Geometry on the set of quantum states and quantum correlations

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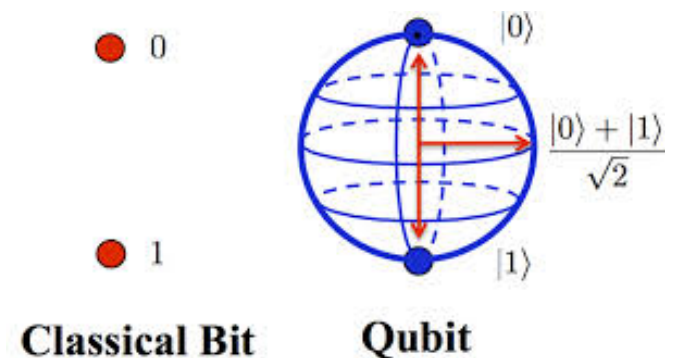
Quantum Correlations & Quantum Information

▷ Quantum Information Theory (QIT) studies quantum systems that can perform information-processing tasks more efficiently than one can do with classical systems:

- *computational tasks* (e.g. factorizing into prime numbers)
- *quantum communication* (e.g. quantum cryptography, ...)

- A quantum computer works with qubits, i.e. two-level quantum systems in linear combinations of $|0\rangle$ and $|1\rangle$.

Classical Bit   Qubit

- **Entanglement** is a **resource** for quantum computation and communication
  [Bennett et al. '96, Josza & Linden '03]

However, **other kinds of "quantum correlations"** differing from entanglement could also explain the quantum efficiencies.

Outlines

- Entangled and non-classical states

- Contractive distances on the set of quantum states

- Geometrical measures of quantum correlations

Basic mathematical objects in quantum mechanics

(1) A Hilbert space $\mathcal{H}$ (in this talk: $n = \dim \mathcal{H} < \infty$).

(2) States ρ are non-negative operators on $\mathcal{H}$ with trace one.

(3) Observables A are self-adjoint operators on $\mathcal{H}$
(in this talk: $A \in \mathrm{Mat}(\mathbb{C}, n)$ finite Hermitian matrices)

(4) An evolution is given by a linear map $\Phi : \mathrm{Mat}(\mathbb{C}, n) \to \mathrm{Mat}(\mathbb{C}, n)$ which is

(TP) trace preserving (so that $\mathrm{tr}(\Phi(\rho)) = \mathrm{tr}(\rho) = 1$)

(CP) Completely Positive, i.e. for any integer $d \geqslant 1$ and any $d \times d$ matrix $(A_{ij})_{i,j=1}^{d} \geqslant 0$ with elements $A_{ij} \in \mathrm{Mat}(\mathbb{C}, n)$, one has $(\Phi(A_{ij}))_{i,j=1}^{d} \geqslant 0$.

Special case: unitary evolution $\Phi(\rho) = U \rho U^*$ with U unitary.

Pure and mixed quantum states

- A pure state is a rank-one projector $\rho_\psi = |\psi\rangle\langle\psi|$ with $|\psi\rangle \in \mathcal{H}$, $\|\psi\| = 1$ (actually, $|\psi\rangle$ belongs to the projective space $P\mathcal{H}$).

  The set $\mathcal{E}(\mathcal{H})$ of all quantum states is a convex cone. Its extremal elements are the pure states.

- A mixed state is a non-pure state. It has infinitely many pure state decompositions

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

  with $p_i \geqslant 0$, $\sum_i p_i = 1$ and $|\psi_i\rangle \in P\mathcal{H}$.

  Statistical interpretation: *the pure states $|\psi_i\rangle$ have been prepared with probability p_i.*

Quantum-classical analogy

Hilbert space $\mathcal{H}$	$\leftrightarrow$	finite sample space Ω
state ρ	$\leftrightarrow$	probability p on $(\Omega, \mathcal{P}(\Omega))$
observable	$\leftrightarrow$	random variable on $(\Omega, \mathcal{P}(\Omega))$
set of quantum states $\mathcal{E}(\mathcal{H})$	$\leftrightarrow$	probability simplex $\mathcal{E}_{\text{class}} = \left\{ p \in \mathbb{R}^n_+; \sum_k p_k = 1 \right\}$
CPTP map Φ	$\leftrightarrow$	stochastic matrices $(\Phi_{kl})_{k,l=1,\ldots,n}$ $\left(\Phi_{kl} \geqslant 0, \sum_k \Phi_{kl} = 1 \ \forall \, l \right)$

