

Mott law for a random walk in a random medium

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Outline of the talk

- The model
- Main results
- Sketch of the proof

Joint work with: A. Faggionato, La Sapienza, Roma

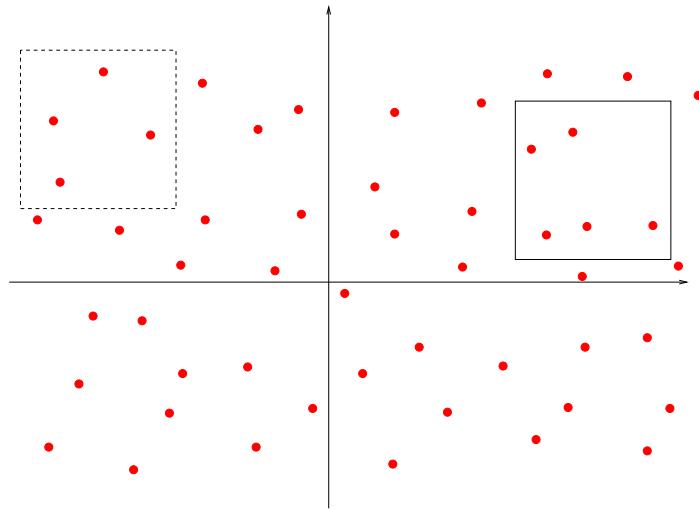
H. Schulz-Baldes, Universität Erlangen

Outline of the talk

- The model

Random environment (1)

- Let x_i be random distinct points in \mathbb{R}^d with a stationary and mixing distribution $\hat{\mathcal{P}}$.



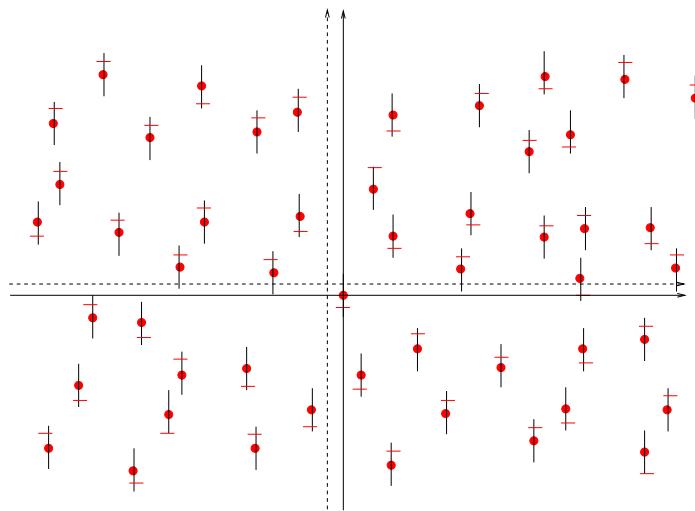
$N_B = \#$ points in $B \subset \mathbb{R}^d$ (B bounded Borel set)

$\mathbb{E}_{\hat{\mathcal{P}}}(N_B) = \rho|B|$, ρ = mean density, $\rho < \infty$.

- EX : (stationary) Poisson process
 - (i) $\hat{\mathcal{P}}(N_B = n) = (\rho|B|)^n e^{-\rho|B|}/n!$
 - (ii) the N_B are independent for disjoint B 's.

Random environment (2)

- To each point x_i is associated a random energy $E_i \in [-1, 1]$.
The E_i are independent and have all the distribution ν .

