

Average entanglement for Markovian quantum trajectories

S. Vogelsberger^{1,*} and D. Spehner^{1,2}¹*Institut Fourier, Université Joseph Fourier and CNRS, BP 74, F-38402 Saint Martin d'Hères, France*²*Laboratoire de Physique et Modélisation des Milieux Condensés, Université Joseph Fourier and CNRS, BP 166, F-38042 Grenoble, France*

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We study the evolution of the entanglement of noninteracting qubits coupled to reservoirs under monitoring of the reservoirs by means of continuous measurements. We calculate the average of the concurrence of the qubits wave function over all quantum trajectories. For two qubits coupled to independent baths subjected to local measurements, this average decays exponentially with a rate depending on the measurement scheme only. This contrasts with the known disappearance of entanglement after a finite time for the density matrix in the absence of measurements. For two qubits coupled to a common bath, the mean concurrence can vanish at discrete times. Our analysis applies to arbitrary quantum jump or quantum state diffusion dynamics in the Markov limit. We discuss the best measurement schemes to protect entanglement in specific examples.

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I. INTRODUCTION

Entanglement is a key resource in quantum information. It can be destroyed or sometimes created by interactions with a reservoir. When the two noninteracting parts of a bipartite system are coupled to independent baths, entanglement typically disappears after a finite time [1–4]. This phenomenon, called “entanglement sudden death” (ESD), occurs for certain initial states only or for all entangled initial states, depending on whether the system relaxes to a steady state belonging to the boundary of the set of separable states (e.g., to a separable pure state for baths at zero temperature) or to its interior (e.g., to a Gibbs state at positive temperature) [5]. A quantum state lies on this boundary if it is separable and an arbitrarily small perturbation makes it entangled; this is the case, for example, for a pure separable state. When the two parts of the system are coupled to a common bath, sudden revivals of entanglement may take place after the state has become separable [6–8].

In this article we consider the loss of entanglement between two noninteracting qubits coupled to one or two baths monitored by continuous measurements. Because of these measurements, the qubits remain at all times in a pure state $|\psi(t)\rangle$, which evolves randomly. To each measurement result (or “realization”) corresponds a quantum trajectory $t \in \mathbb{R}_+ \mapsto |\psi(t)\rangle$ in the Hilbert space \mathbb{C}^4 of the qubits. In the Born-Markov regime, the dynamics is given by the quantum jump (QJ) model [9,10] or, in the case of homodyne and heterodyne detections, by the so-called quantum state diffusion (QSD) models [10–12]. We study how the entanglement of the qubits evolves in time by calculating the average $\overline{C_{\psi(t)}}$ of the Wootters concurrence of $|\psi(t)\rangle$ over all quantum trajectories; $\overline{C_{\psi(t)}}$ differs in general from the concurrence $C_{\rho(t)}$ of the density matrix $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ (here and in what follows the overline denotes the mean over all quantum trajectories) [13,14]. For two qubits coupled to *independent* baths, we find that

$$\overline{C_{\psi(t)}} = C_0 e^{-\kappa t}, \quad (1)$$

where $C_0 = C_{\psi(0)}$ is the initial concurrence and $\kappa \geq 0$ depends on the measurement scheme but not on the initial state $|\psi(0)\rangle$. In particular, if $C_0 > 0$ and $t_{\text{ESD}} \in]0, \infty[$ is the time at which entanglement disappears in the density matrix (assuming that this time is finite), then $C_{\rho(t)} = 0$ at times $t \geq t_{\text{ESD}}$ whereas $\overline{C_{\psi(t)}}$ can only vanish asymptotically. The continuous measurements on the two baths thus protect on average the qubits from ESD. Of course, this does not mean that *all* random wave functions $|\psi(t)\rangle$ remain entangled at all times. But in some cases, such as for pure dephasing or for infinite temperature baths, one can find measurement schemes such that $\kappa = 0$; then, for all trajectories, if the qubits are maximally entangled at $t = 0$ they remain maximally entangled at all times. We show that the best measurement scheme to protect entanglement is in general given by homodyne detection with appropriately chosen laser phases. Related strategies using quantum Zeno effect [15], entanglement distillation [16], quantum feedback [17], and encoding in qutrits [18] have been proposed. It is assumed in this work that the measurements on the baths are performed by perfect detectors. The impact of detection errors has been studied in [18].

When the qubits are coupled to a common bath, we find that $\overline{C_{\psi(t)}}$ has a more complex time behavior than in (1) and may vanish at finite discrete times and, for some initial states, be equal to $C_{\rho(t)}$. It is worthwhile to stress that the formula (1) is valid provided that not only each qubit is coupled to its own bath, but also the two baths are monitored independently from each other by the measurements. This means that the measurements are performed *locally* on each bath. Instead of looking for the measurement scheme maximizing the average concurrence $\overline{C_{\psi(t)}}$ of the two qubits in order to obtain the best entanglement protection, it is also of interest to find a way to perform the measurements such that $\overline{C_{\psi(t)}}$ is minimum and coincides with the concurrence $C_{\rho(t)}$ of the density matrix. This problem has been studied numerically in Ref. [14] and analytically in [19] for specific models of couplings with the two baths. Our result (1) implies that for any Markovian dynamics, if the two qubits are initially entangled and ESD occurs for the density matrix $\rho(t)$, a scheme with the aforementioned property must necessarily involve measurements of nonlocal (joint) observables of the two baths. In the models studied in [14,19], nonlocal measurements are

*vogelsy@ujf-grenoble.fr

indeed used in order to obtain an optimal scheme satisfying $\overline{C_{\psi(t)}} = C_{\rho(t)}$.

The paper is organized as follows. We briefly recall in Sec. II the definition of the concurrence of pure and mixed states and review the quantum jump unraveling of a Lindblad equation for the density matrix in Sec. III. We treat the simple and illustrative case of two two-level atoms coupled to independent baths at zero temperature in Sec. IV, before showing formula (1) in Sec. V for a general quantum jump dynamics. The QSD unravelings are considered in Sec. VI; we obtain the average concurrence for such unravelings as limits of the concurrence for QJ dynamics (corresponding to homodyne and heterodyne detections with intense laser fields). Section VII is devoted to the evolution of the entanglement of two qubits coupled to a common bath at zero temperature. The main conclusions of the work are summarized in Sec. VIII.

