) becomes

$$\frac{dm_{\uparrow\uparrow}}{dt} = -2\Lambda_{\uparrow\uparrow}(t; \underline{h}(t)) m_{\uparrow\uparrow}(t) + 2ig \eta_{\uparrow\uparrow}(t; \underline{h}(t)) \quad (100)$$

with the initial condition  $m_{\uparrow\uparrow}(0) = 0$ . The functions  $\Lambda$  and  $\eta$  are given by

$$\begin{aligned} \Lambda_{\uparrow\uparrow}(t; \underline{h}(t)) &= 4 \int_0^t d\tau \Re K(\tau) \cos(2\mu h_{\uparrow\uparrow}(t)\tau) \\ \eta_{\uparrow\uparrow}(t; \underline{h}(t)) &= \frac{4i}{g} \int_0^t d\tau \Im K(\tau) \sin(2\mu h_{\uparrow\uparrow}(t)\tau) . \end{aligned} \quad (101)$$

with  $h_{\uparrow\uparrow}(t) = \mu^{-1}(Jm_{\uparrow\uparrow}(t)^3 + g)$ . An equation similar to (100) holds for  $s = s' = -1$ .

For  $s = 1$  and  $s' = -1$ , the first and last equations in (94) are equivalent to

$$\begin{cases} \frac{d^2 \zeta_{0,\uparrow\downarrow}}{dt^2} + 2\Lambda_{\uparrow\downarrow}(t; \underline{h}(t)) \frac{d\zeta_{0,\uparrow\downarrow}}{dt} + 4g^2 [1 + \eta_{\uparrow\downarrow}(t; \underline{h}(t))] \zeta_{0,\uparrow\downarrow}(t) = 0 \\ \zeta_{3,\uparrow\downarrow}(t) = -\frac{i}{2g} \frac{d\zeta_{0,\uparrow\downarrow}}{dt} \end{cases} \quad (102)$$

with the initial conditions  $\zeta_{0,\uparrow\downarrow}(0) = 1$  and  $(d\zeta_{0,\uparrow\downarrow}/dt)(0) = 0$ . The solution of (102) with this initial condition satisfies  $\zeta_{0,\uparrow\downarrow}(t) \in \mathbb{R}$  and  $\zeta_{3,\uparrow\downarrow}(t) \in i\mathbb{R}$  for all times  $t$ <sup>15</sup>. Thus

$$m_{\uparrow\downarrow}(t) = m_{\downarrow\uparrow}(t) = 0 \quad (103)$$

$$\begin{aligned} \Lambda_{\uparrow\downarrow}(t; \underline{h}(t)) &= 4 \int_0^t d\tau \Re K(\tau) \cos(2g\tau) \\ \eta_{\uparrow\downarrow}(t; \underline{h}(t)) &= \frac{4}{g} \int_0^t d\tau \Re K(\tau) \sin(2g\tau) . \end{aligned} \quad (104)$$

Note that in absence of coupling with the bath, i.e. for  $\hat{\gamma} = 0$ , (100) and (102) are easily integrated and yields

$$\hat{\rho}_{A,\uparrow\uparrow}(t) = \hat{\rho}_{A,\downarrow\downarrow}(t) = 2^{-N} , \quad \hat{\rho}_{A,\uparrow\downarrow}(t) = 2^{-N} \exp(2igtN\hat{m}_z) . \quad (105)$$

The diagonal contributions remain unchanged and the matrix elements of the off-diagonal contribution  $\hat{\rho}_{A,\uparrow\downarrow}(t)$  oscillates in the joint eigenbasis of the  $\hat{\sigma}_z^{(n)}$  with the period  $\pi(gNm_z)^{-1}$ , where  $Nm_z \neq 0$  is the sum of the eigenvalues of the  $\hat{\sigma}_z^{(n)}$ . As a result,  $\hat{\rho}_{SA}(t_{\text{rec}}^{(j)}) = \hat{\rho}_{SA}(0)$  with  $t_{\text{rec}}^{(j)} = (\pi/g)j$ ,  $j = 1, 2, \dots$ . In absence of apparatus-bath coupling the object and apparatus therefore come back to their initial states at the recurrence times  $t_{\text{rec}}^{(j)}$ .