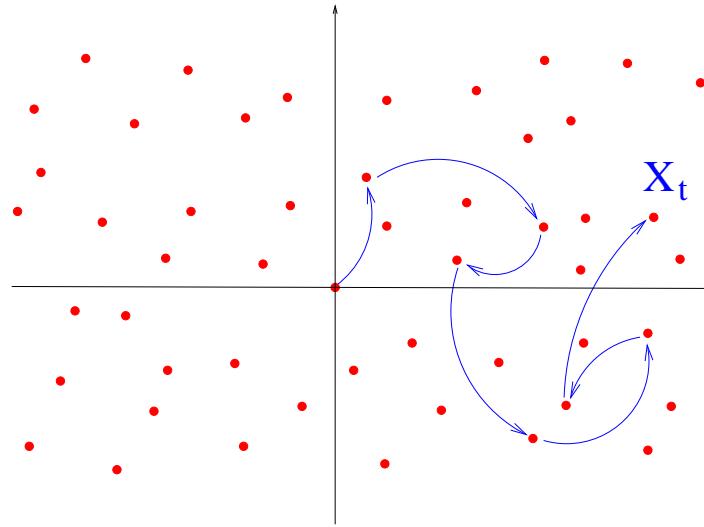


- Pick up (at random) a point among $\{x_i\}$ and choose it as the origin \rightarrow new distribution = Palm distribution $\hat{\mathcal{P}}_0$
EX : $\hat{\mathcal{P}}$ = stat. Poisson process $\rightarrow \hat{\mathcal{P}}_0$ is obtained
by adding 1 (deterministic) point at $x = 0$.

Random walk (1)

Configuration of the environment $\xi = \{x_i, E_i\}$.

A particle located at X_t at time t , starting from $X_0 = 0$, walks randomly on $\{x_i\}$:

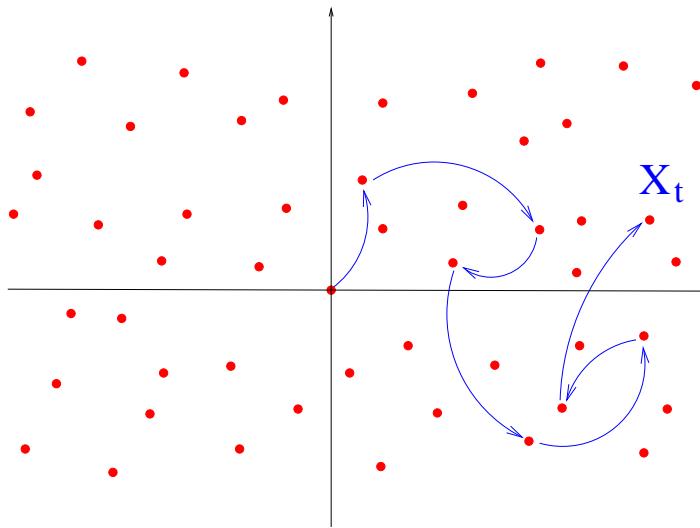


Jumps are possible between any pair of points (x_i, x_j) , with the rate

$$c_{x_i \rightarrow x_j} = e^{-|x_i - x_j|} e^{-\beta(|E_i - E_j| + |E_i| + |E_j|)}$$

β = inverse temperature.

Random walk (2)



For a given configuration $\xi = \{x_i, E_i\}$ of the environment,
let P^ξ be the distribution of the Markov process $(X_t)_{t \geq 0}$.

$\forall x_i \neq x_j, \forall t, t_0 \geq 0,$

$$P^\xi(X_{t_0+t} = x_j | X_{t_0} = x_i) = t c_{x_i \rightarrow x_j}^\xi + \mathcal{O}(t^2).$$

No explosion if $\mathbb{E}_{\hat{\mathcal{P}}_0}(N_B^2) < \infty$.

Physical motivations

Hopping DC electric conduction in disordered solids in the Anderson localization regime

1. add a constant electric field \mathcal{E} : $H_\omega \rightarrow H_\omega + q\mathcal{E}X$,
 $H_\omega = -\Delta + V_\omega$ one-electron RSO, X = position op.
2. couple electrons with a phonon bath at inv. temper. β .

If electron-phonon coupling can be treated within the weak coupling (van Hove) limit, the electron dynamics is given by a classical Markov process, namely, an exclusion process on the set $\{x_i\}$ of localization centers of the eigenfunctions of H_ω , with jump rates given by Fermi's golden rule.

After a mean field approximation, this Markov process reduces to the above RW on $\{x_i\}$ with effective rates $c_{x_i \rightarrow x_j}$.

Einstein relation $\sigma_\beta \propto D_\beta$.