II. ENTANGLEMENT MEASURES FOR QUANTUM TRAJECTORIES

The entanglement of formation of a bipartite quantum system \mathcal{S} in a pure state $|\psi\rangle$ is defined by means of the von Neumann entropy $E_\psi = -\text{tr}(\rho_A \ln \rho_A) = -\text{tr}(\rho_B \ln \rho_B)$ of the density matrices $\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \text{tr}_A(|\psi\rangle\langle\psi|)$ of the two subsystems A and B composing \mathcal{S} [20]. If \mathcal{S} is in a mixed state, E_ρ is the infimum of $\sum p_k E_{\psi_k}$ over all convex decompositions $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ of its density matrix (with $p_k \geq 0$ and $\|\psi_k\| = 1$). When A and B have two-dimensional Hilbert spaces, $E_\rho = f(C_\rho)$ is related to the concurrence [21] C_ρ by a convex increasing function $f: [0, 1] \rightarrow [0, \ln(2)]$; ρ is separable if and only if $C_\rho = 0$ (i.e., $E_\rho = 0$). For a pure state [21],

$$C_\psi = |\langle\sigma_y \otimes \sigma_y T\rangle_\psi|, \quad (2)$$

where $\sigma_y = i(|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|)$ is the y Pauli matrix, $T: |\psi\rangle = \sum_{s,s'} c_{ss'} |s, s'\rangle \mapsto \sum_{s,s'} c_{ss'}^* |s, s'\rangle$ the antiunitary operator of complex conjugation in the canonical basis $\{|s, s'\rangle = |s\rangle \otimes |s'\rangle; s, s' = \uparrow, \downarrow\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$, and $\langle\cdot\rangle_\psi = \langle\psi| \cdot |\psi\rangle$ the quantum expectation in state $|\psi\rangle$.

For quantum trajectories, one has always $\overline{E_{\psi(t)}} \geq E_{\rho(t)}$, this inequality being strict excepted if the decomposition

$$\rho(t) = \overline{|\psi(t)\rangle\langle\psi(t)|} = \int dp[\psi] |\psi(t)\rangle\langle\psi(t)|, \quad (3)$$

realizes the infimum defining $E_{\rho(t)}$. Thanks to the convexity of f , $\overline{E_{\psi(t)}} \geq f(\overline{C_{\psi(t)}})$. Thus Eq. (1) shows that for independent baths and if $C_0 > 0$ then $\overline{E_{\psi(t)}} \geq f(C_0 e^{-\kappa t}) > 0$ whatever the measurement scheme.

It is legitimate to ask which entanglement measure should be averaged, since, for example, $\overline{E_{\psi(t)}} = E_0$ could be constant and $\overline{C_{\psi(t)}}$ time decreasing if $E_0 \neq 0, \ln 2$. The concurrence is a natural candidate as it corresponds for pure states to the supremum over all self-adjoint local observables J_A and J_B with norms less than one of the modulus of the correlation between J_A and J_B ,

$$C_{\psi(t)} = \sup_{\|J_A\|, \|J_B\| \leq 1} | \langle J_A \otimes J_B \rangle_{\psi(t)} - \langle J_A \otimes 1_B \rangle_{\psi(t)} \langle 1_A \otimes J_B \rangle_{\psi(t)} |. \quad (4)$$

Moreover, $\overline{C_{\psi(t)}}$ is easy to calculate in the Markov regime and gives a lower bound on $\overline{E_{\psi(t)}}$.

III. QUANTUM JUMP MODEL

Let us briefly recall the QJ dynamics [9,22,23]. As a result of a measurement on a particle (e.g., a photon) of the bath scattered by the qubits, the qubits wave function suffers a quantum jump,

$$|\psi(t)\rangle \longrightarrow |\psi_{\text{jump}}^{(m,i)}\rangle = \frac{J_m^i |\psi(t)\rangle}{\|J_m^i |\psi(t)\rangle\|}, \quad (5)$$

where the jump operator J_m^i is related to the particle-qubits coupling and the indices m, i label all possible measurement results save for the most likely one, which we call a “no detection.” In the weak coupling limit, the probability that a measurement in the small time interval $[t, t + dt]$ gives the result (m, i) is very small and equal to $dp_m^i(t) = \gamma_m^i \|J_m^i |\psi(t)\rangle\|^2 dt$. The jump rate γ_m^i does not depend on $|\psi(t)\rangle$ and is proportional to the square of the particle-qubit coupling constant. In the no-detection case the wave function of the qubits evolves as

$$|\psi(t + dt)\rangle = \frac{e^{-iH_{\text{eff}} dt} |\psi(t)\rangle}{\|e^{-iH_{\text{eff}} dt} |\psi(t)\rangle\|}, \quad (6)$$

$$H_{\text{eff}} = H_0 - \frac{i}{2} \sum_{m,i} \gamma_m^i J_m^i J_m^{i\dagger},$$

where H_0 is the Hamiltonian of the qubits. The probability that no jump occurs in the time interval $[t_0, t]$ is $p_{\text{nj}}(t_0, t) = \|e^{-iH_{\text{eff}}(t-t_0)} |\psi(t_0)\rangle\|^2$. [This is proven by noting that $p_{\text{nj}}(t_0, t) - p_{\text{nj}}(t_0, t + dt) = \sum_{m,i} dp_m^i(t) p_{\text{nj}}(t_0, t)$, thus $\partial \ln p_{\text{nj}}(t_0, t) / \partial t = -\sum_{m,i} \gamma_m^i \|J_m^i |\psi(t)\rangle\|^2 = (\partial/\partial t) \ln \|e^{-iH_{\text{eff}}(t-t_0)} |\psi(t_0)\rangle\|^2$ by (6).] It is not difficult to show [9] that the density matrix $\rho(t) = \overline{|\psi(t)\rangle\langle\psi(t)|}$ satisfies the Lindblad equation,

$$\frac{d\rho}{dt} = -i[H_0, \rho] + \sum_{m,i} \gamma_m^i \left(J_m^i \rho J_m^{i\dagger} - \frac{1}{2} \{ J_m^{i\dagger} J_m^i, \rho \} \right), \quad (7)$$

where $\{\cdot, \cdot\}$ denotes the anticommutator. It is known that many distinct QJ dynamics unravel the same master equation (7) [22]. For two qubits coupled to independent reservoirs R_A and R_B , the jump operators are *local*, that is, they have the form,

$$J_m^A \otimes 1_B, \quad 1_A \otimes J_m^B, \quad (8)$$

depending on whether the measurements are performed on R_A or R_B . Here J_m^i are 2×2 matrices.

The aforementioned absence of ESD for the mean concurrence of two qubits coupled to independent baths can be traced back to the existence of trajectories for which $|\psi(t)\rangle$ remains entangled at all times. Actually, for a trajectory without jump, $|\psi_{\text{nj}}(t)\rangle \propto e^{-itH_{\text{eff}}} |\psi(0)\rangle$, see (6). By (8), and since the qubits do not interact with each other, $e^{-itH_{\text{eff}}}$ is the tensor product of two local operators acting on each qubit. If $|\psi_{\text{nj}}(t)\rangle$ would be separable at a given time t then, by reversing the dynamics [i.e., by applying $e^{itH_{\text{eff}}}$ to $|\psi_{\text{nj}}(t)\rangle$] one would deduce that $|\psi(0)\rangle$ is separable. Hence $C_{\psi_{\text{nj}}(t)} > 0$ if $C_0 > 0$. But the no-detection probability between times 0 and t is nonzero and thus $\overline{C_{\psi(t)}} > 0$ at all times. Note that this argument does not apply if nonlocal

observables of the two baths are measured or if the two qubits are coupled to a common bath, since then the jump operators are nonlocal.

