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Derivation of Some Translation-Invariant Lindblad Equations for a Quantum Brownian Particle

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Abstract We study the dynamics of a Brownian quantum particle hopping on an infinite lattice with a spin degree of freedom. This particle is coupled to free boson gases via a translation-invariant Hamiltonian which is linear in the creation and annihilation operators of the bosons. We derive the time evolution of the reduced density matrix of the particle in the van Hove limit in which we also rescale the hopping rate. This corresponds to a situation in which both the system-bath interactions and the hopping between neighboring sites are small and they are effective on the same time scale. The reduced evolution is given by a translation-invariant Lindblad master equation which is derived explicitly.

Keywords Out-of-equilibrium quantum statistical physics · Open quantum systems · Weak coupling limit · Singular coupling limit · Quantum Brownian motion

1 Introduction

The irreversible dynamics of a quantum system coupled to infinite baths is often described by determining the time evolution of the reduced density matrix of the system, the latter being obtained by tracing out the bath degrees of freedom in the system + bath state. Under certain approximations (including a Born-Markov approximation), this density matrix is the

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solution of a Lindblad master equation [1, 2]. We are aware of three mathematically well-defined ways to derive such a Lindblad equation starting from the Hamiltonian dynamics of the system and baths [3–5]: the weak coupling limit, the singular coupling limit, and the low density limit. The weak coupling limit goes back to [6] and it was put on a rigorous footing in a series of papers by E.B. Davies [7, 8]. It consists in letting the system-bath coupling constant λ going to zero and rescaling time like $t = \lambda^{-2}\tau$, with $\tau > 0$ fixed. This limit enforces the separation of time scales

$$t_S \ll t_R, \qquad t_B \ll t_R, \tag{1}$$

where t_S (sometimes called the "Heisenberg time") is the time scale on which the system evolves in the absence of coupling with the baths, t_B is the correlation time of the baths, and $t_R \approx t$ is the time scale at which we describe the dynamics, that is, the time scale on which the system evolves under the coupling with the baths (the "relaxation time" for systems converging to stationary states). The first time scale separation in (1) allows to perform the rotating wave approximation. The second time scale separation allows for the Born-Markov approximation. The singular coupling limit is the limit of delta-correlated baths, corresponding to

$$t_B \ll t_S, \qquad t_B \ll t_R. \tag{2}$$

Such a limit, which is a quantum analog of the white noise limit for classical stochastic processes, has been analyzed rigorously in [9–11]. It is physically meaningful in the limit of large bath temperature. Finally, we refer the reader to [12] for a description of the low-density limit.

The weak coupling, singular coupling, and low-density limits have been applied and defined primarily for confined systems (typically atoms) coupled to free fermion or free boson baths. There is a compelling reason for this in the case of the weak coupling limit: the Hamiltonian of a confined system has discrete spectrum, and therefore a well-defined time scale t_S (given by the maximum of the inverse level spacings); in contrast, extended systems may have continuous spectra, corresponding to arbitrarily slow processes in the uncoupled system dynamics, thus invalidating (1) (see, however, [13] for a different approach). A physical example of this is diffusion, where the relevant time scale is set by a spatial scale. In contrast, the singular coupling limit remains well-defined for extended systems, as one can guess from inspection of (2) and as we will illustrate in this article. Note that in the physics literature the dynamics of systems with arbitrarily large t_S are often described by a Bloch-Redfield master equation with a time-dependent generator which is not of the Lindblad form (see [14] and references therein). This equation is perturbative in the system-bath coupling but does not include a rotating wave approximation.

The derivation of the reduced dynamics of extended quantum systems is considerably more involved. In [15], an extended system is studied in the scaling limit $t = \lambda^{-2}\tau$, $x = \lambda^{-2}\xi$, $\lambda \to 0$ (with $\tau > 0$ and $\xi \in \mathbb{R}^d$ fixed) in which both the time t and the position x are rescaled. The scaling of space is dictated by the scaling of time, since on the microscopic scale the particle moves a distance of order λ^{-2} in a time of order λ^{-2}. In this limit, the resulting equation is a linear Boltzmann equation for the Wigner distribution of the particle. This framework has been extended to describe decoherence in position space in [16] and an essentially analogous result, with the weak coupling limit replaced by the low-density limit, was obtained in [17].

Let us also note that quantum systems coupled to infinite baths have been studied in the past fifteen year from another perspective. This approach, due to Jakšić and Pillet [18], uses operator algebras and spectral analysis to describe the dynamics of the system *and* bath at large time and small but finite coupling constant λ. For confined systems, we refer the reader to the lecture notes collected in [19] and the references therein. Extended systems have been analyzed recently from a similar perspective in [20, 21]. Another branch of activity on open quantum systems is the derivation of quantum stochastic equations, see for example [35, 36].