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- The model
- Main results

Convergence to a Brownian motion

Of interest: diffusion constant

$$D_\beta = \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}_0} (\mathbb{E}_{\mathbf{P}^\xi} (X_t^2))$$

Thm 1 :

[Faggionato, Schulz-Baldes & D.S. '04]

- (i) $D_\beta < \infty$ (the limit exists).
- (ii) The process $Y_t = \varepsilon X_{t\varepsilon^{-2}}$ converges weakly in probability as $\varepsilon \rightarrow 0$ to a Brownian motion W_D with covariance $D = D_\beta$.

Bounds on the diffusion constant

$$D_\beta = \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}_0}(\mathbb{E}_{\mathbf{P}^\xi}(X_t^2))$$

The distribution of the E_i is such that $\exists g_\pm > 0$, $\exists \alpha \geq 0$,

$$g_- E^{1+\alpha} \leq \nu(|E_i| \leq E) \leq g_+ E^{1+\alpha}.$$

Thm 2 : [Lower bound: Faggionato, Schulz-Baldes & D.S. '04]
[Upper bound: Faggionato & Mathieu '06]

In dimension $d \geq 2$, for all $\beta \geq \beta_0 > 0$,

$$k_- \beta^\gamma \exp\left\{-c_- \beta^{\frac{\alpha+1}{\alpha+1+d}}\right\} \leq D_\beta \leq k_+ \beta^\kappa \exp\left\{-c_+ \beta^{\frac{\alpha+1}{\alpha+1+d}}\right\}$$

with $c_\pm = r_\pm g_\pm^{-\frac{1}{1+\alpha+d}}$, $r_\pm = \text{const. indep. of } \beta \text{ and } \mathcal{P}_0$

$k_\pm = \text{constant independent of } \beta$

$$\gamma = \frac{(\alpha+1)(2-d)}{\alpha+1+d} < 0 < \kappa = \frac{2(1+\alpha)}{1+\alpha+d}.$$

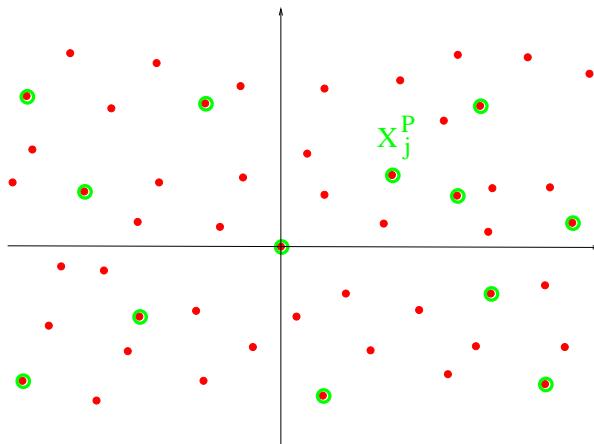
Poisson process

→ shows up naturally in the low temperature limit $\beta \uparrow \infty$.

$$c_{x_i \rightarrow x_j} = e^{-|x_i - x_j|} e^{-\beta(|E_i - E_j| + |E_i| + |E_j|)}$$

Only jumps $x_i \rightarrow x_j$ with $|E_i|, |E_j| \leq E$ are relevant,
with $E \rightarrow 0$ as $\beta \uparrow \infty$

→ random selection of points x_j^p among $\{x_i\}$,
with probability $\nu(|E_i| \leq E) \rightarrow 0$ (thinning).



⇒ $\{x_j^E\}$ converges to a Poisson process
(after an appropriate rescaling).