#### 4.2.4 Interaction-dominated regime

In the time regime  $t \ll t_B = \omega_D^{-1}$ , the apparatus-bath interaction dominates the dynamics (in fact, by our assumption on times scales (79) one has also  $t \ll g^{-1}, J^{-1}$ ). The real part of the correlation function  $K(\tau)$  can be approximated at time  $\tau \ll t_B$  by  $\Re K(0) = \hat{\gamma} \int_0^\infty d\omega \omega \coth(\beta\omega/2) e^{-\omega/\omega_D} / (8\pi)$ , see (104). Similarly,  $\Im K(\tau) \simeq -\hat{\gamma} \int_0^\infty d\omega \omega^2 e^{-\omega/\omega_D} \tau / (8\pi)$ . Using also the limit  $\omega_D^{-1} \ll \beta$  in (79), we get

$$\Re K(\tau) \simeq \frac{\hat{\gamma} \omega_D^2}{8\pi} , \quad \Im K(\tau) \simeq -\frac{\hat{\gamma} \omega_D^3}{4\pi} \tau , \quad \tau \ll \omega_D^{-1} \ll \beta . \quad (106)$$

<sup>15</sup>Actually, the first equation in (102) admits a real solution since if  $\zeta_{0,\uparrow\downarrow}(t) \in \mathbb{R} \forall t$  then  $\zeta_{3,\uparrow\downarrow}(t) \in i\mathbb{R} \forall t$  by the second equation of (102), so that (103) holds true by (99); by virtue of (95), the coefficients  $\Lambda_{\uparrow\downarrow}(t; \underline{h}(t))$  and  $\eta_{\uparrow\downarrow}(t; \underline{h}(t))$  are then real and given by (104).

We may replace  $\cos(2\mu h_{\uparrow\uparrow}(t)\tau)$  and  $\cos(2g\tau)$  by 1,  $\sin(2\mu h_{\uparrow\uparrow}(t)\tau)$  by  $2(Jm_{\uparrow\uparrow}(t)^3 + g)\tau \simeq 2g\tau$  and  $\sin(2g\tau)$  by  $2g\tau$  in (101) and (104) to get

$$\begin{aligned}\Lambda_{\uparrow\uparrow}(t; \underline{h}(t)) &\simeq \Lambda_{\uparrow\downarrow}(t; \underline{h}(t)) \simeq \frac{\hat{\gamma}\omega_D^2}{2\pi} t \\ \eta_{\uparrow\uparrow}(t; \underline{h}(t)) &\simeq -2i \frac{\hat{\gamma}\omega_D^3}{3\pi} t^3 \\ \eta_{\uparrow\downarrow}(t; \underline{h}(t)) &\simeq \frac{\hat{\gamma}\omega_D^2}{2\pi} t^2\end{aligned}, \quad t \ll \omega_D^{-1} \ll \beta, g^{-1}, J^{-1}.$$

Eq. (100) is then easily integrated and gives for  $t \ll \omega_D^{-1}$

$$m_{\uparrow\uparrow}(t) \simeq -m_{\downarrow\downarrow}(t) \simeq \frac{\hat{\gamma}\omega_D^3 g}{3\pi} t^4. \quad (107)$$

One thus finds that  $m_{\uparrow\uparrow}(t)$  is positive and increases with time, whereas  $m_{\downarrow\downarrow}(t)$  is negative and time decreasing.