A bipartite system AB is composed of two subsystems A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. It has Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

For instance, A and B can be the polarizations of two photons localized far from each other $\Rightarrow \mathcal{H}_{AB} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$ (2 qubits):
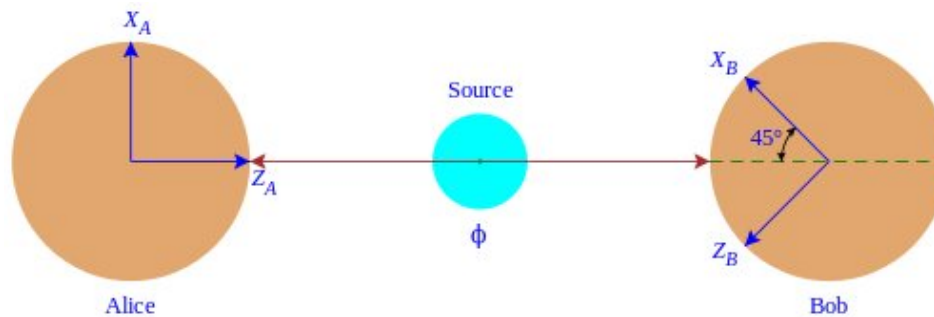

★ A pure state $|\Psi\rangle$ of AB is separable if it is a product state $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$ with $|\psi\rangle \in P\mathcal{H}_A$ and $|\phi\rangle \in P\mathcal{H}_B$.

★ A mixed state ρ is separable if it admits a pure state decomposition $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ with $|\Psi_i\rangle = |\psi_i\rangle \otimes |\phi_i\rangle$ separable for all i.

★ Nonseparable states are called entangled. Entanglement is

↪ *the most specific feature of Quantum Mechanics.*

↪ *used as a resource in Quantum Information (e.g. quantum cryptography, teleportation, high precision interferometry...).*

★ **Examples of entangled & separable states:** let $\mathcal{H}_A \simeq \mathcal{H}_B \simeq \mathbb{C}^2$ (qubits) with canonical basis $\{|0\rangle, |1\rangle\}$. The pure states $|\Psi^{\pm}_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}\Big(|0 \otimes 0\rangle \pm |1 \otimes 1\rangle\Big)$ are maximally entangled.

↪ *lead to the maximal violation of the Bell inequalities observed experimentally [Aspect et al '82]* $\Rightarrow$ *nonlocality of QM*

In contrast, the mixed state $\rho = \dfrac{1}{2}|\Psi^+_{\text{Bell}}\rangle\langle\Psi^+_{\text{Bell}}| + \dfrac{1}{2}|\Psi^-_{\text{Bell}}\rangle\langle\Psi^-_{\text{Bell}}|$

is separable ! (indeed, $\rho = \dfrac{1}{2}|0 \otimes 0\rangle\langle 0 \otimes 0| + \dfrac{1}{2}|1 \otimes 1\rangle\langle 1 \otimes 1|$).

Classical states

★ A state ρ of AB is **classical** if it has a **spectral decomposition** $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ with product $\perp$ states $|\Psi_k\rangle = |\alpha_k\rangle \otimes |\beta_k\rangle$. Classicality is equivalent to separability for pure states only.

★ A state ρ is A-**classical** if $\rho = \sum_i q_i |\alpha_i\rangle\langle\alpha_i| \otimes \rho_{B|i}$ with $\{|\alpha_i\rangle\}$ orthonormal basis of $\mathcal{H}_A$ and $\rho_{B|i}$ arbitrary states of B.

★ The set $\mathcal{C}_{AB}$ (resp. $\mathcal{C}_A$) of all (A-)classical states is **not convex**. Its convex hull is the **set of separable states** $\mathcal{S}_{AB}$.
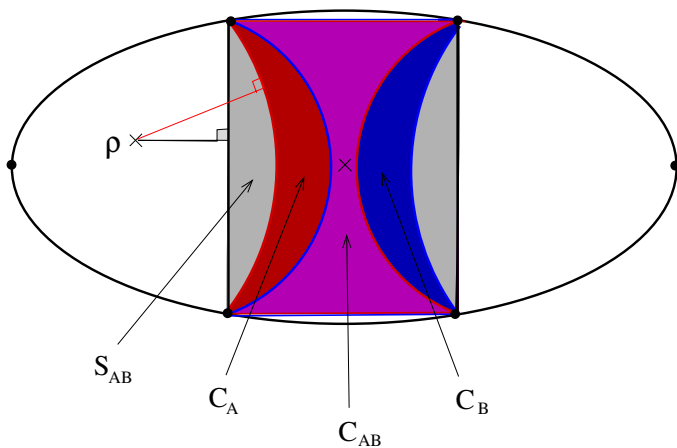
◇ Some tasks impossible to do classically can be realized using **separable non-classical mixed states.**

◇ Such states are easier to produce and presumably more robust to a coupling with an environment.