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Step 1 : “environment viewed from the particle”

Action of the \mathbb{R}^d -translations on $\xi = \sum_i \delta_{(x_i, E_i)}$:

$$S_x \xi = \sum_i \delta_{(\textcolor{red}{x_i} - x, \textcolor{red}{E_i})} , \quad x \in \mathbb{R}^d .$$

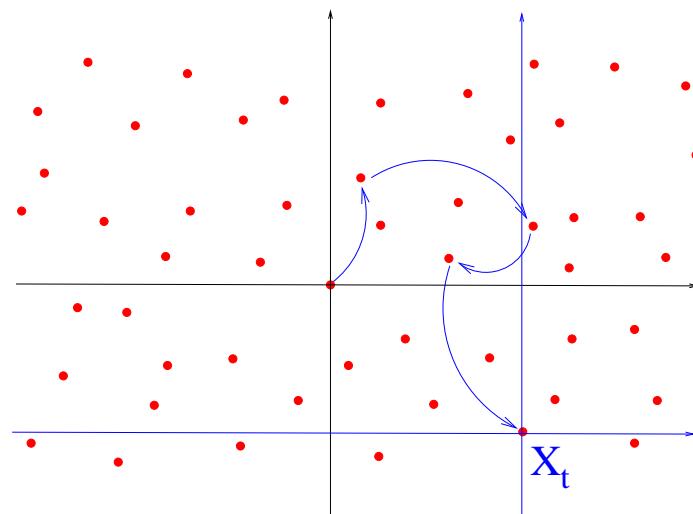
The following properties:

1. symmetry of the jump rates $c_{x_i \rightarrow x_j}^\xi = c_{x_j \rightarrow x_i}^\xi$
2. covariance $c_{x_i \rightarrow x_j}^{S_x \xi} = c_{x_i + x \rightarrow x_j + x}^\xi$
3. ergodicity of $\hat{\mathcal{P}}$

imply that the Markov process

$$\xi_t = S_{X_t} \xi$$

is **reversible** and **ergodic** for the Palm distribution \mathcal{P}_0 .



Step 2 : variational formula for D_β

An application of the results:

[De Masi, Ferrari, Goldstein & Wick '89]

[Kipnis & Varadhan '86]

yields the convergence to a Brownian motion W_{D_β} with

$$D_\beta = \inf_{f \in L^\infty} \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_0^{\xi} \left(x^{(1)} + f(S_x \xi) - f(\xi) \right)^2$$

where $\hat{\xi} = \sum_i \delta_{x_i}$.

Consequences:

(1) $D_\beta \geq \tilde{D}_\beta$ = diffusion constant a RW on $\{x_i\}$ with jump rates $\tilde{c}_{x_i \rightarrow x_j} \leq c_{x_i \rightarrow x_j}$.

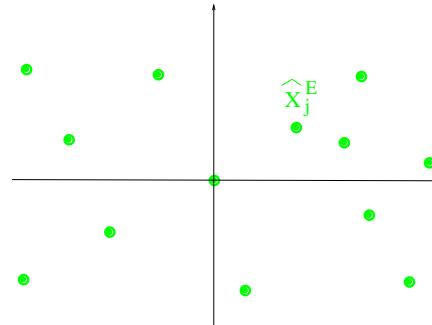
(2) $D_\beta \leq \int \mathcal{P}_0(\xi) \dots$ for a (well-chosen) test function f .

Step 3 : thinning and rescaling

Let $0 < E < 1$ and $R > 0$

- **Thinning :**

$$\{x_j^E\} = \{\textcolor{red}{x}_i : |E_i| \leq E\}$$



- **Rescaling** $x_j^E \rightarrow \widehat{x}_j^E = (r/R) x_j^E$ such that

$$\underbrace{\mathbb{E}_{\mathcal{P}_0} \left(\# \text{ points } \widehat{x}_j^E \in \text{cube of side 1} \right)}_{= \rho(R/r)^d \nu(|E_i| \leq E)} = \mathcal{O}(1)$$

for some fixed $r > 0$ independent of E .

