Geometric quantum discords

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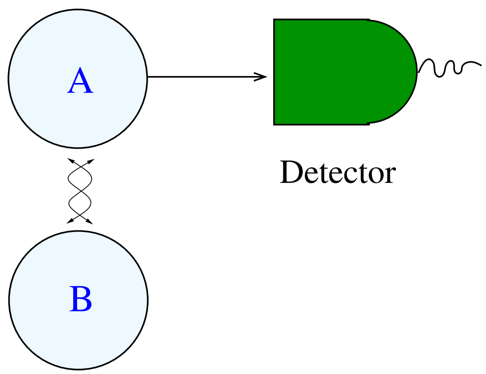
Quantum vs classical correlations

- **Total correlations** between two parties A and B:

$$I_{A:B}(\rho) = S(\rho_A) + S(\rho_B) - S(\rho) \text{ mutual information}$$

- **Quantum discord** $=$ *mutual information not accessible by local measurements on subsystem A*



$$\delta_A(\rho) = \min_{\{\pi_i^A\}} \Big\{ I_{A:B}(\rho) - I_{A:B}\Big(\sum_i \pi_i^A \otimes 1\, \rho\, \pi_i^A \otimes 1 \Big) \Big\}$$

$\pi_i^A =$ rank-one orthogonal projectors for A

[Ollivier & Zurek PRL ('01), Henderson & Vedral JPA ('01)]

- The A-**classical** (=classical-quantum) states are the states

$$\boxed{\sigma_{A\text{-cl}} = \sum_i q_i |\alpha_i\rangle\langle\alpha_i| \otimes \rho_{B|i} \quad \Leftrightarrow \quad \delta_A(\sigma_{A\text{-cl}}) = 0}$$

with $\{|\alpha_i\rangle\} =$ orthonormal basis for A.

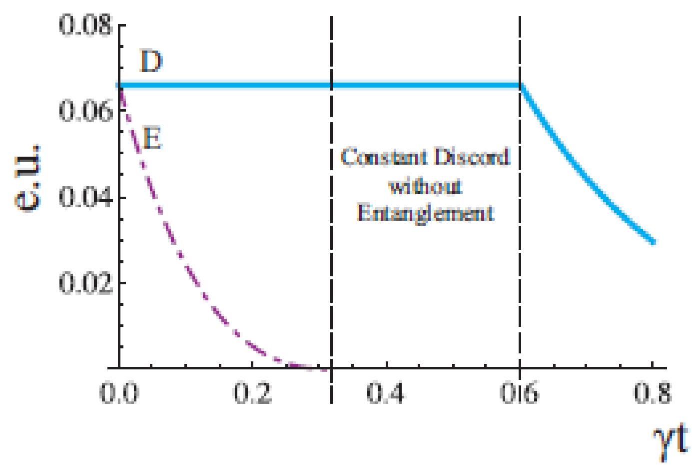
Quantum correlations & entanglement

- Certain mixed separable states are neither A- nor B-classical, and thus have QCs different from entanglement.

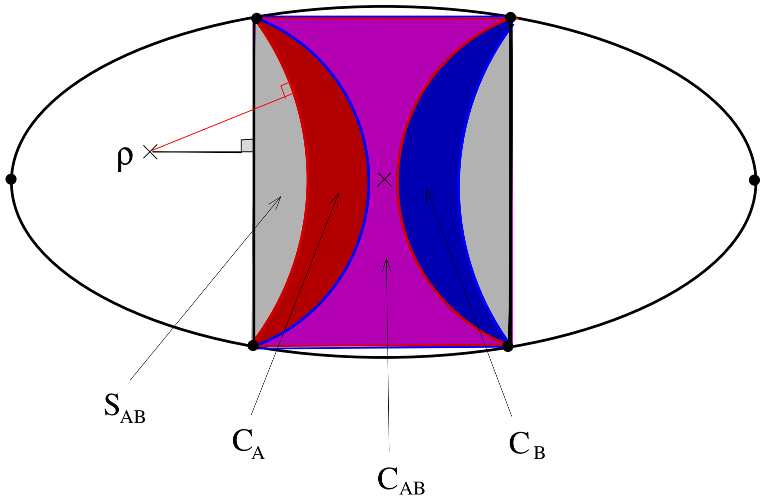
  $\hookrightarrow$ *such states may be useful as resources for quantum computation or quantum communication*