IV. PHOTON COUNTING

Let us illustrate the random dynamics described previously on a simple and experimentally relevant example [24]. Each qubit is a two-level atom coupled resonantly to the electromagnetic field initially in the vacuum (zero-temperature photon bath). The two atoms are far from each other and thus interact with independent field modes. Two perfect photon counters D_i make a click when a photon is emitted by qubit i ($i = A, B$), whatever the direction of the emitted photon. Doing the rotating wave approximation, the jump operators are $J_-^i = \sigma_-^i = |\downarrow\rangle\langle\uparrow|$. For simplicity we take $H_0 = 0$. By (6), if no photon is detected in the time interval $[0, t]$ the qubits state is

$$|\psi(t)\rangle = \mathcal{N}(t)^{-1} \sum_{s,s'=\uparrow,\downarrow} c_{ss'} e^{-\gamma_{ss'} t/2} |s, s'\rangle, \quad (9)$$

with $\gamma_{\uparrow\uparrow} = \gamma_A + \gamma_B$, $\gamma_{\uparrow\downarrow} = \gamma_A$, $\gamma_{\downarrow\uparrow} = \gamma_B$, $\gamma_{\downarrow\downarrow} = 0$ (γ_i being the jump rate for detector D_i), $c_{ss'} = \langle s, s' | \psi(0) \rangle$, and $\mathcal{N}(t)^2 = \sum_{s,s'} |c_{ss'}|^2 e^{-\gamma_{ss'} t}$. The concurrence (2) of $|\psi(t)\rangle$ is $C(t) = C_0 \mathcal{N}(t)^{-2} e^{-(\gamma_A + \gamma_B)t/2}$ with $C_0 = 2|c_{\uparrow\uparrow}c_{\downarrow\downarrow} - c_{\uparrow\downarrow}c_{\downarrow\uparrow}|$. If a photon is detected at time t_j by, say, the photon counter D_A , the qubits are just after the jump (5) in the separable state $|\psi(t_j+)\rangle \propto |\downarrow\rangle \otimes (c_{\uparrow\uparrow} e^{-\gamma_{\uparrow\uparrow} t_j/2} |\uparrow\rangle + c_{\uparrow\downarrow} e^{-\gamma_{\uparrow\downarrow} t_j/2} |\downarrow\rangle)$. Since neither a jump nor the interjump dynamics can create entanglement [the jump operators (8) being local], $|\psi(t)\rangle$ remains separable at all times $t \geq t_j$, even if more photons are subsequently detected. Thus $C(t) = 0$ if at least one photon is detected in the time interval $[0, t]$. Averaging over all realizations of the quantum trajectories and using the probability $p_{\text{nj}}(0, t) = \mathcal{N}(t)^2$ that no photon is detected in $[0, t]$, one finds $\overline{C(t)} = C_0 e^{-(\gamma_A + \gamma_B)t/2}$.

This argument is easily extended to baths at positive temperatures by adding two jump operators $J_+^i = \sigma_+^i$ with rates $\gamma_+^i \leq \gamma_-^i$. The mean concurrence is then $\overline{C(t)} = C_0 e^{-(\gamma_+^A + \gamma_-^A + \gamma_+^B + \gamma_-^B)t/2}$. It is compared in Fig. 1 with the concurrence of the density matrix obtained by solving the master equation (7), which shows ESD for all initial states.

V. GENERAL QUANTUM JUMP DYNAMICS

We now consider a general QJ dynamics with jump operators given by (8). The Hamiltonian of the qubits has the form $H_0 = H_A \otimes 1_B + 1_A \otimes H_B$. Let $K = K_A \otimes 1_B + 1_A \otimes K_B$ with

$$K_i = \frac{1}{2} \sum_m \gamma_m^i J_m^{i\dagger} J_m^i, \quad (10)$$

γ_m^i being the jump rates for the detector D_i ($i = A, B$). We first assume that no jump occurs between t and $t + dt$. By expanding the exponential in (6), one gets

$$C(t + dt) = p_{\text{nj}}(t, t + dt)^{-1} |\langle \sigma_y \otimes \sigma_y T \rangle_{\psi(t)} + i dt \langle H_{\text{eff}}^\dagger \sigma_y \otimes \sigma_y T + \sigma_y \otimes \sigma_y T H_{\text{eff}} \rangle_{\psi(t)} + O(dt)^2, \quad (11)$$

where $p_{\text{nj}}(t, t + dt) = \langle 1 - 2K dt + O(dt)^2 \rangle_{\psi(t)}$ is the probability that no jump occurs between t and $t + dt$. Now, for any

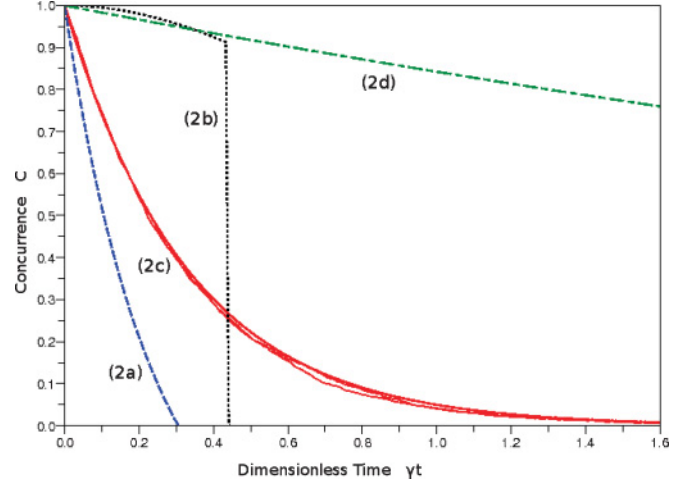


FIG. 1. (Color online) Concurrences of two qubits coupled to independent baths at positive temperature as a function of γt for $\gamma_+^i = \gamma_-^i/2 = \gamma$. The initial state is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - i|\downarrow\downarrow\rangle)$: (2a) $C_{\rho(t)}$ for the density matrix (blue dashed line); (2b) $C_{\psi(t)}$ for a single trajectory (black dotted line); (2c) $\overline{C_{\psi(t)}}$ averaged over 1500 trajectories and from Eq. (1) (red solid lines); (2d) $\overline{C_{\psi(t)}}$ for the best measurement scheme (see text).

local operator \mathcal{O}_i acting on qubit i , one has

$$\langle \mathcal{O}_i \sigma_y \otimes \sigma_y T \rangle_{\psi(t)} = \langle \sigma_y \otimes \sigma_y T \mathcal{O}_i^\dagger \rangle_{\psi(t)} = \frac{C(t)}{2} \text{tr}_{\mathbb{C}^2}(\mathcal{O}_i), \quad (12)$$

with

$$C(t) = \langle \sigma_y \otimes \sigma_y T \rangle_{\psi(t)} = 2[c_{\uparrow\downarrow}^*(t)c_{\downarrow\uparrow}^*(t) - c_{\uparrow\uparrow}^*(t)c_{\downarrow\downarrow}^*(t)], \quad (13)$$

and $c_{ss'}(t) = \langle s, s' | \psi(t) \rangle$. Since $C(t) = |C(t)|$ and $H_{\text{eff}} = \sum_i (H_i - iK_i)$, this gives

$$C(t + dt) p_{\text{nj}}(t, t + dt) = C(t) \left[1 - \text{tr}_{\mathbb{C}^4}(K) \frac{dt}{2} + O(dt^2) \right]. \quad (14)$$