Assumption : $\nu(|E_i| \leq E) E^{-(1+\alpha)} \in [g_-, g_+]$

$$\hookrightarrow R = r E^{-\frac{(1+\alpha)}{d}} .$$

Step 4 : new random walk

Let \tilde{D}_E be the diffusion constant of the RW with jump rates

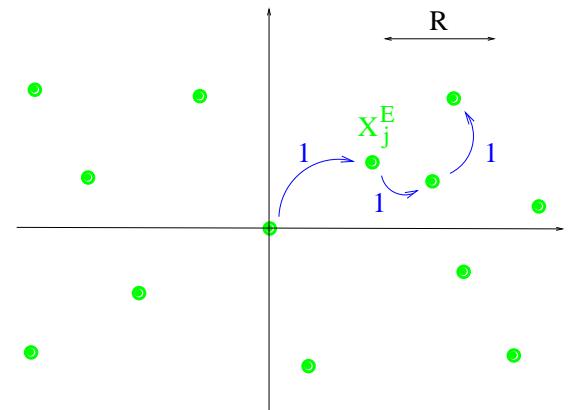
$$\tilde{c}_{x_i \rightarrow x_j} = \begin{cases} 1 & \text{if } |E_i|, |E_j| \leq E \text{ and } |x_i - x_j| \leq R \\ 0 & \text{otherwise.} \end{cases}$$

→ RW on $\{x_j^E\} \subset \{\textcolor{red}{x}_i\}$.

$$\begin{aligned} c_{x_i \rightarrow x_j} &= e^{-|x_i - x_j|} e^{-\beta(|E_i - E_j| + |E_i| + |E_j|)} \\ &\geq e^{-R - 4\beta E} \tilde{c}_{x_i \rightarrow x_j} \end{aligned}$$

D_β is monotonous

$$\Rightarrow D_\beta \geq e^{-R - 4\beta E} \tilde{D}_E .$$



Step 5 : lower bound on D_β

\hat{D}_r = diffusion constant of the RW on $\{\hat{x}_j^E\}$ with jump rates

$$\tilde{c}_{\hat{x}_j \rightarrow \hat{x}_k} = \begin{cases} 1 & \text{if } |\hat{x}_j^E - \hat{x}_k^E| \leq r \\ 0 & \text{otherwise.} \end{cases}$$

$$D_\beta \geq e^{-R-4\beta E} \tilde{D}_E \geq e^{-R-4\beta E} \left(\frac{R}{r}\right)^2 \nu(|E_0| \leq E) \hat{D}_r.$$

Optimization w.r.t. E (Mott's argument).

Constraint : $R = r E^{-\frac{(1+\alpha)}{d}}$

$$\hookrightarrow E = \text{const. } \beta^{-\frac{d}{1+\alpha+d}}, \quad R = \text{const. } \beta^{\frac{1+\alpha}{1+\alpha+d}}$$

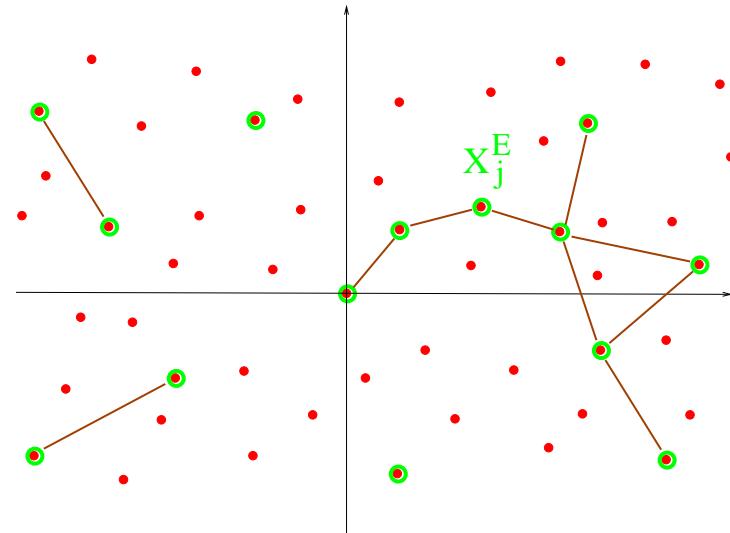
$$D_\beta \geq \hat{D}_r \beta^{-\frac{(\alpha+1)(d-2)}{\alpha+1+d}} \exp \left\{ -c_- \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} \text{ QED? ...if } \hat{D}_r > 0!$$