Quantum vs classical correlations

◇ **Central question in Quantum Information theory:** identify (and try to protect) the Quantum Correlations responsible for the exponential speedup of quantum algorithms.

classical correlations

quantum correlations

◇ For mixed states, two (at least) kinds of QCs

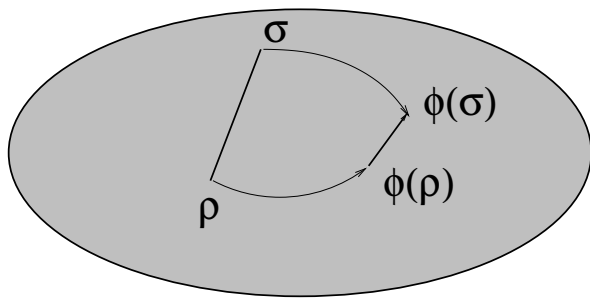↗ **entanglement** *[Schrödinger '36]*

↘ **nonclassicality** (quantum discord)

[Ollivier, Zurek '01, Henderson, Vedral '01]

Outlines

✓ Entangled and non-classical states

- Contractive distances on the set of quantum states

Contractive distances

CONTRACTIVE DISTANCE

• The set $\mathcal{E}_{\mathsf{AB}}$ of all quantum states of a bipartite system AB (i.e. , operators $\rho \geqslant 0$ on $\mathcal{H}_{\mathsf{AB}}$ with $\operatorname{tr}\rho = 1$) can be equipped with many distances d.

• From a QI point of view, interesting distances must be contractive under CPTP maps, i.e. for any such map Φ on $\mathcal{E}_{\mathsf{AB}}$, $\forall \rho, \sigma \in \mathcal{E}_{\mathsf{AB}}$,
$$d(\Phi(\rho), \Phi(\sigma)) \leqslant d(\rho, \sigma)$$
Physically: *irreversible evolutions can only decrease the distance between two states.*

• A contractive distance is in particular unitarily invariant, i.e. $d(U\rho U^*, U\sigma U^*) = d(\rho, \sigma)$ for any unitary U on $\mathcal{H}_{\mathsf{AB}}$

• The L^p-distances $d_p(\rho, \sigma) = \|\rho - \sigma\|_p = (\operatorname{tr}|\rho - \sigma|^p)^{1/p}$ are *not* contractive excepted for $p = 1$ (trace distance) *[Ruskai '94]*.

Petz's characterization of contractive distances

- **Classical setting:** there exists a unique (up to a multiplicative factor) contractive Riemannian distance d_{clas} on the probability simplex $\mathcal{E}_{\mathrm{clas}}$, with Fisher metric $ds^2 = \sum_k dp_k^2/p_k$ *[Cencov '82]*

- **Quantum generalization:** any Riemannian contractive distance on the set of states $\mathcal{E}(\mathcal{H})$ with $n = \dim \mathcal{H} < \infty$ has metric

$$\mathrm{d}s^2 = g_\rho(\mathrm{d}\rho, \mathrm{d}\rho) = \sum_{k,l=1}^{n} c(p_k, p_l) |\langle k|\mathrm{d}\rho|l\rangle|^2$$

where p_k and $|k\rangle$ are the eigenvalues and eigenvectors of ρ,

$$c(p,q) = \frac{pf(q/p) + qf(p/q)}{2pqf(p/q)f(q/p)}$$

and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitary operator-monotone function such that $f(x) = xf(1/x)$ *[Morozova & Chentsov '90, Petz '96]*