In this work, we consider an extended system coupled to bosonic baths. We are interested in the dynamics of the reduced density matrix of this system at long times and for weak couplings. The system is a quantum particle moving on an infinite lattice $\mathbb{Z}^d$, which has some internal degrees of freedom acting on a finite-dimensional Hilbert space. In the simplest case, the Hamiltonian of the particle is the sum of the discrete Laplacian $-\Delta$ on $\ell^2(\mathbb{Z}^d)$ and of a self-adjoint Hamiltonian S describing the internal degrees of freedom. One may also think of more general Hamiltonians coupling the position and internal degrees of freedom. The particle interacts with free boson gases via a translation-invariant Hamiltonian, assumed to be linear in the creation and annihilation operators of the bosons. We consider the following scaling limit: (i) the time is rescaled as $t = \lambda^{-2}\tau$, where λ is the particle-boson coupling constant and $\tau > 0$ is fixed, (ii) the particle Hamiltonian is rescaled as $H_P = -\lambda^2\Delta + S$, (iii) one takes the limit $\lambda \to 0$. This scaling combines the weak coupling and the singular coupling limits: if the translational degrees of freedom are frozen, it reduces to the weak coupling limit for the internal state, whereas if one ignores the internal degrees of freedom it amounts to a singular coupling limit for the motional state, as will be explained below (see [3, 11]). In the latter case, however, the master equation is trivial. Indeed, by energy conservation the particle can only absorb or emit bosons with a vanishing frequency in the limit $\lambda \to 0$; since such bosons have also a vanishing momentum and the total momentum is conserved, one has no momentum transfers between the particle and the baths; thus the coupling to the baths has no effect on the particle, except possibly for decoherence in the momentum basis. This is in fact the main reason why we consider a particle with internal degrees of freedom even though we are primarily concerned with its motion on the lattice. Note that since the hopping strength is of order λ^2, we do not need a space rescaling as in [15–17].

The above model allows for a tractable rigorous analysis in spite of the fact that we deal with a spatially extended system. Our main result states that, in dimension $d \geq 2$, the reduced density matrix of the particle converges in the aforementioned scaling limit to the solution of a Lindblad master equation which is determined explicitly. This equation contains the physics of dissipative extended systems, in particular diffusion (whose analysis is, however, not treated in this work, we refer the reader to [20, 21] for results in this direction). Its derivation requires much less mathematical complications than in the works [15–17].

Some related models have been studied in [22–25]. In particular, Vacchini and coworkers considered in a series of non rigorous works [23, 24] similar models but in a low density limit; they argue that the evolution of the particle density matrix (that is, not only of the associated Wigner transform) is governed in this limit by a Lindblad master equation. In [25], drift and diffusion of an electron moving on a one-dimensional lattice and submitted to a static electric field have been studied in a model in which the coupling to the bath is simulated by repeated interactions with two level systems. Finally, we point out that our model can be viewed as a continuous version of a dissipative quantum walk [26].

The paper is organized as follows. We introduce the model in Sect. 2, first at finite volume for the particle and baths and then by considering the infinite volume limit. Our results are presented and discussed in Sect. 3, together with two important examples. The last Sect. 4 contains the proofs and some technical results are proven in the Appendix.

2 The Model

2.1 The Quantum Particle

Our model consists of a quantum particle on the lattice $\mathbb{Z}^d$ coupled to free boson fields. In this subsection and the three following ones, we describe the model at finite volume.[1] We thus restrict the lattice to a finite hypercube with periodic boundary conditions and consider $\Lambda = \mathbb{Z}^d/(2L\mathbb{Z})^d$ with $1 \le L < \infty$, d being the space dimension. We will often identify Λ with $]-L, L]^d \cap \mathbb{Z}^d \subset \mathbb{Z}^d$. The infinite volume limit $L \to \infty$ will be taken in Sect. 2.5. The particle has translational degrees of freedom $x \in \Lambda$ and an internal degree of freedom $s = 1, \dots, N < \infty$, which may correspond to a spin or to an internal state of an atom or a molecule. The Hilbert space of the particle, $\mathcal{H}_P^\Lambda = \ell^2(\Lambda) \otimes \mathbb{C}^N$, has finite dimension $|\Lambda| \times N$, where $|\Lambda| = (2L)^d$ is the cardinality of Λ. The particle Hamiltonian H_P^Λ consists of a hopping term acting on $\ell^2(\Lambda)$ plus a term governing the internal dynamics given by a self-adjoint operator S acting on $\mathbb{C}^N$,