Step 6 : choice of the test function f

Graph

→ vertices $\{x_j^E\} \subset \{x_i\}$

→ edges : $x_j^E \sim x_k^E$ iff
 $|x_j^E - x_k^E| \leq R$



W_E^ξ = connected component of the graph containing $x = 0$,

$$f(\xi) = \min_{y \in W_E^\xi} \{y^{(1)}\} \underbrace{\chi(|W_E^\xi| \leq N)}_{\text{cut-off function}} \in L^\infty$$

Properties:

- $x_i \in W_E^\xi$ et $|W_E^\xi| \leq N \Rightarrow f(\xi) - f(S_{x_i}\xi) = x_i^{(1)}$
 (comes from the covariance $W_E^{S_{x_i}\xi} = W_E^\xi - x_i$).
- $x_i \notin W^\xi \Rightarrow c_{0 \rightarrow x} = e^{-|x_i|} e^{-\beta(|E_0 - E_i| + |E_i| + |E_0|)}$
 $\leq e^{-\min\{R, \beta E\}}$.

Step 7 : upper bound on D_β (1)

- $|f(\xi)| \leq \left| \min_{y \in W_E^\xi} \{y^{(1)}\} \right| \leq R|W_E^\xi| \quad , \quad |W_E^\xi| \equiv \text{card}(W_E^\xi)$

Let $k \in [0, 1[$. Recall that $c_{0 \rightarrow x}^\xi \leq e^{-\min\{R, \beta E\}}$ if $x \notin W_E^\xi$.

$$\begin{aligned}
 D_\beta &= \inf_{f \in L^\infty} \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0 \rightarrow x}^\xi \underbrace{(x^{(1)} + f(S_x \xi) - f(\xi))^2}_{=0 \text{ if } x \in W_E^\xi \text{ and } |W_E^\xi| \leq N} \\
 &\leq e^{-k \min\{R, \beta E\}} \times \\
 &\quad \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) (c_{0 \rightarrow x}^\xi)^{1-k} \left(2|x|^2 + 8R^2|W_E^\xi|^2 \right) \\
 &\quad + \text{terms going to 0 as } N \rightarrow \infty.
 \end{aligned}$$

By Hölder ineq. and the exponential decay of $c_{0 \rightarrow x}^\xi$, if $p > 1$

$$D_\beta \leq e^{-k \min\{R, \beta E\}} \left(C_1 + C_2 R^2 \mathbb{E}_{\mathcal{P}_0}(|W_E^\xi|^{2p}) \right).$$

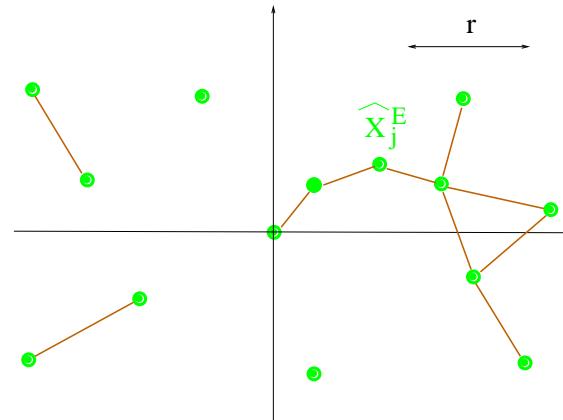
Step 7 : upper bound on D_β (2)

Graph after rescaling

→ vertices $\{\widehat{x}_j^E\}$

→ edges : $\widehat{x}_j^E \sim \widehat{x}_k^E$ iff

$$|\widehat{x}_j^E - \widehat{x}_k^E| \leq r$$



$\widehat{W}^\xi =$ connected component of the graph containing $x = 0$,

$|\widehat{W}^\xi| \equiv \text{card}(W^\xi) = |W_E^\xi|$.

$\Rightarrow \mathbb{E}_{\mathcal{P}_0}(|W_E^\xi|^{2p}) \leq \mathbb{E}(|\hat{W}|^{2p})$ independent of E .