If detector D_i gives the result m in the time interval $[t, t + dt]$, the concurrence is by virtue of (5)

$$C_{\text{jump}}^{(m,i)}(t + dt) = \frac{\gamma_m^i dt}{dp_m^i(t)} C(t) |\det_{\mathbb{C}^2}(J_m^i)|, \quad (15)$$

where we have used the identity,

$$\langle \mathcal{O}_i^\dagger \sigma_y \otimes \sigma_y T \mathcal{O}_i \rangle_{\psi(t)} = C(t) \det_{\mathbb{C}^2}(\mathcal{O}_i^\dagger), \quad (16)$$

valid for any local operator \mathcal{O}_i acting on qubit i . Collecting the previous formulas and using the Markov property of the jump process, one gets $\overline{C(t + dt)} = \overline{C(t)} [1 - \kappa_{\text{QJ}} dt + O(dt^2)]$ with

$$\kappa_{\text{QJ}} = \frac{1}{2} \text{tr}_{\mathbb{C}^4}(K) - \sum_{m,i} \gamma_m^i |\det_{\mathbb{C}^2}(J_m^i)|. \quad (17)$$

Letting dt go to zero, one obtains $d\overline{C(t)}/dt = -\kappa_{\text{QJ}} \overline{C(t)}$. The solution (1) of this differential equation has the exponential decay claimed previously. To show that $\kappa_{\text{QJ}} \geq 0$,

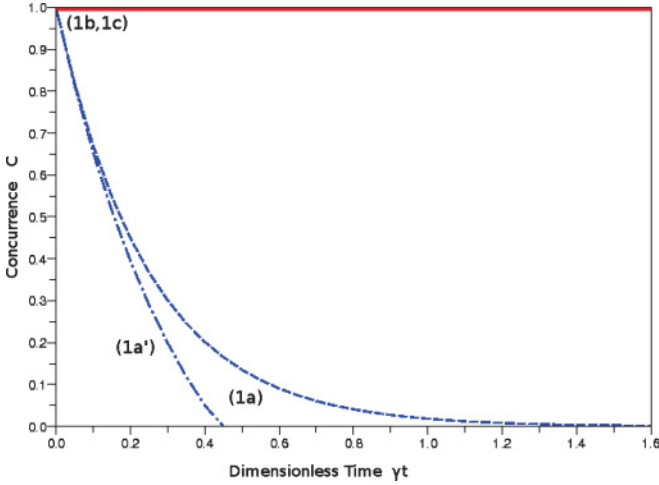


FIG. 2. (Color online) Same as in Fig. 1 for pure dephasing and the initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + e^{-i\varphi}|\downarrow\downarrow\rangle)$: (1a, 1a') $C_{\rho(t)}$ with $\varphi = \frac{\pi}{2}$ and $\varphi = 0$ (blue dashed and dotted-dashed lines); (1b, 1c) $C_{\psi(t)} = \overline{C_{\psi(t)}}$ (red solid line).

let $2\theta_m^i$ be the argument of $\det_{\mathbb{C}^2}(J_m^i)$. We write $\kappa_{\text{QJ}} = \sum_{m,i} \gamma_m^i [\text{tr}_{\mathbb{C}^2}(J_m^{i\dagger} J_m^i) - 2\text{Re}\{e^{-2i\theta_m^i} \det_{\mathbb{C}^2}(J_m^i)\}]/2$ as

$$\kappa_{\text{QJ}} = \sum_{m,i} \frac{\gamma_m^i}{2} (|\langle \uparrow | \tilde{J}_m^i | \uparrow \rangle - \langle \downarrow | \tilde{J}_m^{i\dagger} | \downarrow \rangle|^2 + |\langle \uparrow | 2\text{Re} \tilde{J}_m^i | \downarrow \rangle|^2), \quad (18)$$

with $\tilde{J}_m^i = e^{-i\theta_m^i} J_m^i$ and $2\text{Re} \tilde{J}_m^i = \tilde{J}_m^i + \tilde{J}_m^{i\dagger}$. Thus $\kappa_{\text{QJ}} \geq 0$.

Note that $\kappa_{\text{QJ}} = 0$ if all matrices J_m^i are self-adjoint and traceless (then $\theta_m^i = \pi/2$ and $\text{Re} \tilde{J}_m^i = 0$). We show in Fig. 2 the concurrence of the density matrix given by solving (7) for a pure dephasing with $J^i = e^{i\pi/4} \sigma_-^i + e^{-i\pi/4} \sigma_+^i$. One has ESD for all initial states save for $|\psi(0)\rangle = (|\uparrow\uparrow\rangle \pm i|\downarrow\downarrow\rangle)/\sqrt{2}$. Since $\kappa_{\text{QJ}} = 0$, (1) implies $\overline{C(t)} = C_0$. If the two qubits are maximally entangled at $t = 0$, then $C_{\psi(t)} = \overline{C(t)} = C_0 = 1$ for all quantum trajectories at any time $t \geq 0$. Therefore, for pure dephasing one can protect perfectly the qubits by measuring continuously and locally the two independent baths.

We can now give the optimal measurement scheme to protect the entanglement of two qubits coupled to independent baths at positive temperatures. Let us replace the photon-counting jump operators $J_{\pm} = \sigma_{\pm}$ by $J_{\mu}^i = \sum_{m=\pm} (\gamma_m^i/\gamma_{\mu}^i)^{\frac{1}{2}} u_{\mu m}^i \sigma_m^i$ where $U_i = (u_{\mu m}^i)_{\mu=1,\dots,N}^{m=\pm}$ are unitary $2 \times N$ matrices. This corresponds to a rotation of the measurement basis and gives another unraveling of the master equation (7). Less us stress that the new jump operators J_{μ}^i still act locally on each qubit. By (17), the new rate is $\kappa = \sum_{\mu,i} (\sqrt{\gamma_-^i} |u_{\mu-}^i| - \sqrt{\gamma_+^i} |u_{\mu+}^i|)^2/2$. By using $\sum_{\mu} |u_{\mu\pm}^i|^2 = 1$ and optimizing over all unitaries U_i , one finds that the smallest disentanglement rate arises when, for example, $u_{1\pm}^i = \pm u_{2\pm}^i = 1/\sqrt{2}$ ($N = 2$) and is given by

$$\kappa_{\text{QJ}}^{\text{opt}} = \frac{1}{2} \sum_{i=A,B} (\sqrt{\gamma_-^i} - \sqrt{\gamma_+^i})^2. \quad (19)$$

Note that $\kappa_{\text{QJ}}^{\text{opt}} = \kappa_{\text{QJ}}$ at zero temperature and $\kappa_{\text{QJ}}^{\text{opt}} = 0$ (perfect protection) at infinite temperature. The decay of $\overline{C(t)}$ for this

optimal measurement is shown in Fig. 1 (green dashed-dotted line).