$$H_P^\Lambda = \lambda^\alpha H_{\text{hop}}^\Lambda + S \tag{3}$$

where we identified S with $1_{\ell^2(\Lambda)} \otimes S$. We have introduced in front of the hopping term a small parameter λ^α playing the role of a hopping strength or of an inverse mass; $\lambda > 0$ will be chosen below to be the particle-boson coupling constant and α is a positive scaling exponent. Our most interesting result will correspond to $\alpha = 2$ and $d \ge 2$. Note that for $\alpha = \infty$ and $\lambda < 1$, i.e., $\lambda^\alpha = 0$, the translation degrees of freedom can be dropped altogether and the dynamics takes place on $\mathbb{C}^N$. In the simplest setup, H_{hop}^Λ is (up to a minus sign) the discrete Laplacian on $\ell^2(\Lambda)$,

$$H_{\text{hop}}^\Lambda = -\Delta = -\sum_{x,y\in\Lambda,\, |x-y|_\Lambda = 1} \big(|x\rangle\langle y| - |x\rangle\langle x|\big) \otimes 1_{\mathbb{C}^N} \tag{4}$$

where $\{|x\rangle;\ x \in \Lambda\}$ is the canonical basis of $\ell^2(\Lambda)$, the Dirac notation $|x\rangle\langle y|$ refers to the operator $\psi \mapsto \psi(y)|x\rangle$ from $\ell^2(\Lambda)$ to $\ell^2(\Lambda)$, and $|x-y|_\Lambda = \sum_{i=1}^d \min_{k\in\mathbb{Z}} |x_i - y_i + 2kL|$, i.e., we use periodic boundary conditions. In a more elaborate setup, H_{hop}^Λ will be modified such that the propagation of the particle may couple to the internal state. An Hamiltonian that accommodates this idea is presented in Sect. 3.3.

An important property of our model is invariance under space translations. These translations are represented on $\mathcal{H}_P^\Lambda$ by unitary operators $U_P^\Lambda(x)$ defined by $U_P^\Lambda(x)|y\rangle \otimes |\phi\rangle = |x+y\rangle \otimes |\phi\rangle$ for any $x, y \in \Lambda$ and $|\phi\rangle \in \mathbb{C}^N$. We state some conditions on H_{hop}^Λ, which are in particular satisfied by the Hamiltonian (4).

(A1) *The hopping Hamiltonian H_{hop}^Λ has the form*

$$H_{\text{hop}}^\Lambda = \sum_{x\in\Lambda} U_P^\Lambda(-x) h_{\text{hop}} U_P^\Lambda(x) = \sum_{x,y,z\in\Lambda} |y-x\rangle\langle z-x| \langle y|h_{\text{hop}}|z\rangle \tag{5}$$

[1] In the mathematical literature on open quantum systems, it is common to work from the beginning with baths in the thermodynamical limit; the bath state and dynamics are defined with the help of (a representation of) an abstract CCR algebra (see e.g. [19]). We shall not use such an approach here and treat together the infinite lattice limit for the particle and the thermodynamical limit for the bath.

where h_{hop} is a Λ-independent self-adjoint operator on $\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^N$ satisfying

$$\langle y|h_{\mathrm{hop}}|z\rangle = 0 \quad \textit{whenever } |y|>R \textit{ or } |z|>R \tag{6}$$

for some $R<\infty$. In particular, this implies that H^Λ_{hop} is translation-invariant, $U^\Lambda_P(-x)H^\Lambda_{\mathrm{hop}}U^\Lambda_P(x)=H^\Lambda_{\mathrm{hop}}$ for any $x\in\Lambda$, and has a finite range independent of Λ.

One has $h_{\mathrm{hop}}=(1/2)\sum_{|z|=1}(2|0\rangle\langle 0|-|0\rangle\langle z|-|z\rangle\langle 0|)\otimes 1_{\mathbb{C}^N}$ in the case of the discrete Laplacian (4).

By using the Combes-Thomas estimate, which can be applied independently of S since the latter operator does not act on the translation degrees of freedom, one can show that the finite range condition (6) implies the propagation bound

$$\left\|\langle x|\mathrm{e}^{-\mathrm{i}tH^\Lambda_P}|y\rangle\right\|\le \mathrm{e}^{\kappa\lambda^\alpha|t|}\mathrm{e}^{-|x-y|_\Lambda} \tag{7}$$

for some positive and Λ-independent constant κ. Here and in what follows, $\|A\|$ denotes the operator norm of the operator A (acting either on $\mathbb{C}^N$, $\mathcal{H}^\Lambda_P$, or another space); in (7), the quantity inside the norm is a $N\times N$ matrix acting on the internal degrees of freedom of the particle.