  $\hookrightarrow$ *they are presumably less fragile than entangled states.*

Time evolution of the entanglement and discord for 2 qubits subject to pure dephasing
[Mazzola, Piilo & Maniscalco PRL ('10)]



Geometrical view of QCs



$$C_A = \{A\text{-classical states}\},$$
$$C_B = \{B\text{-classical states}\}$$
$$\subset S_{AB} = \{\text{separable states}\}$$

Geometric discord: [Dakic, Vedral & Bruckner PRL **105** ('01)]

$$\boxed{D_A(\rho) = \min_{\sigma_{A\text{-cl}} \in C_A} d(\rho, \sigma_{A\text{-cl}})^2}$$

Similarly, **geometric measure of entanglement**

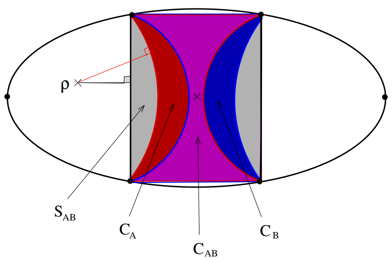
[Vedral *et al* PRL **78** ('97)] [Vedral & Plenio PRA **57** ('98)]

$$\boxed{E_{AB}(\rho) = \min_{\sigma_{\text{sep}} \in S_{AB}} d(\rho, \sigma_{\text{sep}})^2}$$

Advantages of the geometric approach

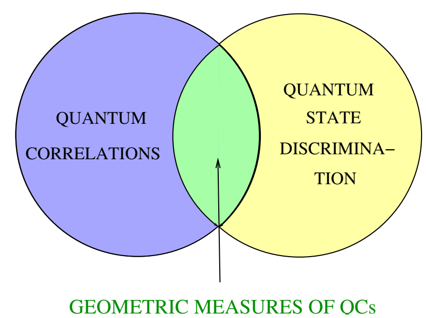
$\hookrightarrow$ 

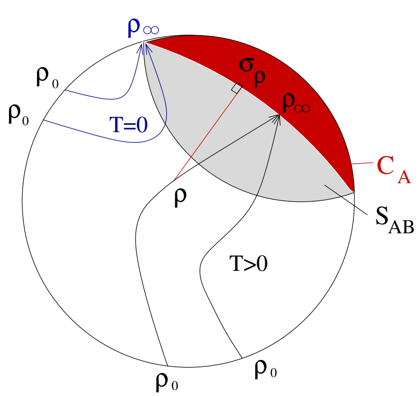
D_A is typically easier to compute than the usual discord δ_A.

$\hookrightarrow$ 

Geometric entanglement is always smaller or equal to the geometric discord, $E_{AB}(\rho) \leqslant D_A(\rho)$ (not true for δ_A and E_{EoF}).

$\hookrightarrow$ Operational interpretation of D_A related to the distinguishability of quantum states



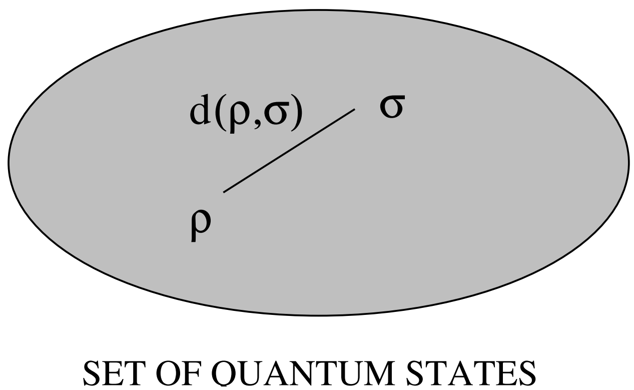
$\hookrightarrow$ 

Useful geometrical information on ρ given by the closest A-classical state(s) σ_ρ to ρ.