VI. HOMODYNE AND HETERODYNE DETECTION

Let us come back to our example of two atoms coupled to the electromagnetic field initially in the vacuum. If homodyne photodetection is used instead of photon counting, the jump operators become $J_{\pm\alpha}^i = \sigma_{\pm}^i \pm \alpha_i$, α_i being the amplitude of a classical laser field (there are now four jump operators since each homodyne detector involves two photon counters) [11]. Assuming that the two photon beams emitted by the atoms are combined with the two laser fields via 50% beam splitters, the jump rates associated with $J_{\pm\alpha}^i$ are equal, $\gamma_{\pm\alpha}^i = \gamma_i/2$. Thanks to (17), one easily finds that the disentanglement rate for the new QJ dynamics, $\kappa_{\text{QJ}}(\alpha) = (\gamma_A + \gamma_B)/2$, is the same as for photon counting.

In contrast, $\kappa_{\text{QJ}}(\alpha)$ depends on the laser amplitudes for pure dephasing (jump operators $J_{\pm\alpha}^i = \mathbf{v}_i \cdot \sigma \pm \alpha_i$ with $\mathbf{v}_i \in \mathbb{R}^3$, $\|\mathbf{v}_i\| = 1$, and σ the vector formed by the Pauli matrices σ_x , σ_y , and σ_z): then $\kappa_{\text{QJ}}(\alpha) = 2 \sum_i \gamma_i \min\{\alpha_i^2, 1\}$ for real α_i 's. One reaches perfect entanglement protection [$\overline{C(t)} = C_0$] only for vanishing laser intensities α_i^2 . In the case of two qubits coupled to two baths at positive temperatures, a general choice of jump operators such that the density matrix (3) satisfies the master equation (7) with the four Lindblad operators σ_{\pm}^i , $i = A, B$, is $J_{\mu,\pm\alpha}^i = J_{\mu}^i \pm \alpha_{\mu}^i$ with the jump rates $\gamma_{\mu,\pm\alpha}^i = \gamma_{\mu}^i/2$, laser amplitudes $\alpha_{\mu}^i \in \mathbb{C}$, and $J_{\mu}^i = \sum_{m=\pm} (\gamma_m^i/\gamma_{\mu}^i)^{\frac{1}{2}} u_{\mu m}^i \sigma_m^i$ for an arbitrary unitary matrix $(u_{\mu m}^i)_{\mu=1,\dots,N}^{m=\pm}$ (see the discussion in the preceding section). The corresponding disentanglement rate, $\kappa_{\text{QJ}}(\alpha) = \sum_{\mu,i} \gamma_{\mu}^i [\text{tr}_{\mathbb{C}^2}(J_{\mu}^{i\dagger} J_{\mu}^i) + 2|\alpha_{\mu}^i|^2 - 2|\det_{\mathbb{C}^2}(J_{\mu}^i)| + (\alpha_{\mu}^i)^2]/2$, is equal to $\kappa_{\text{QJ}}(0)$ if $\det(J_{\mu}^i) = 0$ or for complex laser amplitudes $\alpha_{\mu}^i = |\alpha_{\mu}^i| e^{i\theta_{\mu}^i}$ satisfying $2\theta_{\mu}^i = \arg[\det(J_{\mu}^i)]$; otherwise, $\kappa_{\text{QJ}}(\alpha)$ is larger than $\kappa_{\text{QJ}}(0)$. We can conclude that the smallest disentanglement rate is given by (19) and the best unravelings to protect the entanglement of the qubits are either the QJ model with jump operators $J_1^i \propto (\gamma_+^i)^{\frac{1}{2}} \sigma_+^i + (\gamma_-^i)^{\frac{1}{2}} \sigma_-^i$ and $J_2^i \propto (\gamma_+^i)^{\frac{1}{2}} \sigma_+^i - (\gamma_-^i)^{\frac{1}{2}} \sigma_-^i$ or the corresponding homodyne unraveling with laser phases $\theta_1^i = \pi/2$ and $\theta_2^i = 0$.

Let us now consider a general QJ model with jump operators J_m^i . A new unraveling of (7) is obtained from the QJ model with jump operators $J_{m,\pm\alpha}^i = J_m^i \pm \alpha_m^i$ and rates $\gamma_{m,\pm\alpha}^i = \gamma_m^i/2$. For large positive laser amplitudes $\alpha_m^i \gg 1$, this dynamics converges after an appropriate coarse graining in time to the QSD model described by the stochastic Schrödinger equation [11,25]:

$$|d\psi\rangle = \left[(-iH_0 - K)dt + \sum_{m,i} \gamma_m^i \left(\text{Re}\langle J_m^i \rangle_{\psi} J_m^i - \frac{1}{2} (\text{Re}\langle J_m^i \rangle_{\psi})^2 \right) dt + \sum_{m,i} \sqrt{\gamma_m^i} (J_m^i - \text{Re}\langle J_m^i \rangle_{\psi}) dw_m^i \right] |\psi\rangle, \quad (20)$$

where dw_m^i are the Itô differentials for independent real Wiener processes satisfying the Itô rules $dw_m^i dw_n^j = \delta_{ij} \delta_{mn} dt$. One

can determine the mean concurrence for the QSD model (20) by taking the limit of the mean concurrence for the QJ dynamics with jump operators $J_{m,\pm\alpha}^i$. This gives again the exponential decay (1) but with a new rate,

$$\kappa_{\text{ho}} = \frac{\text{tr}_{\mathbb{C}^4}(K)}{2} - \sum_{m,i} \gamma_m^i \left(\text{Re det}_{\mathbb{C}^2}(J_m^i) + \frac{1}{2} [\text{Im tr}_{\mathbb{C}^2}(J_m^i)]^2 \right). \quad (21)$$

In fact, if $2\theta_{m,\pm\alpha}^i$ is the argument of $\det(J_m^i \pm \alpha_m^i) = (\alpha_m^i)^2 \pm \alpha_m^i \text{tr}(J_m^i) + O(1)$ then $e^{2i\theta_{m,\pm\alpha}^i} \sim 1 \pm i \text{Im tr}(J_m^i)/\alpha_m^i$ for $\alpha_m^i \gg 1$, $\alpha_m^i > 0$. Using (18), a short calculation gives (21).