2.2 The Bosonic Baths

The particle is coupled to one or several bosonic baths labelled by $i\in I$ (I is a finite set). Let $\mathbb{T}^d$ be the d-dimensional torus, identified with the hypercube $]-\pi,\pi]^d$. Let $\Lambda^*=(\pi/L)\mathbb{Z}^d\cap\mathbb{T}^d\setminus\{0\}$ be the dual of the lattice Λ after having removed the origin.[2] The frequency $\nu_i(q)$ of a boson with a (nonzero) quantized momentum $q\in\Lambda^*$ is the value at q of a function $\nu_i: q\in\mathbb{T}^d\mapsto\nu_i(q)\in\mathbb{R}_+$ (dispersion relation of the bath i). We assume that ν_i is continuous on $\mathbb{T}^d$, C^∞ on $\mathbb{T}^d\setminus\{0\}$, and it satisfies

$$\nu_i(q)>0 \quad \text{for } q\neq 0. \tag{8}$$

The Hilbert space of bath i is the symmetric bosonic Fock space built on $\mathfrak{h}_i=\ell^2(\Lambda^*)$,

$$\mathcal{H}^\Lambda_{B,i}=\Gamma_s(\mathfrak{h}_i)=\mathbb{C}\oplus\mathfrak{h}_i\oplus(\mathfrak{h}_i\otimes_{\mathrm{s}}\mathfrak{h}_i)\oplus\cdots \tag{9}$$

where $\otimes_{\mathrm{s}}$ stands for the symmetrized tensor product (see [27] for details and background). The full bath space is then given by $\mathcal{H}^\Lambda_B=\bigotimes_{i\in I}\mathcal{H}^\Lambda_{B,i}$. The boson Hamiltonian H^Λ_B, acting on $\mathcal{H}^\Lambda_B$, is

$$H^\Lambda_B=\sum_{i\in I}\sum_{q\in\Lambda^*}\nu_i(q)a^*_{i,q}a_{i,q} \tag{10}$$

where $a^*_{i,q}$ and $a_{i,q}$ are the creation and annihilation operators for bosons with momentum q in the bath i. We recall that $a^*_{i,q}$ and $a_{i,q}$ are unbounded operators on $\mathcal{H}^\Lambda_B$, acting trivially on $\mathcal{H}^\Lambda_{B,j}$ with $j\neq i$, which satisfy the canonical commutation relations $[a_{i,q},a^*_{j,q'}]=\delta_{i,j}\delta_{q,q'}$ and $[a_{i,q},a_{j,q'}]=0$.

[2] In our finite-volume setup, bosons with zero quasi-momentum $q=0$ do not play any role and should be removed in order that all expressions in this section be well-defined.

2.3 Coupling Between the Particle and Baths

The particle and baths are coupled via a translation-invariant interaction Hamiltonian acting on $\mathcal{H}^\Lambda = \mathcal{H}_P^\Lambda \otimes \mathcal{H}_B^\Lambda$,

$$H_{\text{int}}^\Lambda = \frac{1}{\sqrt{|\Lambda|}} \sum_{i\in I} \sum_{q\in\Lambda^*} g_{0,i}(q) W_i \otimes \mathrm{e}^{\mathrm{i}q\cdot X} \otimes \left(a_{i,q}^* + a_{i,-q}\right) \tag{11}$$

where X is the position operator (acting on $\ell^2(\Lambda)$ as a multiplication by x and acting trivially on $\mathbb{C}^N$), W_i is an Hermitian $N \times N$-matrix that models the interaction with the internal degree of freedom, and $g_{0,i}(q)$ are momentum-dependent coupling constants. Having in mind the thermodynamical limit which will be considered below, we assume that $g_{0,i}(q)$ are the values at the quantized momenta $q \in \Lambda^*$ of some continuous functions $g_{0,i} : \mathbb{T}^d \mapsto \mathbb{C}$. These functions are called the form factors in the sequel. They must satisfy $g_{0,i}(q) = \overline{g_{0,i}(-q)}$, $q \in \mathbb{T}^d$, in order that H_{int}^Λ be self-adjoint. Introducing the field operators

$$\Phi_i^\Lambda(\varphi) = |\Lambda|^{-1/2} \sum_{q\in\Lambda^*} \left(\varphi(q) a_{i,q}^* + \overline{\varphi(q)} a_{i,q}\right) \tag{12}$$

for $\varphi \in \ell^2(\Lambda^*)$, one may rewrite H_{int}^Λ as

$$H_{\text{int}}^\Lambda = \sum_{i\in I} \sum_{x\in\Lambda} W_i \otimes |x\rangle\langle x| \otimes \Phi_i^\Lambda(g_{x,i}) \tag{13}$$

where we have set

$$g_{x,i}(q) = \mathrm{e}^{\mathrm{i}q\cdot x} g_{0,i}(q). \tag{14}$$

Up to the freedom in the form factors $g_{0,i}(q)$, the choice of the Hamiltonian (11) is dictated by the requirement that it must be invariant under space translations and linear in the creation and annihilation operators of the bosons. Space translations are represented on the bosonic Fock space of the bath i by unitary operators $U_i^\Lambda(x)$ satisfying $U_i^\Lambda(-x) a_{i,q} U_i^\Lambda(x) = \mathrm{e}^{\mathrm{i}q\cdot x} a_{i,q}$. One easily checks that for any $x \in \mathbb{Z}^d$,

$$U^\Lambda(-x) H_{\text{int}}^\Lambda U^\Lambda(x) = H_{\text{int}}^\Lambda \tag{15}$$

with $U^\Lambda(x) = U_P^\Lambda(x) \otimes \bigotimes_{i\in I} U_i^\Lambda(x)$. For instance, electrons in solids are coupled to low-energy acoustic phonons via an Hamiltonian of the form (11), see [28, 29].