Distance associated to the von Neumann entropy

▷ Quantum analog of the Shannon entropy: von Neumann entropy

$$S(\rho) = -\operatorname{tr}(\rho \ln \rho)$$

▷ Since S is concave, the **physically most natural metric** is

$$\mathrm{d}s^2 = g_S(\mathrm{d}\rho, \mathrm{d}\rho) = -\left.\frac{\mathrm{d}^2 S(\rho + t\mathrm{d}\rho)}{\mathrm{d}t^2}\right|_{t=0} = \left.\frac{\mathrm{d}^2 F(X + s\mathrm{d}X)}{\mathrm{d}s^2}\right|_{s=0}$$

[Bogoliubov; Kubo & Mori; Balian, Alhassid & Reinhardt, '86, Balian '14].
with $F(X) = \ln \operatorname{tr}(e^X)$ and $\rho = e^{X - F(X)} = e^X / \operatorname{tr}(e^X)$.

▷ $\mathrm{d}s^2$ has the Petz form with $f(x) = \frac{x-1}{\ln x}$

↪ the corresponding distance is contractive.

▷ **Loss of information when mixing** the neighboring equiprobable
states $\rho_{\pm} = \rho \pm \frac{1}{2}\mathrm{d}\rho$: $\mathrm{d}s^2/8 = S(\rho) - \frac{1}{2}S(\rho_+) - \frac{1}{2}S(\rho_-)$

Bures distance and Uhlmann fidelity

▷ **Fidelity** (generalizes $F=|\langle\psi|\phi\rangle|^2$ for mixed states) *[Uhlmann '76]*

$$F(\rho,\sigma) = \left\{\mathrm{tr}[\sqrt{\sigma}\rho\sqrt{\sigma}]^{1/2}\right\}^2 = F(\sigma,\rho)$$

▷ **Bures distance:** $d_{\mathrm{Bu}}(\rho,\sigma) = \left(2 - 2\sqrt{F(\rho,\sigma)}\right)^{\frac{1}{2}}$ *[Bures '69]*

   ↪ has metric of the Petz form with $f(x) = \frac{x+1}{2}$

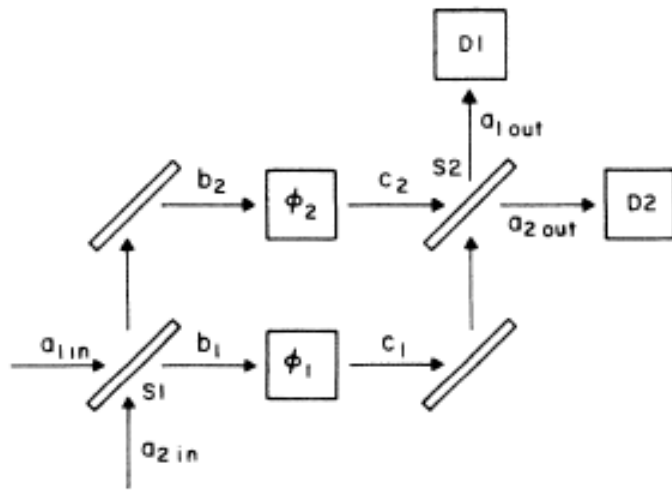   ↪ **smallest** contractive Riemannian distance   *[Petz '96]*

   ↪ coincides with the Fubiny-Study metric on $P\mathcal{H}$ for pure states

   ↪ $d_{\mathrm{Bu}}(\rho,\sigma)^2$ is jointly convex in (ρ,σ)

▷ $d_{\mathrm{Bu}}(\rho,\sigma) = \sup d_{\mathrm{clas}}(p,q)$ with **sup over all measurements** giving outcome k with proba p_k (for state ρ) and q_k (for state σ)

                                                        [Fuchs '96]

Bures distance and Fisher information

In quantum metrology, the goal is to estimate an unknown parameter ϕ by measuring the output states

$$\rho_{\text{out}}(\phi) = e^{-\mathrm{i}\phi H}\, \rho\, e^{\mathrm{i}\phi H}$$

and using a statistical estimator depending on the measurement results

(e.g. in quantum interferometry: estimate the phase shift $\phi_1 - \phi_2$)

$\hookrightarrow$ precision $\Delta\phi = \left\langle \left(\left| \frac{\partial \langle \phi_{\text{est}} \rangle_\phi}{\partial \phi} \right|^{-1} \phi_{\text{est}} - \phi \right)^2 \right\rangle_\phi^{1/2}$