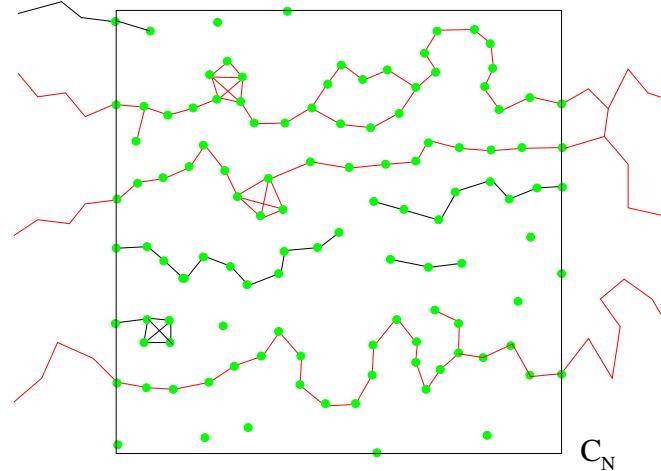
With the optimal choice for E and R (same as for upper b.),

$$\begin{aligned} D_\beta &\leq e^{-k \min\{R, \beta E\}} \left(C_1 + C_2 R^2 \mathbb{E}_{\mathcal{P}_0}(|W_E^\xi|^{2p}) \right) \\ &\leq C (1 + \beta^\kappa) \exp \left\{ -c_+ \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} \end{aligned}$$

QED ...if $\mathbb{E}(|\hat{W}|^{2p}) < \infty$!

Step 8 : percolation theory (1)

Points \hat{x}_j^E separated by a distance smaller than r are connected by edges.



$r > r_c$: \exists ! infinite cluster of connected points (a.s.)

$r < r_c$ (or dim. $d = 1$) : no infinite cluster (a.s.)

→ in order that the RW on $\{\hat{x}_j^E\}$ with jump rates

$\tilde{c}_{\hat{x} \rightarrow \hat{y}} = \chi(|\hat{x} - \hat{y}| \leq r)$ have a diffusion constant $\hat{D}_r > 0$,

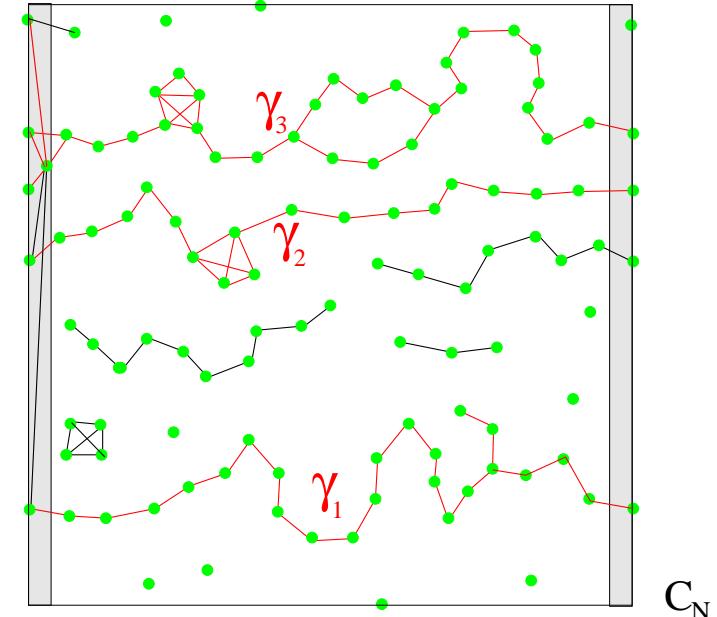
it is necessary that $r > r_c$ and $d \geq 2$.

→ in order that $\mathbb{E}(\text{card}(\hat{W})^{2p}) < \infty$, it is sufficient that $r < r_c \Rightarrow$ QED lower bound.