The total Hamiltonian of the coupled system, acting on $\mathcal{H}^\Lambda$, is

$$H_{\text{tot}}^\Lambda = \left(\lambda^\alpha H_{\text{hop}}^\Lambda + S\right) \otimes 1_{\mathcal{H}_B^\Lambda} + 1_{\mathcal{H}_P^\Lambda} \otimes H_B^\Lambda + \lambda H_{\text{int}}^\Lambda \tag{16}$$

where we have introduced the dimensionless coupling constant λ in front of H_{int}^Λ. Using $\nu_i(q) > 0$ and the finiteness of Λ, one can apply the Kato-Rellich theorem to conclude that H_{tot}^Λ is self-adjoint on the domain of H_B^Λ.

2.4 Initial State

We assume that the particle and bosons are in a product state $\rho_P^\Lambda \otimes \rho_B^\Lambda$ initially, where ρ_P^Λ is the initial density matrix of the particle and ρ_B^Λ the initial density matrix of the bosons (i.e., ρ_P^Λ and ρ_B^Λ are positive operators on $\mathcal{H}_P^\Lambda$ and $\mathcal{H}_B^\Lambda$ with trace one). We now specify our assumptions on the boson initial density matrix ρ_B^Λ. To this end, we consider the n-point correlation functions

$$\operatorname{tr}\bigl(\rho_B^\Lambda a_{i_1,q_1}^{\#_1} a_{i_2,q_2}^{\#_2}\cdots a_{i_n,q_n}^{\#_n}\bigr) \tag{17}$$

where $a_{i,q}^{\#}$ stands for $a_{i,q}$ or $a_{i,q}^*$.

(B1) *The bath density matrix ρ_B^Λ is translation-invariant and stationary with respect to the free dynamics generated by H_B^Λ:*

$$U_i^\Lambda(x)\rho_B^\Lambda U_i^\Lambda(-x) = \mathrm{e}^{-\mathrm{i}tH_B^\Lambda}\rho_B^\Lambda \mathrm{e}^{\mathrm{i}tH_B^\Lambda} = \rho_B^\Lambda \tag{18}$$

for any $x\in\Lambda$, $i\in I$, and $t\in\mathbb{R}$.

(B2) *ρ_B^Λ is quasi-free. That is, the correlation functions (17) exist for any $q_1,\dots,q_n\in\Lambda^*$, they vanish if the number of creators is distinct from the number of annihilators*[3] *(in particular, if n is odd) and they satisfy the following Gaussian property (also known as Wick's identity): for n even,*

$$\operatorname{tr}\bigl(\rho_B^\Lambda a_{i_1,q_1}^{\#_1}\cdots a_{i_n,q_n}^{\#_n}\bigr) = \sum_{\text{parings } \underline{\pi} \text{ of } (1,\dots,n)} \prod_{m=1}^{n/2} \operatorname{tr}\bigl(\rho_B^\Lambda a_{i_{\iota_m},q_{\iota_m}}^{\#_{\iota_m}} a_{i_{\sigma_m},q_{\sigma_m}}^{\#_{\sigma_m}}\bigr). \tag{19}$$

The sum in (19) runs over all pairing of $(1,\dots,n)$, that is, over all sets $\underline{\pi} = \{(\iota_1,\sigma_1),\dots,(\iota_{n/2},\sigma_{n/2})\}$ of *$n/2$ pairs of distinct indices such that $1=\iota_1<\iota_2<\cdots<\iota_{n/2}$, $\iota_m<\sigma_m$* for any *$m=1,\dots,n/2$, and the union $\{\iota_1,\dots,\iota_{n/2}\}\cup\{\sigma_1,\dots,\sigma_{n/2}\}$ is equal to $\{1,\dots,n\}$.*

(B3) *$\rho_B^\Lambda = \bigotimes_{i\in I}\rho_{B,i}^\Lambda$ is a product of quasi-free density matrices $\rho_{B,i}^\Lambda$ on $\mathcal{H}_{B,i}^\Lambda$. In particular, the two-point correlation function* $\operatorname{tr}(\rho_B^\Lambda a_{i,q_1}^{\#_1} a_{j,q_2}^{\#_2})$ *vanishes if $i\neq j$.*