The smallest precision is given by the **quantum Crámer-Rao bound**

$$(\Delta\phi)_{\text{best}} = \frac{1}{\sqrt{N}\sqrt{\mathcal{F}(\rho, H)}}\,, \quad \mathcal{F}(\rho, H) = 4 d_{\text{Bu}}(\rho, \rho + \mathrm{d}\rho)^2\,, \quad \mathrm{d}\rho = -\mathrm{i}[H, \rho]$$

$N =$ number of measurements

$\mathcal{F}(\rho, H) =$ **quantum Fisher information**   *[Braunstein & Caves '94]*

Summary

CONTRACTIVE RIEMANNIAN METRICS:

Classical		*Quantum*	*Interpretation*
		Bures $\mathrm{d}s^2_{\mathrm{Bu}}$	Q. metrology
unique:	$\nearrow$	$\vdots$	(Fisher information)
$\mathrm{d}s^2_{\mathrm{clas}} = \sum_k \dfrac{\mathrm{d}p_k^2}{p_k}$	$\longrightarrow$	$\mathrm{d}s^2_S = -\mathrm{d}^2 S$	Loss of information
(Fisher)	$\searrow$	$\vdots$	when merging 2 states
		Hellinger $\mathrm{d}s^2_{\mathrm{Hel}}$	Q. state discrimination
		$\vdots$	with many copies

Outlines

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✓ Contractive distances on the set of quantum states

• Geometrical measures of quantum correlations

Geometric approach of quantum correlations
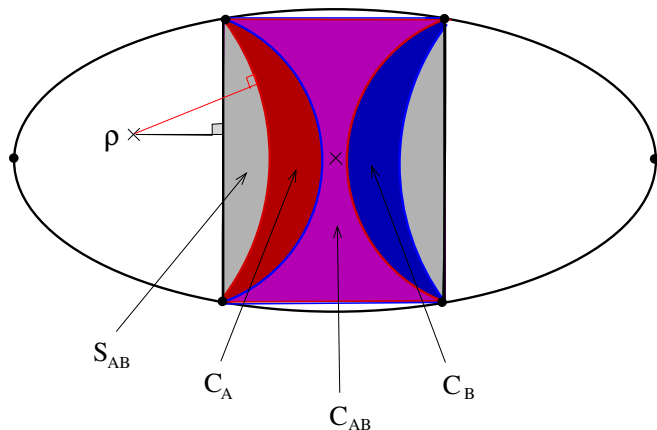
Geometric entanglement:

$$E(\rho) = \min_{\sigma_{\text{sep}} \in \mathcal{S}_{AB}} d(\rho, \sigma_{\text{sep}})^2$$

Geometric quantum discord :

$$D_A(\rho) = \min_{\sigma_{A\text{-cl}} \in \mathcal{C}_A} d(\rho, \sigma_{A\text{-cl}})^2$$

Properties:

✓ $E(\rho_\Psi) = D_A(\rho_\Psi)$ for pure states ρ_Ψ $\quad \leftarrow$ for Bures distance

✓ E is convex $\qquad\qquad\qquad\qquad\qquad\quad \leftarrow$ if d^2 is jointly convex

✓ Entanglement monotonicity: $E(\Phi_A \otimes \Phi_B(\rho)) \leqslant E(\rho)$ for any TPCP maps Φ_A and Φ_B acting on A and B (also true for D_A but only when $\Phi_A(\rho_A) = U_A \rho_A U_A^*$). $\qquad \leftarrow$ if d is contractive

Bures geometric measure of entanglement

$$E_{\mathrm{Bu}}(\rho) = d_{\mathrm{Bu}}(\rho, \mathcal{S}_{\mathsf{AB}})^2 = 2 - 2\sqrt{F(\rho, \mathcal{S}_{\mathsf{AB}})}$$

with $F(\rho, \mathcal{S}_{\mathsf{AB}}) = \max_{\sigma_{\mathrm{sep}} \in \mathcal{S}_{\mathsf{AB}}} F(\rho, \sigma_{\mathrm{sep}})$

$= $ maximal fidelity between ρ and a separable state.

$\longrightarrow$ **Main physical question:** determine $F(\rho, \mathcal{S}_{\mathsf{AB}})$ explicitely.