Unlike κ_{QJ} , κ_{ho} changes when the operators J_m^i in (20) acquire a phase factor, $J_m^i \rightarrow e^{-i\theta_m^i} J_m^i$. This arises for homodyne detection with complex laser amplitudes $\alpha_m^i = |\alpha_m^i| e^{i\theta_m^i}$, $|\alpha_m^i| \gg 1$. Minimizing over the laser phases θ_m^i yields

$$\kappa_{\text{ho}}^{\text{opt}} = \frac{1}{2} \text{tr}_{\mathbb{C}^4}(K) - \sum_{m,i} \gamma_m^i \left(\left| \det_{\mathbb{C}^2}(J_m^i) - \frac{1}{4} [\text{tr}_{\mathbb{C}^2}(J_m^i)]^2 \right| + \frac{1}{4} |\text{tr}_{\mathbb{C}^2}(J_m^i)|^2 \right). \quad (22)$$

It is easy to show that $\kappa_{\text{ho}}^{\text{opt}} \leq \kappa_{\text{QJ}}$, this inequality being strict excepted if the two eigenvalues of J_m^i have the same modulus for all (m,i) . Thus optimal homodyne detection protects entanglement better than—or, if the aforementioned condition is fulfilled, as well as—photon counting. Let us stress that the optimal measurements (in particular, the laser phases θ_m^i minimizing the rate κ_{ho}) only depend on the Lindblad operators J_m^i in the master equation (7) and are thus the same for all initial states of the qubits.

Let us now discuss the case of heterodyne detection. The corresponding jump operators $J_{m,\pm\alpha}^i(t_q) = J_m^i \pm \alpha_m^i e^{i\Omega_m^i t_q}$ depend on the time t_q of the q th jump due to the oscillations of the laser amplitudes [23]. The associated rates are $\gamma_{m,\pm\alpha}^i = \gamma_m^i/2$ as for homodyne detection. We assume here that $\alpha_m^i > 0$. In the limit $(\alpha_m^i)^2 \gg \Omega_m^i/\gamma_m^i \gg 1$ of large laser intensities and rapidly oscillating laser amplitudes, the QJ dynamics with jump operators $J_{m,\pm\alpha}^i(t_q)$ converges to the QSD model given by the stochastic Schrödinger equation [22],

$$|d\psi\rangle = \left\{ (-iH_0 - K)dt + \frac{1}{2} \sum_{m,i} \gamma_m^i \left(\langle J_m^i \rangle_\psi^* J_m^i - \frac{1}{2} |\langle J_m^i \rangle_\psi|^2 \right) dt + \sum_{m,i} \sqrt{\gamma_m^i} \left[\left(J_m^i - \frac{1}{2} \langle J_m^i \rangle_\psi \right) d\xi_m^i - \frac{1}{2} \langle J_m^i \rangle_\psi^* (d\xi_m^i)^* \right] \right\} |\psi\rangle, \quad (23)$$

where $d\xi_m^i$ are the Itô differential of independent complex Wiener processes satisfying the Itô rules $d\xi_m^i d\xi_n^j = 0$ and $d\xi_m^i (d\xi_n^j)^* = \delta_{ij} \delta_{mn} dt$. Equation (23) describes the coarse-grained evolution of the normalized wave function $|\psi(t)\rangle$ on a time scale Δt such that (i) many jumps and many laser

amplitude oscillations occur in a time interval of length Δt and (ii) $|\psi(t)\rangle$ does not change significantly on such a time interval. These conditions are satisfied when $(\alpha_m^i)^2 \gamma_m^i \Delta t \gg \Omega_m^i \Delta t \gg 1$ and $\gamma_m^i \Delta t \ll 1$. We now show that the mean concurrence for the QSD model (23) is given by (1) and determine the rate κ of its exponential decay. This can be done by calculating the derivative $d\overline{C}(t)/dt$ in a similar way as in Sec. V, using (23) and the Itô rules. It turns out to be simpler to estimate directly the average concurrence of the QJ model for heterodyne detection in the aforementioned limits, in analogy with our previous analysis for homodyne detection. Let us first remark that the results of Sec. V remain valid if the jump operators $J_m^i(t)$ vary slowly in time, on a time scale $(\Omega_m^i)^{-1}$ much larger than the mean time $(\alpha_m^i)^{-2}/\gamma_m^i$ between consecutive jumps. Hence $d\overline{C}/dt = -\kappa_{\text{het}}(t) \overline{C}(t)$ and thus $\overline{C}(t) = C_0 e^{-\int_0^t dt' \kappa_{\text{het}}(t')}$ with a time-dependent rate $\kappa_{\text{het}}(t)$ given by (18). To simplify notations, we temporarily omit the sum in (18) and do not write explicitly the lower and upper indices m and i . Let us set $\tau = \text{tr}(J)/2 = |\tau| e^{i\varphi}$ and $\delta = \det(J) = e^{2i\theta} |\delta|$. Let $2\theta_{\pm\alpha}(t)$ denote the argument of $\det(J \pm \alpha e^{i\Omega t})$. Generalizing the calculation outlined above for homodyne detection, one gets $e^{2i\theta_{\pm\alpha}(t)} \sim e^{2i\Omega t} (1 \pm 2i \text{Im}\{\tau e^{-i\Omega t}\}/\alpha)$ as $\alpha \gg 1$. By (18) this yields

$$\begin{aligned} \kappa_{\text{het}}(t) &= \frac{\gamma}{2} (|\langle \uparrow | J | \uparrow \rangle - e^{2i\Omega t} \langle \downarrow | J^\dagger | \downarrow \rangle - 2i e^{i\Omega t} \text{Im}\{\tau e^{-i\Omega t}\}|^2 \\ &\quad + |\langle \uparrow | (J + e^{2i\Omega t} J^\dagger) | \downarrow \rangle|^2) \\ &= \frac{\text{tr}_{\mathbb{C}^4}(K)}{2} - \gamma [|\delta| \cos(2\theta - 2\Omega t) \\ &\quad + 2|\tau|^2 \sin^2(\varphi - \Omega t)], \end{aligned}$$

up to terms of order α^{-1} . By neglecting the oscillatory integral $\int_0^t dt' \cos(2\theta - 2\Omega t')$ (which is of order $\Omega^{-1} \ll \Delta t \leq t$) and approximating $\int_0^t dt' \sin^2(\varphi - \Omega t')$ by $t/2$, one obtains $\int_0^t dt' \kappa_{\text{het}}(t') \simeq t(\text{tr}_{\mathbb{C}^4}(K)/2 - \gamma |\tau|^2)$. Putting together the previous results, this shows that $\overline{C}(t) \rightarrow C_0 e^{-\kappa_{\text{het}} t}$ in the limit $\alpha^2 \gg \Omega/\gamma \gg 1$ and $\Omega \gg (\Delta t)^{-1} \gg \gamma$, with

$$\kappa_{\text{het}} = \frac{\text{tr}_{\mathbb{C}^4}(K)}{2} - \frac{1}{4} \sum_{m,i} \gamma_m^i |\text{tr}(J_m^i)|^2. \quad (24)$$

We note that $\kappa_{\text{het}} \geq \kappa_{\text{ho}}^{\text{opt}}$. For given jump operators J_m^i , the measurement scheme which better protects the qubits against disentanglement is thus given by homodyne detections with optimally chosen laser phases. In this scheme, the average concurrence decays exponentially with the rate (22).