Percolation theory (2)

Restriction to a finite cube C_N :

$\hat{D}_{r,N}$ = diffusion const. of the
RW on $\{\hat{x}_j^E\} \cap C_N$
with jumping rates
 $\tilde{c}_{\hat{x} \rightarrow \hat{y}}$ and periodic
boundary conditions



$$\hat{D}_r \geq \limsup_{N \rightarrow \infty} \hat{D}_{r,N}$$

$M_N = \# \text{ points } \hat{x}_j^E \in C_N$

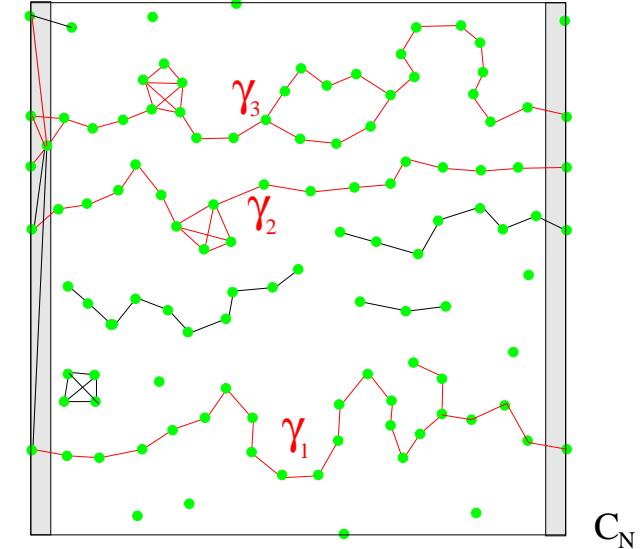
$$M_N \sim \hat{\rho} N^d \quad \text{as} \quad N \rightarrow \infty \quad (\text{ergodicity})$$

$\hat{\rho} = \rho(R/r)^d \nu(|E_i| \leq E) = \text{ mean density of } \{\hat{x}_j^E\}.$

Percolation theory (3)

$G_N = \#$ disjoints paths γ
connecting the left and
right sides of C_N

L_γ = smallest $\#$ points one must
visit on γ to cross C_N .



Poisson process $\{\hat{x}_j^E\}$, $r > r_c \Rightarrow \exists b > 0$,

$\hat{\mathcal{P}}_0(G_N \geq bN^{d-1}) \rightarrow 1$ exponentially as $N \rightarrow \infty$

$$\begin{aligned} \hat{D}_r &\geq \limsup_{N \rightarrow \infty} \hat{D}_{r,N} \geq \limsup_{N \rightarrow \infty} \mathbb{E}_{\mathcal{P}_0} \left(\frac{1}{G_N} \underbrace{\sum_{\gamma} \frac{\text{const. } N}{L_\gamma}}_{\geq \text{const. } N G_N^2 M_N^{-1}} \right) \\ &\geq b' \limsup_{N \rightarrow \infty} \mathbb{E}_{\mathcal{P}_0} \left(\chi(G_N \geq bN^{d-1}) \frac{N^d}{M_N} \right) > 0 \quad \text{QED u.b.} \end{aligned}$$

Main result again

$$D_\beta = \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}_0}(\mathbb{E}_{\mathbf{P}^\xi}(X_t^2))$$

The distribution of the E_i is such that $\exists g_\pm > 0, \exists \alpha \geq 0,$

$$g_- E^{1+\alpha} \leq \nu(|E_i| \leq E) \leq g_+ E^{1+\alpha}.$$

Thm 2 : [Lower bound: Faggionato, Schulz-Baldes & D.S. '04]
[Upper bound: Faggionato & Mathieu '06]

In dimension $d \geq 2$, for all $\beta \geq \beta_0 > 0$,

$$k_- \beta^\gamma \exp\left\{-c_- \beta^{\frac{\alpha+1}{\alpha+1+d}}\right\} \leq D_\beta \leq k_+ \beta^\kappa \exp\left\{-c_+ \beta^{\frac{\alpha+1}{\alpha+1+d}}\right\}$$

with $c_\pm = r_\pm g_\pm^{-\frac{1}{1+\alpha+d}}$, $r_\pm = \text{const. indep. of } \beta \text{ and } \mathcal{P}_0$

$k_\pm = \text{constant independent of } \beta$

$$\gamma = \frac{(\alpha+1)(2-d)}{\alpha+1+d} < 0 < \kappa = \frac{2(1+\alpha)}{1+\alpha+d}.$$