Let us seek a solution of (102) in the limit  $t \ll \omega_D^{-1}$  of the form  $\zeta_{0,\uparrow\downarrow}(t) = e^{-\chi(t)} \cos(\theta(t))$  with  $\chi(t)$  an even real function such that  $\chi(0) = 0$  and  $\theta(t)$  an odd real function. Setting  $\chi(t) = at^2 + bt^4 + \mathcal{O}(t^6)$  and  $\theta(t) = dt + et^3 + \mathcal{O}(t^5)$  and calculating the real coefficients  $a, b, d$  and  $e$ , one easily obtain

$$\begin{aligned}\chi(t) &= \frac{\hat{\gamma}\omega_D^2 g^2}{2\pi} t^4 + \mathcal{O}(t^6) \\ \theta(t) &= 2gt \left(1 - \frac{\hat{\gamma}\omega_D^2}{6\pi} t^2\right)\end{aligned} \quad (108)$$

We note that  $d\theta/dt = 2g - (d\chi/dt)/\theta(t) + \mathcal{O}(t^4)$ . Disregarding terms of order  $(\omega_D t)^5$ , one gets

$$\zeta_{0,\uparrow\downarrow}(t) \simeq e^{-\chi(t)} \cos(\theta(t)) \quad , \quad \zeta_{3,\uparrow\downarrow}(t) \simeq ie^{-\chi(t)} \sin(\theta(t)) \quad , \quad t \ll \omega_D^{-1}. \quad (109)$$

In view of (87), (93), (97), (99), (102), (108) and (109), the object-apparatus density matrix reads at in time  $t \ll \omega_D^{-1}$

$$\begin{aligned}\rho_{A,ss}(t) &\simeq \left(\frac{1}{2} + \frac{m_{ss}(t)}{2} \sigma_3\right)^{\otimes N} \\ \rho_{A,\uparrow\downarrow}(t) &\simeq \exp\left\{-\left(\frac{t}{t_{\text{dec}}}\right)^4\right\} \frac{1}{2^N} \exp(iN\theta(t)\hat{m}_z)\end{aligned} \quad (110)$$

with

$$t_{\text{dec}} = \left(\frac{2\pi}{\hat{\gamma}N}\right)^{1/4} \sqrt{\frac{1}{\omega_D g}}. \quad (111)$$

It is interesting to compare (110) with the result (105) in absence of apparatus-bath coupling. In presence of such a coupling, the diagonal contributions  $\rho_{A,ss}(t)$ ,  $s = \uparrow, \downarrow$ , start to evolve at time  $t > 0$  towards the corresponding ferromagnetic equilibrium of the apparatus coupled the object spin value  $s$ . The off-diagonal contribution  $\rho_{A,\uparrow\downarrow}(t)$  decays to zero due to the decoherence

factor  $e^{-(t/t_{\text{dec}})^4}$  which has the same form as in the model studied in lecture 2, see (47). Formula (111) makes sense provided that  $t_{\text{dec}} \ll \omega_D^{-1}$ , i.e.,

$$N \gg \hat{\gamma}^{-1} \left( \frac{\omega_D}{g} \right)^2. \quad (112)$$

Note that  $t_{\text{dec}}$  is much smaller than the recurrence time  $t_{\text{rec}}^{(1)} = \pi/g$ . This means that the information about the object coherences which is spread out among the  $N$  spins by the object-apparatus entanglement process is rapidly lost in the bath, so that this information cannot come back to the object at a later time, unlike in the case  $\hat{\gamma} = 0$ .

#### 4.2.5 The Markov regime

At times  $t$  much larger than the bath correlation time  $T_B = \beta$ , the Redfield equation (80) can be approximated by a Markovian master equation. This is done by replacing by  $+\infty$  the upper bound  $t$  in the integral in the r.h.s. (this is justified if  $t \gg T_B$  since  $K(\tau) \simeq 0$  for  $\tau \gg T_B$ ). Replacing the upper bound in the time integrals defining  $\Lambda$  in (101), one gets in view of (78)