Outline

- Which distance on the set of quantum states ?
- Geometric discord with Bures distance
- Geometric distance with quantum Hellinger distance
- Conclusions and perspectives

Distances on the set of quantum states



⋆ **Quantum Hellinger distance**

$$d_{\mathrm{He}}(\rho,\sigma) = \left(2 - 2\,\mathrm{tr}\,\sqrt{\rho}\sqrt{\sigma}\right)^{1/2}$$

⋆ **Bures distance and Uhlmann fidelity** [Bures ('69), Ulhmann ('76)]

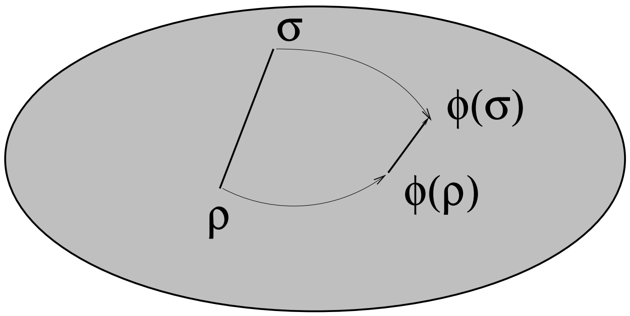
$$d_{\mathrm{Bu}}(\rho,\sigma) = \left(2 - 2\sqrt{F(\rho,\sigma)}\right)^{1/2}, \; F(\rho,\sigma) = \left(\mathrm{tr}\,\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^{2}$$

Physically, the Bures metric (=quantum Fisher information) allows to estimate the best phase precision in quantum interferometry. [Braunstein & Caves PRL ('94)]

Contractive distances

PROPERTIES OF THE BURES & HELLINGER DISTANCES

- *contractive under Completely Positive trace-preserving maps* Φ
- *reduce to the Fisher statistical distance for commuting matrices*



CONTRACTIVE DISTANCE

- *for pure states,* d_{Bu} = *Fubiny-Study distance* $\neq d_{\mathrm{He}}$
- *possess Riemannian metrics* (i.e. $ds^2 = g_\rho(\dot{\rho}, \dot{\rho})dt^2$)
- $d_{\mathrm{Bu}}(\rho, \sigma) \leqslant d_{\mathrm{He}}(\rho, \sigma)$.

Other possible distances: trace distance (contractive but not Riemannian), Hilbert-Schmidt distance (not contractive), Kubo-Mori distance (contractive and Riemannian),...

Outline

✓ Which distance on the set of quantum states ?

- Geometric discord with Bures distance

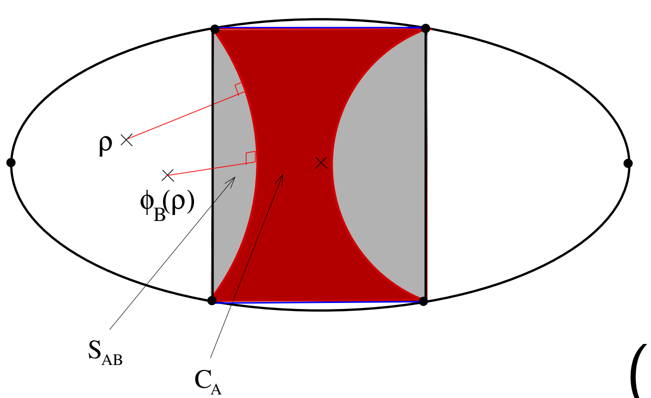
Bures geometric discord

As the quantum discord, the Bures geometric discord is a *bona fide* measure of QCs, namely [Spehner, J. Math. Phys. **55** ('14)]

(i) $D_A(\rho) = 0 \;\Leftrightarrow\; \rho$ is A-classical *(true for any distance)*

(ii) D_A is invariant under local unitary transformations

(iii) D_A is monotonous under CP trace-preserving local maps acting on subsystem B *(true for contractive distances)*