Although (23) is different from the QSD equation for the normalized wave function introduced by Gisin and Percival [12], the quantum trajectories $t \mapsto |\psi(t)\rangle$ for the two dynamics are the same up to a random fluctuating phase [22] which does not affect the concurrence $C_{\psi(t)}$. More generally, one can show that the mean concurrence for the QSD model with correlated complex noises satisfying the Itô rules $d\xi_m^i d\xi_n^j = u_{mn}^{ij} dt$ and $d\xi_m^i (d\xi_n^j)^* = \delta_{ij} \delta_{mn} dt$ [26], which gives back the model of Gisin and Percival when $u_{mn}^{ij} = 0$, decays exponentially as in

(1) if the two baths are independent (i.e., if $u_{mn}^{AB} = 0$ for any m, n).

VII. QUBITS COUPLED TO A COMMON BATH

We focus here on a specific model of two qubits with equal frequencies coupled resonantly to the same modes of the electromagnetic field initially in the vacuum. A photon counter D makes a click when a photon is emitted by qubit A or B . The jump operator in the rotating wave approximation, $J = \sigma_- \otimes 1_B + 1_A \otimes \sigma_-$, is now nonlocal. We take $H_0 = 0$. Proceeding as for independent baths, the contribution to the mean concurrence of quantum trajectories without jump between 0 and t is $p_{\text{nj}}(0, t) C_{\text{nj}}(t) = |\langle \sigma_y \otimes \sigma_y T \rangle e^{-tK} |\psi(0)\rangle|$ and can be determined with the help of (13). By calculating the exponential of $K = \gamma J^\dagger J / 2$, one finds $e^{-(t-t_0)K} |\psi(t_0)\rangle = \sum_{s,s'} c_{ss'}(t) |s, s'\rangle$ with $c_{\uparrow\uparrow}(t) = e^{-\gamma(t-t_0)} c_{\uparrow\uparrow}(t_0)$, $2c_{ss'}(t) = (e^{-\gamma(t-t_0)} + 1)c_{ss'}(t_0) + (e^{-\gamma(t-t_0)} - 1)c_{s's}(t_0)$ for $ss' = \uparrow\downarrow$ or $\downarrow\uparrow$, and $c_{\downarrow\downarrow}(t) = c_{\downarrow\downarrow}(t_0)$. Quantum trajectories having one jump in $[0, t]$ give a nonzero contribution. The probability density that the jump occurs at time $t_j \in [0, t]$ is given by $\gamma p_{\text{nj}}(t_j, t) \|J|\psi(t_j)\rangle\|^2 p_{\text{nj}}(0, t_j) = \gamma \mathcal{N}_{1j, t_j}(t)^2$ with $\mathcal{N}_{1j, t_j}(t) = \|e^{-(t-t_j)K} J e^{-t_j K} |\psi(0)\rangle\|$ [this follows from the formula $p_{\text{nj}}(t_0, t) = \|e^{-(t-t_0)K} |\psi(t_0)\rangle\|^2$; see Sec. III]. The contribution of trajectories having one jump in $[0, t]$ is then obtained by multiplying this density by $C_{1j, t_j}(t) = 2\mathcal{N}_{1j, t_j}(t)^{-2} e^{-2\gamma t} |c_{\uparrow\uparrow}|^2$ and integrating over t_j . After two clicks, $|\psi(t)\rangle = |\downarrow\downarrow\rangle$ is in an invariant separable state. Therefore, trajectories with more than one jump do not contribute to the mean concurrence. Setting $c_{\pm} = c_{\uparrow\downarrow} \pm c_{\downarrow\uparrow}$, one gets

$$\overline{C(t)} = \frac{1}{2} |c_-^2 - c_+^2 e^{-2\gamma t} + 4c_{\uparrow\uparrow} c_{\downarrow\downarrow} e^{-\gamma t}| + 2|c_{\uparrow\uparrow}|^2 \gamma t e^{-2\gamma t}. \quad (25)$$

The time behavior of the concurrence (25) depends strongly on the initial state. Unlike in the case of independent baths, $\overline{C(t)}$ may vanish at nonzero finite discrete times t_0 . A necessary and sufficient condition for this loss of entanglement (immediately followed by a revival) is $c_{\uparrow\uparrow} = 0$ and $\arg(c_{\downarrow\downarrow}) = \arg(c_{\downarrow\uparrow})$ (i.e., $c_+/c_- \in [-\infty, -1] \cup [1, \infty]$). If this condition is fulfilled, $\overline{C(t)}$ vanishes at time $t_0 = \gamma^{-1} \ln(|c_+/c_-|)$; see Fig. 3. It is not difficult to show by solving the master equation (7) with $J = \sigma_- \otimes 1_B + 1_A \otimes \sigma_-$ that, for any initial state containing at most one excitation (i.e., such that $c_{\uparrow\uparrow} = 0$), $\overline{C(t)} = |c_-^2 - c_+^2 e^{-2\gamma t}|/2$ coincides at all times with the concurrence $C_{\rho(t)}$ for the density matrix. In contrast, if $c_{\uparrow\uparrow} \neq 0$ then $\overline{C(t)}$ increases at small times whereas $C_{\rho(t)}$ decreases, as shown in the inset of Fig. 3. For any initial state, $\overline{C(t)}$ converges at large times $t \gg \gamma^{-1}$ to the same asymptotic value $C_\infty = |c_-|^2/2$ as the concurrence $C_{\rho(t)}$ [15,27].

A nonlocal measurement scheme depending on the initial state $|\psi(0)\rangle$ and such that $\overline{C(t)} = C_{\rho(t)}$ at all times $t \in [0, t_{\text{EDS}}]$ has been found recently [19] for two qubits coupled to two baths at zero temperature in the rotating-wave approximation. If one neglects the Hamiltonian of the qubits, this scheme is time independent. The corresponding quantum trajectories are given by a QSD equation [26] for homodyne detection with two jump operators J_1 and J_2 similar to the jump

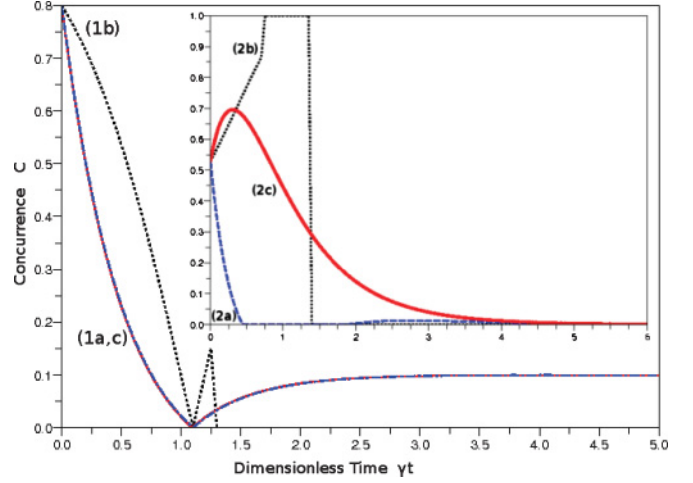


FIG. 3. (Color online) Concurrence of two qubits coupled to a common bath versus γt for $|\psi(0)\rangle = \frac{2}{\sqrt{5}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{5}}|\downarrow\uparrow\rangle$: (1a) $C_{\rho(t)}$ (blue dashed line); (1b) $C_{\psi(t)}$ for a single trajectory (black dotted line); (1c) $\overline{C(t)}$ given by (25) (red line superimposed on the blue line). Inset (2) is the same for $|\psi(0)\rangle = \frac{7i}{\sqrt{53}}|\uparrow\uparrow\rangle + \frac{2i}{\sqrt{53}}|\downarrow\downarrow\rangle$.