$$\begin{aligned} \Lambda_{\uparrow\uparrow}(\infty; h_{\uparrow\uparrow}(t)) &= \hat{\gamma} \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \omega \coth\left(\frac{\beta\omega}{2}\right) e^{-|\omega|/\omega_D} \lim_{\varepsilon \rightarrow 0^+} \int_0^{\infty} d\tau e^{i\tau(\omega+i\varepsilon)} \cos(2\mu h_{\uparrow\uparrow}(t)\tau) \\ &= \frac{\hat{\gamma}}{2} \mu h_{\uparrow\uparrow}(t) \coth(\beta\mu h_{\uparrow\uparrow}(t)) \exp(-2\mu h_{\uparrow\uparrow}(t)/\omega_D). \end{aligned} \quad (113)$$

The last exponential is almost equal to 1 by (79) and (90). Similarly, one finds

$$\eta_{\uparrow\uparrow}(\infty; h_{\uparrow\uparrow}(t)) \simeq -i \frac{\hat{\gamma}}{2g} \mu h_{\uparrow\uparrow}(t). \quad (114)$$

Hence (100) takes the following form in the Markov regime  $t \gg T_B$ :

$$\frac{dm_{\uparrow\uparrow}}{dt} = \hat{\gamma} \mu h_{\uparrow\uparrow}(t) \left( 1 - \frac{m_{\uparrow\uparrow}(t)}{\tanh(\beta\mu h_{\uparrow\uparrow}(t))} \right), \quad h_{\uparrow\uparrow}(t) = \mu^{-1}(Jm_{\uparrow\uparrow}(t)^3 + g). \quad (115)$$

Not surprisingly, the stationary solutions of (115) correspond to extrema of the free energy  $F$ . They are given by the implicit equation (73).

### 4.3 Discussion

By analogy with the model of lecture 2, we may define the entanglement time  $t_{\text{ent}}$  as the time at which the difference between the magnetisations  $m_{\uparrow\uparrow}(t)$  and  $m_{\downarrow\downarrow}(t)$  in the  $\uparrow\uparrow$  and  $\downarrow\downarrow$  sectors equals the fluctuation  $\Delta m_z$ . The latter is easily found to be initially  $\Delta m_z = \text{tr}(\hat{m}_z^2 \hat{\rho}_A(0))^{1/2} = 1/N$ . Using (107), one gets

$$t_{\text{ent}} = \left( \frac{3\pi}{2\hat{\gamma}N} \right)^{1/4} (\omega_D^3 g)^{-1/4} \quad (116)$$

provided that  $N \gg \hat{\gamma}^{-1}(\omega_D/g)$ . Like in the model of lecture 2, one has  $t_{\text{ent}} \ll t_{\text{dec}}$  in the interaction-dominated regime (actually, comparing (111) and (116) one finds  $t_{\text{dec}}/t_{\text{ent}} = (\omega_D/g)^{1/4} \gg 1$ ). The apparatus states  $\rho_{A,\uparrow\uparrow}(t)$  and  $\rho_{A,\downarrow\downarrow}(t)$  tied up with the  $\mathcal{S}$ -spin  $s = \uparrow$  and  $s = \downarrow$  become macroscopically distinguishable at the much larger time

$$t_{\text{int}} = \hat{\gamma}^{-1} \max\{J^{-1}, g^{-1}\} \gg t_{\text{ent}}. \quad (117)$$

At time  $t \gtrsim t_{\text{int}}$ , the magnetisations  $m_{\uparrow\uparrow}(t)$  and  $m_{\downarrow\downarrow}(t)$  have relaxed to their equilibrium value<sup>16</sup>, so that  $m_{\uparrow\uparrow}(t) \simeq m_1 \simeq 1$  and  $m_{\downarrow\downarrow}(t) \simeq -m_1 \simeq -1$  for  $g > g_c(\beta, J)$  and  $J\beta \gg 1$ . In this case, if the object-apparatus interaction is switched off at time  $t_{\text{int}}$ , the apparatus states  $\rho_{A,\uparrow\uparrow}$  and  $\rho_{A,\downarrow\downarrow}$  will remain a ferromagnetic state having the  $N$  spins pointing in the  $\uparrow$  and  $\downarrow$  directions, respectively, up to small thermal fluctuations (see Sec. 4.1.3). In the spin model under study, one does not need an amplification mechanism like in the model of lecture 2, thanks to the existence of the ferromagnetic phase transition.