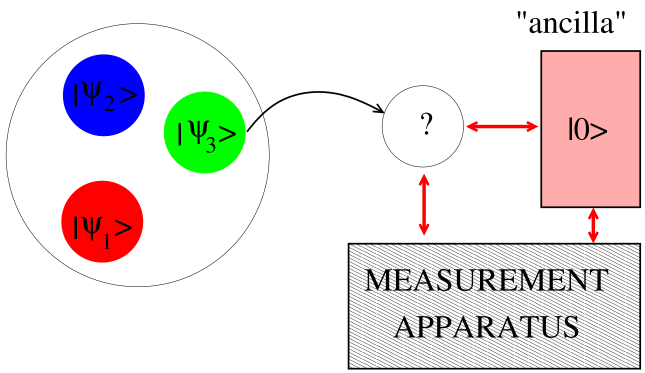
(iv) the restriction of D_A to pure states is entanglement monotone,

$$\boxed{D_A(|\Psi\rangle) = E_{AB}(|\Psi\rangle) = 2 - 2\sqrt{\mu_{\max}}}$$

$\mu_{\max} =$ max. Schmidt coefficient of $|\Psi\rangle$

(v) if $\dim \mathcal{H}_A \leqslant \dim \mathcal{H}_B$, then $D_A(\rho)$ is maximum $\Leftrightarrow \rho$ is maximally entangled.

Quantum State Discrimination (QSD)



- A receiver gets a state ρ_i randomly chosen with probability η_i among a known set $\{\rho_1, \cdots, \rho_n\}$.
- To determine the state he has in hands, he performs a measurement on it.

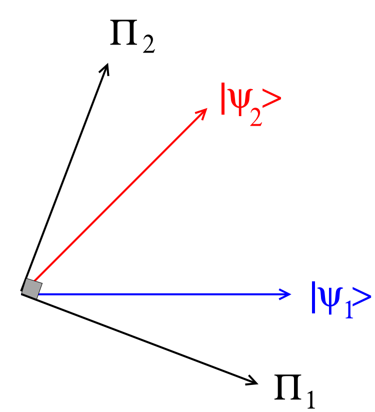
$\hookrightarrow$ **Applications in** quantum communication, cryptography,...

◇ If the ρ_i are $\perp$, one can discriminate them unambiguously

◇ Otherwise one succeeds with probability

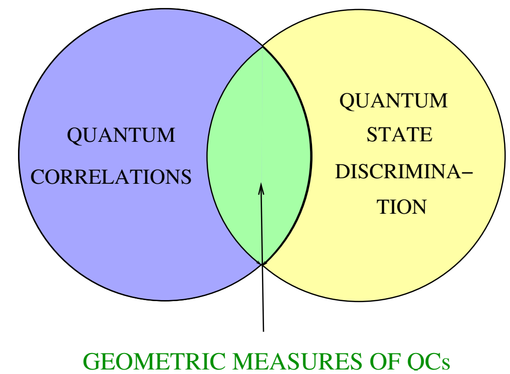
$$\boxed{P_S = \sum_i \eta_i \, \mathrm{tr}\, \rho_i \Pi_i}$$

Π_i = projectors (or POVM) of the von Neumann (or generalized) measurement.



Geometric discord & QSD

- The Bures geometric discord is linked with a quantum state discrimination task: [Spehner & Orszag NJP **15** ('13)]



$$\boxed{D_A(\rho) = 2 - 2 \max_{\{|\alpha_i\rangle\}} \sqrt{P_S^{\rm opt}}}$$

$P_S^{\rm opt}$ = optimal success probability in discriminating the states

$$\boxed{\rho_i = \eta_i^{-1} \sqrt{\rho}\, |\alpha_i\rangle\langle\alpha_i| \otimes 1 \sqrt{\rho} \quad \text{with proba} \quad \eta_i = \langle\alpha_i|\rho_A|\alpha_i\rangle}$$

$\{|\alpha_i\rangle\}$ = orthonormal basis for A.