$\blacklozenge$ **pb:** *it is not easy to find the geodesics for the Bures distance!*

$\triangleright$ The closest separable state to a pure state ρ_Ψ is a pure product state, so that $F(\rho_\Psi, \mathcal{S}_{\mathsf{AB}}) = \max_{|\varphi\rangle, |\chi\rangle} |\langle \varphi \otimes \chi | \Psi \rangle|^2$ $\quad \to$ *easy!*

$\triangleright$ For mixed states ρ, $F(\rho, \mathcal{S}_{\mathsf{AB}})$ coincides with the convex roof

[Streltsov, Kampermann and Bruß'10]

$$F(\rho, \mathcal{S}_{\mathsf{AB}}) = \max_{\{|\Psi_i\rangle, \eta_i\}} \sum_i p_i F(\rho_{\Psi_i}, \mathcal{S}_{\mathsf{AB}})$$

$\to$ *not easy!*

max. over all pure state decompositions $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ of ρ.

The two-qubit case

Assume that both subsystems A and B are qubits, $\mathcal{H}_\mathsf{A} \simeq \mathcal{H}_\mathsf{B} \simeq \mathbb{C}^2$.

- **Concurrence:** *[Wootters '98]*

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

with $\lambda_1^2 \geqslant \lambda_2^2 \geqslant \lambda_3^2 \geqslant \lambda_4^2$ the eigenvalues of $\rho\,\sigma_y \otimes \sigma_y\,\overline{\rho}\,\sigma_y \otimes \sigma_y$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \text{Pauli matrix}$$
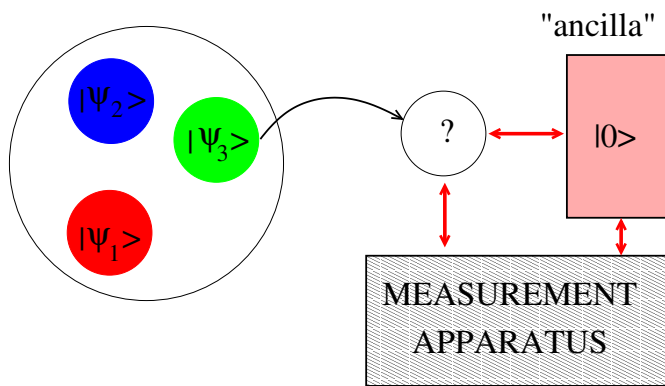
$\overline{\rho} = $ complex conjugate of ρ in the canonical (product) basis.

- Then *[Wei and Goldbart '03, Streltsov, Kampermann and Bruß'10]*

$$F(\rho, \mathcal{S}_\mathsf{AB}) = \tfrac{1}{2}\big(1 + \sqrt{1 - C(\rho)^2}\big)$$

Quantum State Discrimination

- A receiver gets a state ρ_i randomly chosen with probability η_i among a known set of states $\{\rho_1, \cdots, \rho_m\}$.

- To determine the state he has in hands, he performs a measurement on it.

$\hookrightarrow$ **Applications :** quantum communication, cryptography,...

$\diamond$ If the ρ_i are $\perp$, one can discriminate them unambiguously

$\diamond$ Otherwise one succeeds with probability

$$P_S = \sum_i \eta_i \operatorname{tr}(M_i \rho_i)$$

M_i = non-negative operators describing the measurement, $\sum_i M_i = 1$.
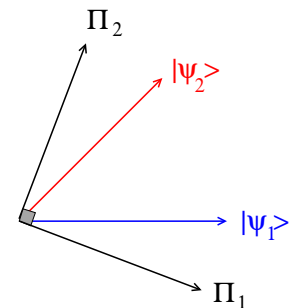
Open pb (for $n > 2$): *find the optimal measurement $\{M_i^{\mathrm{opt}}\}$ and highest success probability P_S^{opt}.*

Bures geometric quantum discord

The square Bures distance $D_\mathsf{A}(\rho) = d_\mathrm{Bu}(\rho, \mathcal{C}_\mathsf{A})^2$ to the set $\mathcal{C}_\mathsf{A}$ of A-classical states is a geometric analog of the **quantum discord** characterizing the "quantumness" of states

(actually, the A-classical states are the states with zero discord)

- $P_S^\mathrm{opt}(|\alpha_i\rangle) = $ optimal success proba. in discriminating the states

$$\rho_i = \eta_i^{-1} \sqrt{\rho} \, |\alpha_i\rangle\langle\alpha_i| \otimes 1 \, \sqrt{\rho}$$

with proba $\eta_i = \langle\alpha_i| \, \mathrm{tr}_\mathsf{B}(\rho)|\alpha_i\rangle$, where $\{|\alpha_i\rangle\} = $ orthonormal basis of $\mathcal{H}_\mathsf{A}$.