All the above calculations rely on the DMFA and on a perturbative treatment of the apparatus-bath interaction with the help of the Bloch-Redfield equation (80). The DMFA is expected to be accurate for large  $N$ . As pointed out earlier, the time-dependent Bloch-Redfield equation describes accurately the exact dynamics if  $K(0)^2 T_B^3 t_{\text{int}} \ll 1$ . Using (106), (117) and  $T_B = \beta$ , this condition can be rewritten as

$$\hat{\gamma} \ll \frac{J}{\omega_D(\omega_D\beta)^3}, \frac{g}{\omega_D(\omega_D\beta)^3}. \quad (118)$$

At time  $t \gg t_{\text{dec}}$ , the off-diagonal contributions  $\rho_{A,\uparrow\downarrow}(t)$  and  $\rho_{A,\downarrow\uparrow}(t)$  are vanishingly small by virtue of (110). The object-apparatus state can be approximated by

$$\rho_{SA}(t) \simeq \sum_{s=\uparrow,\downarrow} |c_s|^2 |s\rangle\langle s| \otimes \left( \frac{1}{2} + \frac{m_{ss}(t)}{2} \right)^{\otimes N} \quad (119)$$

in analogy with the model of lecture 2 and in agreement with (3). The whole discussion of Sec. 3.7 applies here as well. With probability  $|c_\uparrow|^2$  (resp.  $|c_\downarrow|^2$ ), the object is in state  $|\uparrow\rangle$  (resp.  $|\downarrow\rangle$ ) and the apparatus is in the ferromagnetic state with magnetisation  $m_{\uparrow\uparrow} \simeq 1$  (resp.  $m_{\downarrow\downarrow} \simeq -1$ ). Like in the model of lecture 2, under the simultaneous action of the object-apparatus and apparatus-bath couplings the measurement process does never produce a Schrödinger cat state as an intermediate step.

**Remark:** According to Allahverdyan, Balian and Nieuwenhuizen, if the number  $N$  of spins in the apparatus is very large the object-apparatus coherences actually decay like Gaussian on a time scale  $t_{\text{collapse}}$  smaller than  $t_{\text{dec}}$ . In fact, the authors argue that, approximating  $(\cos(\theta(t)))^N$  by  $\exp(-2g^2 N t^2)$  in (109),

$$\zeta_{0,\uparrow\downarrow}(t) \simeq \exp \left\{ - \left( \frac{t}{t_{\text{dec}}} \right)^4 \right\} \exp \left\{ - \left( \frac{t}{t_{\text{collapse}}} \right)^2 \right\}, \quad t \ll \omega_D^{-1} \ll \beta, g^{-1}, J^{-1}, \quad (120)$$

with

$$t_{\text{collapse}} = \frac{1}{g\sqrt{2N}} \quad (121)$$

and  $t_{\text{collapse}} \ll t_{\text{dec}}$  for  $N \gg \hat{\gamma}(\omega_D/g)^2$ . However, such a decay is not present if one considers e.g. the matrix element  $\langle s = \uparrow; \uparrow \cdots \uparrow | \hat{\rho}_{SA}(t) | s' = \downarrow; \uparrow \cdots \uparrow \rangle = c_\uparrow c_\downarrow^* \exp\{- (t/t_{\text{dec}})^4\} \exp\{iN\theta(t)\}$ . We therefore think that the physical decay of the object-pointer coherence is given by  $t_{\text{dec}}$ .

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<sup>16</sup>It is clear on (115) that the time scale for this relaxation process is of order (117).

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