- The closest A-classical state(s) σ_ρ to ρ is given in terms of the optimal von Neumann measurement for discriminating the ρ_i.
- Closed formula for $D_A(\rho)$ and σ_ρ for Bell-diagonal 2-qubit states [Spehner & Orszag JPA **47** ('14)], [Aaronson, Franco & Adesso PRA **47** ('13)]

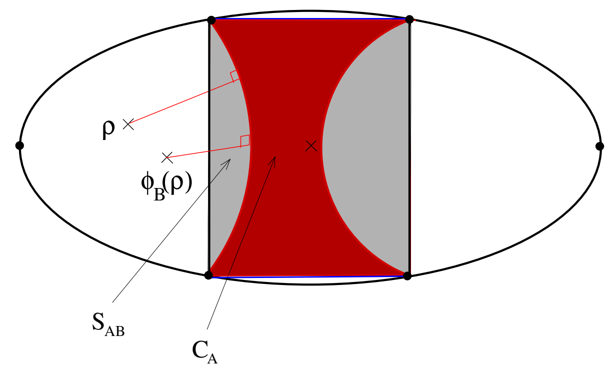
Outline

- ✓ Which distance on the set of quantum states ?
- ✓ Geometric discord with Bures distance
- Geometric distance with quantum Hellinger distance

Quantum Hellinger geometric discord

[Roga, Spehner & Illuminati ('14)]

◇ The geometric discord for the Hellinger distance is a *bona fide* measure of QCs like in the Bures case. For pure states,



$$\boxed{D_A^{\rm He}(|\Psi\rangle) = 2 - 2K^{-1/2}}$$

$K = (\sum \mu_i^2)^{-1}$ Schmidt number.

◇ $D_A^{\rm He}(\rho)$ is simply related to the Hilbert-Schmidt geometric discord of the square root of ρ,

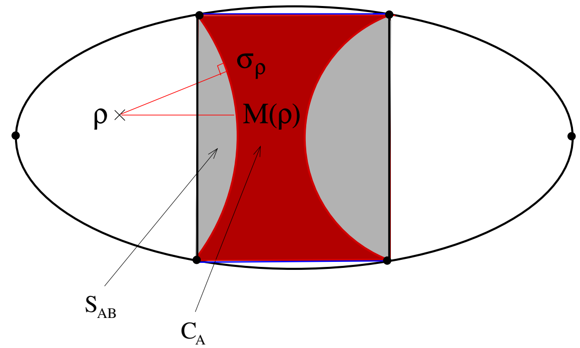
$$\boxed{D_A^{\rm He}(\rho) = 2 - 2\Big(1 - D_A^{\rm HS}(\sqrt{\rho})\Big)^{1/2}}$$

$\hookrightarrow D_A^{\rm He}$ is (almost) as easy to compute as the Hilbert-Schmidt discord $D_A^{\rm HS}$, but unlike $D_A^{\rm HS}$ it is a proper measure of QCs!

Other kinds of geometric discords

★ **Measurement-induced discord** [Luo & Fu PRA **82** ('10)]

$$\boxed{\mathcal{D}_{\rm meas}(\rho) = \min_{\{\pi_i^A\}} d\big(\rho, M_A(\rho)\big)^2}$$



$M_A(\rho) = \sum_i \pi_i^A \otimes 1 \rho\, \pi_i^A \otimes 1$

– for the Hilbert-Schmidt distance $\mathcal{D}_{\rm meas}(\rho) = D_A(\rho)$

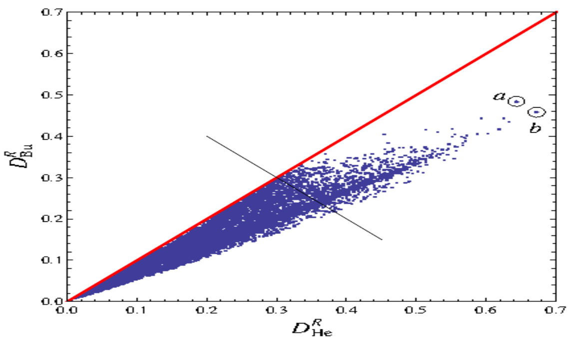
– for the Bures and Hellinger dist. $\mathcal{D}_{\rm meas}(\rho) \neq D_A(\rho)$

★ **Discord of response** [Roga, Giampoalo & Illuminati JPA ('14)]

$$\boxed{\mathcal{D}_R(\rho) = \min_{U_A} d\big(\rho, U_A \otimes 1 \rho\, U_A \otimes 1\big)^2}$$

with minimum over all unitaries U_A on A with non-degenerate spectrum given by the roots of unity.