Note that, according to assumptions (B1) and (B2), $\operatorname{tr}(\rho_B^\Lambda a_{i,q_1}^{\#_1} a_{j,q_2}^{\#_2})$ also vanishes if $q_1\neq q_2$. Assumption (B3) means that the baths are not correlated initially. Assumptions (B1)–(B3) imply that ρ_B^Λ is completely determined by the set $\{\zeta_i^\Lambda(q);\ q\in\Lambda^*\}$ of occupation numbers $\zeta_i^\Lambda(q) = \operatorname{tr}(\rho_B^\Lambda a_{i,q}^* a_{i,q})\in\mathbb{R}_+$ of bosons with momentum q in bath i. In particular,

$$\begin{aligned} &\operatorname{tr}\bigl(\rho_B^\Lambda \Phi_i^\Lambda(\varphi_1)\Phi_i^\Lambda(\varphi_2)\bigr) \\ &\quad = \frac{1}{|\Lambda|}\sum_{q\in\Lambda^*}\Bigl(\zeta_i^\Lambda(q)\varphi_1(q)\overline{\varphi_2(q)} + \bigl(1+\zeta_i^\Lambda(q)\bigr)\overline{\varphi_1(q)}\varphi_2(q)\Bigr) \end{aligned} \tag{20}$$

for any $\varphi_1,\varphi_2\in\ell^2(\Lambda^*)$. When taking the thermodynamic limit we will need the additional hypothesis:

(B4) *$\zeta_i^\Lambda(q)$ are the values at the quantized momenta $q\in\Lambda^*$ of some continuous function $\zeta_i:\mathbb{T}^d\setminus\{0\}\to\mathbb{R}_+$ such that $|g_{0,i}|^2\zeta_i\in L^1(\mathbb{T}^d)$.*

The prime example of an initial state satisfying (B1)–(B3) is

$$\rho_B^\Lambda = \bigotimes_{i\in I}\rho_{\beta_i,i}^\Lambda \quad \text{with } \rho_{\beta_i,i}^\Lambda = \frac{\mathrm{e}^{-\beta_i H_{B,i}^\Lambda}}{\operatorname{tr}(\mathrm{e}^{-\beta_i H_{B,i}^\Lambda})} \tag{21}$$

[3]This property is sometimes called gauge invariance since it follows from the invariance of the state under the gauge transformation $a_{i,q}\to \mathrm{e}^{\mathrm{i}\theta}a_{i,q}$.

the Gibbs state at inverse temperature $\beta_i > 0$. This situation corresponds to a particle coupled to thermal baths (which may have different temperatures). Then $\zeta_i(q) = (\mathrm{e}^{\beta_i \nu_i(q)} - 1)^{-1}$ is the Bose-Einstein distribution. If the form factors are such that $|g_{0,i}|^2/\nu_i \in L^1(\mathbb{T}^d)$ then the last assumption (B4) holds true.

2.5 Thermodynamic Limit

To observe irreversible phenomena, we have to consider the baths at the thermodynamic limit, that is, send $\Lambda \nearrow \mathbb{Z}^d$ keeping the boson densities fixed. By $\Lambda \uparrow \mathbb{Z}^d$ we mean the limit $L \to \infty$, L being the size of the hypercube Λ; in this limit the motion of the particle takes place on the infinite lattice $\mathbb{Z}^d$.

Let H_{hop} be the bounded self-adjoint operator on $\mathcal{H}_P = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ defined by $\langle x|H_{\mathrm{hop}}^\Lambda|y\rangle \to \langle x|H_{\mathrm{hop}}|y\rangle$ as $\Lambda \nearrow \mathbb{Z}^d$ for any $x, y \in \mathbb{Z}^d$. This Hamiltonian is given formally by $H_{\mathrm{hop}} = \sum_{x\in\mathbb{Z}^d} U_P(-x) h_{\mathrm{hop}} U_P(x)$. It describes the hopping of the particle between the lattice sites in the infinite volume limit. We identify all operators on $\ell^2(\Lambda) \otimes \mathbb{C}^N$ as finite-rank operators on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$. By the finite range condition (A1), $\langle x|H_{\mathrm{hop}}|y\rangle = 0$ for $|x - y| > 2R$ and $H_{\mathrm{hop}}^\Lambda \to H_{\mathrm{hop}}$ strongly as $\Lambda \nearrow \mathbb{Z}^d$. Since H_{hop}^Λ (and thus H_{hop}) are bounded, it then follows, e.g. by the Duhamel formula, that

$$\mathrm{e}^{-\mathrm{i}tH_P^\Lambda} \to \mathrm{e}^{-\mathrm{i}tH_P} \quad \text{strongly as } \Lambda \nearrow \mathbb{Z}^d \tag{22}$$

with $H_P = \lambda^\alpha H_{\mathrm{hop}} + S$.