operator J introduced in this section, combined with intense laser fields via 50% beam splitters, as described in Sec. VI (the main difference between $J_{1,2}$ and J comes from the presence of appropriately chosen phase factors in front of σ_- and σ_+ making $J_{1,2}$ nonsymmetric under the exchange of the two qubits). It is striking that we also find in our model that $\overline{C(t)} = C_{\rho(t)}$ for specific initial states, even though the dynamics in the absence of measurements—and thus the density matrix concurrence $C_{\rho(t)}$ —are not the same in the two models (here the two qubits are coupled to a common bath, whereas they are coupled to distinct baths in Ref. [19]).

VIII. CONCLUSION

We have found explicit formulas for the mean concurrence $\overline{C(t)}$ of quantum trajectories and have shown that the measurements on the baths may be used to protect the entanglement of two qubits. These results shed new light on the phenomenon of entanglement sudden death. For independent baths, $\overline{C(t)}$ is either constant in time or vanishes exponentially with a rate depending on the measurement scheme only, whereas for a common bath $\overline{C(t)}$ depends strongly on the initial state and may coincide with the concurrence $C_{\rho(t)}$ of the density matrix for some initial states. A constant $\overline{C(t)}$ implies a perfect protection of maximally entangled states for all quantum trajectories. In the case of pure dephasing and for Jaynes-Cummings couplings at infinite temperature, we have found measurement schemes independent of the initial state of the qubits which lead to such a perfect entanglement protection. Despite obvious analogies, this way to protect entanglement differs from the strategy based on the quantum Zeno effect proposed in Ref. [15]. In fact, in the QJ and QSD models considered here the time interval between consecutive measurements is not arbitrarily small with respect to the damping constant γ^{-1} . In the QJ model this time interval dt must be chosen such that the jump probability $dp(t) \propto \gamma dt$ is very small but one cannot let γdt go to zero since this would

amount to replacing $dp(t)$ by 0 and $e^{-iH_{\text{eff}}dt}$ by e^{-iH_0dt} in (6). In contrast, a perfect entanglement protection is reached in [15] in the idealized limit $\gamma dt \rightarrow 0$ (i.e., when the measurements completely prevent the decay of the superradiant state [28]).

For independent baths, $\overline{C(t)}$ is strictly greater than $C_{\rho(t)}$ if the latter concurrence vanishes after a finite time. Therefore, if there exists a measurement scheme such that the mean entanglement of formation $\overline{E(t)}$ is equal to the entanglement of formation of the density matrix [which would imply $\overline{C(t)} \leq C_{\rho(t)}$], this scheme must necessarily involve measurements of nonlocal (joint) observables of the two baths.

Let us finally note that it should be possible to check our findings experimentally by using similar optical devices as in Ref. [4].

Note added in proof. Recently, we learned that related results have been obtained in [29].

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- [1] L. Diósi, in *Irreversible Quantum Dynamics*, edited by F. Benatti and R. Floreanini, Lecture Notes in Physics Vol. 622 (Springer, Berlin, 2003), p. 157.
 - [2] P. J. Dodd and J. J. Halliwell, *Phys. Rev. A* **69**, 052105 (2004).
 - [3] T. Yu and J. H. Eberly, *Phys. Rev. Lett.* **93**, 140404 (2004).
 - [4] M. P. Almeida *et al.*, *Science* **316**, 579 (2007).
 - [5] M. O. Terra Cunha, *New J. Phys.* **9**, 237 (2007).
 - [6] D. Braun, *Phys. Rev. Lett.* **89**, 277901 (2002).
 - [7] Z. Ficek and R. Tanás, *Phys. Rev. A* **74**, 024304 (2006).
 - [8] L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway, *Phys. Rev. A* **79**, 042302 (2009).
 - [9] J. Dalibard, Y. Castin, and K. Mølmer, *Phys. Rev. Lett.* **68**, 580 (1992).
 - [10] H. J. Carmichael, *An Open System Approach to Quantum Optics*, Lecture Notes in Physics, Series M18 (Springer, Berlin, 1993).
 - [11] H. M. Wiseman and G. J. Milburn, *Phys. Rev. A* **47**, 642 (1993).
 - [12] N. Gisin and I. C. Percival, *J. Phys. A: Math. Gen.* **25**, 5677 (1992).
 - [13] H. Nha and H. J. Carmichael, *Phys. Rev. Lett.* **93**, 120408 (2004).
 - [14] A. R. R. Carvalho, M. Busse, O. Brodier, C. Viviescas, and A. Buchleitner, *Phys. Rev. Lett.* **98**, 190501 (2007).
 - [15] S. Maniscalco, F. Francica, R. L. Zaffino, N. Lo Gullo, and F. Plastina, *Phys. Rev. Lett.* **100**, 090503 (2008).
 - [16] D. M. Mundarain and M. Orszag, *Phys. Rev. A* **79**, 052333 (2009).
 - [17] A. R. R. Carvalho and J. J. Hope, *Phys. Rev. A* **76**, 010301(R) (2007).
 - [18] E. Mascarenhas, B. Marques, D. Cavalcanti, M. T. Cunha, and M. F. Santos, *Phys. Rev. A* **81**, 032310 (2010).
 - [19] C. Viviescas, I. Guevara, A. R. R. Carvalho, M. Busse, and A. Buchleitner, *Phys. Rev. Lett.* **105**, 210502 (2010).
 - [20] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
 - [21] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
 - [22] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
 - [23] M. B. Plenio and P. L. Knight, *Rev. Mod. Phys.* **70**, 101 (1998).
 - [24] S. Haroche and J.-M. Raimond, *Exploring the Quantum: Atoms, Cavities and Photons* (Oxford University Press, Oxford, 2006).
 - [25] D. Spehner and M. Orszag, *J. Math. Phys.* **43**, 3511 (2002).
 - [26] H. M. Wiseman and L. Diósi, *Chem. Phys.* **268**, 91 (2001).
 - [27] M. Orszag and M. Hernandez, *Adv. Opt. Photon.* **2**, 229 (2010).
 - [28] M. C. Fischer, B. Gutierrez-Medina, and M. G. Raizen, *Phys. Rev. Lett.* **87**, 040402 (2001); P. E. Toscheck and C. Wunderlich, *Eur. Phys. J. D* **14**, 387 (2001).
 - [29] E. Mascarenhas, D. Cavalcanti, V. Vedral, and M. França Santos, e-print [arXiv:1006.1233v1](https://arxiv.org/abs/1006.1233v1) [quant-ph].