Comparison of the geometric discords



[Roga, Spehner & Illuminati ('14)]

Distribution of discords of response $\mathcal{D}_R^{\mathrm{He}}$, $\mathcal{D}_R^{\mathrm{Bu}}$ with Hellinger and Bures distances for random two-qubit states.

- If subsystem A is a qubit ($n_A = 2$), for both Bures and Hellinger

$$\boxed{\mathcal{D}_R(\rho) = 2\sin^2\left(\frac{\pi}{n_A}\right)\left(D_A(\rho) - \frac{1}{2}D_A(\rho)^2\right)}$$

also true for qutrits ($n_A = 3$) in the Hellinger case.

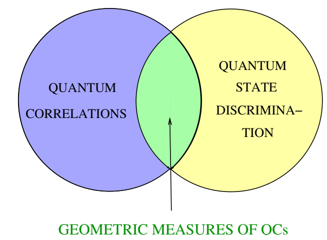
- General bound (valid for any space dimensions of A and B)

$$\boxed{D_A^{\mathrm{Bu}}(\rho) \leqslant \underbrace{D_A^{\mathrm{He}}(\rho), \mathcal{D}_{\mathrm{meas}}^{\mathrm{Bu}}(\rho)}_{= \text{ for pure states}} \leqslant 2D_A^{\mathrm{Bu}}(\rho) - \frac{1}{2}D_A^{\mathrm{Bu}}(\rho)^2}$$

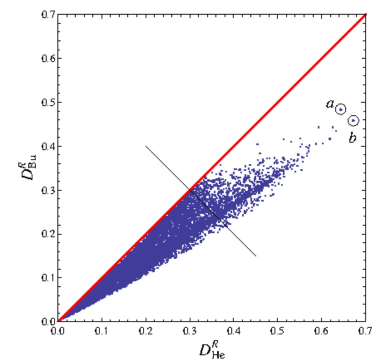
Proof: use known bounds on the success proba P_S^{opt}.

Conclusions and perspectives

- The geometric discord for the Bures distance is related to an ambiguous quantum state discrimination task.



- The geometric discord for the Hellinger distance provides the 1st instance of a *fully computable* and *physically reliable* measure of QCs.
- These geometric discords are not equivalent but useful general bounds between them exist.



PERPECTIVES

★ Evolution of QCs and closest classical states using the geometric discords in concrete dissipative or dephasing channels

★ Closest classical states of the correlated ground states near a quantum phase transition in condensed matter systems.

THAT'S ALL!

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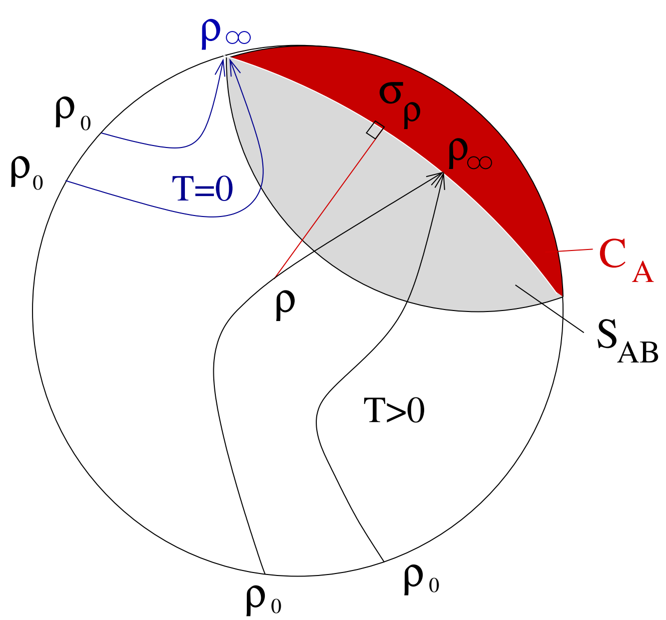

THANK YOU FOR YOU ATTENTION!