In what follows, we will denote by $\mathcal{B}(\mathcal{H}_P)$ (respectively $\mathcal{B}_1(\mathcal{H}_P)$) the Banach space of bounded (trace-class) operators on $\mathcal{H}_P$ (recall that the norm on $\mathcal{B}_1(\mathcal{H}_P)$ is the trace norm $\|A\|_1 = \mathrm{tr}(|A|)$), and by $\mathcal{S}_P$ the convex cone of density matrices on $\mathcal{B}(\mathcal{H}_P)$ (i.e., positive operators in $\mathcal{B}_1(\mathcal{H}_P)$ with trace one). We must clearly assume that the finite volume initial state of the particle converges as $L \to \infty$ in the trace-norm topology.

(A2) $\rho_P^\Lambda \underset{\Lambda\nearrow\mathbb{Z}^d}{\to} \rho_P$ *in* $\mathcal{B}_1(\mathcal{H}_P)$.

It is easy to show from the commutation relations of the $a_{i,q}^\#$'s that the field operator (12) evolves under the Hamiltonian (10) as

$$\mathrm{e}^{\mathrm{i}tH_B^\Lambda} \Phi_i^\Lambda(\varphi) \mathrm{e}^{-\mathrm{i}tH_B^\Lambda} = \Phi_i^\Lambda\big(\mathrm{e}^{\mathrm{i}t\nu_i}\varphi\big). \tag{23}$$

The following space-and-time bath correlation functions

$$f_i^\Lambda(x, y; t, s) = \mathrm{tr}\big(\rho_B^\Lambda \Phi_i^\Lambda\big(\mathrm{e}^{\mathrm{i}t\nu_i} g_{x,i}\big)\Phi_i^\Lambda\big(\mathrm{e}^{\mathrm{i}s\nu_i} g_{y,i}\big)\big) \tag{24}$$

will play an important role in what follows. By translation invariance and stationarity of the bath initial state ρ_B^Λ (assumption (B1)), $f_i^\Lambda(x, y; t, s) = f_i^\Lambda(x - y, t - s)$ only depends on the position difference $x - y$ and time difference $t - s$, where, according to (20),

$$f_i^\Lambda(x, t) = \frac{1}{|\Lambda|} \sum_{q\in\Lambda^*} \big|g_{0,i}(q)\big|^2 \big(\zeta_i^\Lambda(q)\mathrm{e}^{\mathrm{i}q\cdot x}\mathrm{e}^{\mathrm{i}t\nu_i(q)} + \big(1 + \zeta_i^\Lambda(q)\big)\mathrm{e}^{-\mathrm{i}q\cdot x}\mathrm{e}^{-\mathrm{i}t\nu_i(q)}\big). \tag{25}$$

By assumption (B4), $f_i^\Lambda(x, t)$ converges as $\Lambda \nearrow \mathbb{Z}^d$ to

$$f_i(x, t) = \int_{\mathbb{T}^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \big|g_{0,i}(q)\big|^2 \big(\zeta_i(q)\mathrm{e}^{\mathrm{i}q\cdot x}\mathrm{e}^{\mathrm{i}t\nu_i(q)} + \big(1 + \zeta_i(q)\big)\mathrm{e}^{-\mathrm{i}q\cdot x}\mathrm{e}^{-\mathrm{i}t\nu_i(q)}\big) \tag{26}$$

uniformly in t (recall that ν_i is bounded).

2.6 Reduced Density Matrix of the Particle

The reduced density matrix $\rho_P^\Lambda(t)$ of the particle at time $t \geq 0$ is the partial trace over $\mathcal{H}_B^\Lambda$ of the time-evolved density matrix of the total "particle + bosons" system,

$$\rho_P^\Lambda(t) = \mathrm{tr}_B\left(\mathrm{e}^{-\mathrm{i}tH_{\mathrm{tot}}^\Lambda}\rho_P^\Lambda \otimes \rho_B^\Lambda \mathrm{e}^{\mathrm{i}tH_{\mathrm{tot}}^\Lambda}\right). \tag{27}$$

The following proposition states that it is well defined in the thermodynamic limit under the assumptions described in the preceding subsections.

Proposition 1 *Assume that* (A1)–(A2) *and* (B1)–(B4) *are satisfied. Then for each* $t \geq 0$ *and* $\lambda > 0$, *the reduced density matrix* (27) *converges as* $L \to \infty$,

$$\rho_P^\Lambda(t) \underset{\Lambda \nearrow \mathbb{Z}^d}{\to} \mathcal{Z}_{t,\lambda}(\rho_P) \tag{28}$$

in the trace-norm topology, where $\mathcal{Z}_{t,\lambda}$ *is a completely positive and trace-preserving map acting on* $\mathcal{B}_1(\mathcal{H}_P)$.

This proposition will be proven in Sect. 4.6.