- The geometric quantum discord is given by solving a state
  discrimination problem   *[Spehner and Orszag '13]*

$$D_\mathsf{A}(\rho) = 2 - 2\max_{\{|\alpha_i\rangle\}} \sqrt{P_S^\mathrm{opt}(|\alpha_i\rangle)}$$

Closest A-classical states to a state ρ

- The closest A-classical states to ρ are

$$\sigma_\rho = \frac{1}{F(\rho, \mathcal{C}_A)} \sum_i |\alpha_i^{\mathrm{opt}}\rangle\langle\alpha_i^{\mathrm{opt}}| \otimes \langle\alpha_i^{\mathrm{opt}}|\sqrt{\rho}\,\Pi_i^{\mathrm{opt}}\sqrt{\rho}\,|\alpha_i^{\mathrm{opt}}\rangle$$

[Spehner and Orszag '13]

where $\{\Pi_i^{\mathrm{opt}}\}$ is the optimal von Neumann measurement and $\{|\alpha_i^{\mathrm{opt}}\rangle\}$ the orthonormal basis of $\mathcal{H}_A$ maximizing P_S^{opt}, i.e.

$$F(\rho, \mathcal{C}_A) = \sum_{i=1}^{n_A} \eta_i^{\mathrm{opt}}\,\mathrm{tr}(M_i^{\mathrm{opt}}\rho_i^{\mathrm{opt}}).$$

- ρ can have either a unique or an infinity of closest A-classical states.

The qubit case

- If A is a qubit, $\mathcal{H}_A \simeq \mathbb{C}^2$, and $\dim \mathcal{H}_B = n_B$, then

$$F(\rho, \mathcal{C}_A) = \frac{1}{2} \max_{\|\mathbf{u}\|=1} \left\{ 1 - \operatorname{tr} \Lambda(\mathbf{u}) + 2 \sum_{l=1}^{n_B} \lambda_l(\mathbf{u}) \right\}$$
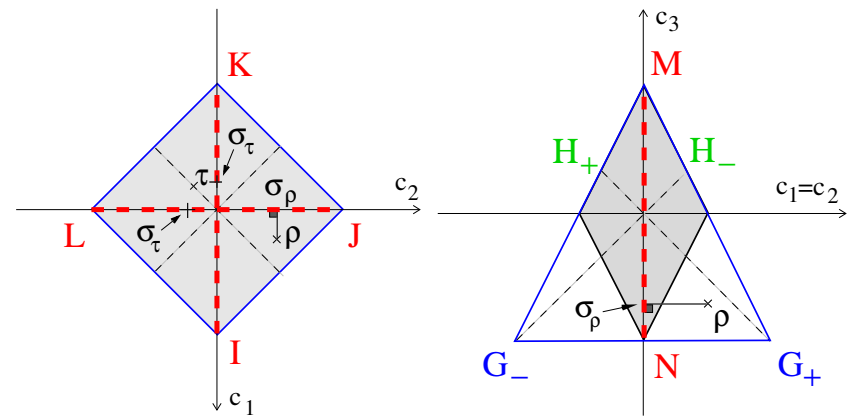
[Spehner and Orszag '14]

$\lambda_1(\mathbf{u}) \geqslant \cdots \geqslant \lambda_{2n_B}(\mathbf{u})$ eigenvalues
  of the $2n_B \times 2n_B$ matrix

$\Lambda(\mathbf{u}) = \sqrt{\rho}\, \sigma_{\mathbf{u}} \otimes 1 \sqrt{\rho}$
  with $\mathbf{u} \in \mathbb{R}^3$, $\|\mathbf{u}\| = 1$, and

$\sigma_{\mathbf{u}} = u_1 \sigma_1 + u_2 \sigma_2 + u_3 \sigma_3$ with σ_i
  Pauli matrices.

Conclusions & perspectives

- **Conclusions:**

  ▷ Contractive Riemannian distances on the set of quantum states provide useful tools for measuring quantum correlations in bipartite systems.

  ▷ **Major challenges are**

    ↪ compute the geometric measures for simple systems
    ↪ compare the measures obtained from different distances and look for universal properties

- **References:**

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  - *R. Roga, D. Spehner, F. Illuminati*, arXiv:1510.06995