3 Results and Discussion

3.1 The Scaling Limit

To obtain rigorously a kinetic equation for $\rho_P(t) = \mathcal{Z}_{t,\lambda}(\rho_P)$, we perform a van Hove limit by setting $t = \lambda^{-2}\tau$ and letting $\lambda \to 0$ while keeping the rescaled time $\tau > 0$ fixed. In the interaction picture with respect to the internal Hamiltonian S, the reduced density matrix of the particle is in the scaling limit

$$\rho_{\mathrm{sl}}(\tau) = \lim_{\lambda\to 0, t=\lambda^{-2}\tau\to\infty} \mathrm{e}^{\mathrm{i}tS}\mathcal{Z}_{t,\lambda}(\rho_P)\mathrm{e}^{-\mathrm{i}tS} \tag{29}$$

(since we never consider objects on the bath space in this limit we write $\rho_{\mathrm{sl}}(\tau)$ instead of $\rho_{P,\mathrm{sl}}(\tau)$). Note that the infinite volume limit (28) has been taken first, before letting $\lambda \to 0$. Our main result is the existence and characterization of the limit (29). Recall that λ appears both in front of the interaction Hamiltonian H_{int} and of the hopping term H_{hop} in the total Hamiltonian (16). Hence hopping between the lattice sites goes to zero as $\lambda \to 0$ and the motion induced by the Hamiltonian H_{hop} becomes effective only at large times $t \approx \lambda^{-\alpha}\tau$. It is then intuitively clear that for $\alpha > 2$ the hopping will be absent in our scaling limit, whereas for $\alpha = 2$ both hopping and dissipative effects due to boson absorptions and emissions should be contained in the kinetic equation for $\rho_{\mathrm{sl}}(\tau)$.

Before stating the result, let us introduce some notation. In the following, $\{A, B\} = AB + BA$ denotes the anticommutator of two operators A and B on $\mathcal{H}_P$, $\{|s\rangle;\ s = 1, \ldots, N\}$ is the orthonormal basis of $\mathbb{C}^N$ diagonalizing the internal Hamiltonian S, and $E_s \in \sigma(S)$ are the eigenvalues of S, that is, $S|s\rangle = E_s|s\rangle$. For any Bohr frequency $\omega \in \sigma([S, \cdot]) = \sigma(S) - \sigma(S)$, we define the $N \times N$ matrix

$$W_{i,\omega} = \sum_{s,s'=1,\ldots,N} \delta_{E_s - E_{s'},\omega}\langle s|W_i|s'\rangle|s\rangle\langle s'| \tag{30}$$

and the *spectrally averaged* hopping Hamiltonian

$$H_{\mathrm{hop}}^{\natural} = \sum_{s,s'=1,\dots,N} \delta_{E_s,E_{s'}} \langle s|H_{\mathrm{hop}}|s'\rangle |s\rangle\langle s'| \tag{31}$$

where $\delta_{a,b}$ is the Kronecker delta symbol (equal to unity if $a=b$ and zero otherwise). Finally, let us recall that a quantum dynamical semigroup (QDS) on $\mathcal{B}_1(\mathcal{H}_P)$ is a semigroup $(\mathcal{T}_\tau)_{\tau\geq 0}$ of maps $\mathcal{T}_\tau : \mathcal{B}_1(\mathcal{H}_P) \to \mathcal{B}_1(\mathcal{H}_P)$ which are completely positive and trace-preserving, such that $\tau \in \mathbb{R}_+ \mapsto \mathcal{T}_\tau$ is $*$-weakly continuous. Lindblad [1] has derived the general form of the generators of norm-continuous QDS (see also [2] and an extension to unbounded generators in [30]).

Theorem 1 *Let assumptions* (A1)–(A2) *and* (B1)–(B4) *be satisfied and let* $\alpha \geq 2$. *Assume moreover that the infinite-volume correlation functions* $f_i(x,t)$ *satisfy*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\infty \mathrm{d}t \sup_{x\in\mathbb{Z}^d, i\in I} \left|f_i(x,t)\right| \mathrm{e}^{-\frac{|x|}{n}} = 0. \tag{32}$$

Then for any $\tau > 0$ *and any* $\rho_P \in \mathcal{S}_P$, *the limit* (29) *exists in the trace-norm topology and is equal to* $\rho_{\mathrm{sl}}(\tau) = \mathrm{e}^{\tau\mathcal{L}^\natural}\rho_P$, *where* $(\mathrm{e}^{\tau\mathcal{L}^\natural})_{\tau\geq 0}$ *is a norm-continuous quantum dynamical semigroup on* $\mathcal{B}_1(\mathcal{H}_P)$. *If* $\alpha = 2$ *the generator* $\mathcal{L}^\natural$ *of this semigroup is given by*

$$\mathcal{L}^\natural(\rho) = -\mathrm{i}\big[H_{\mathrm{hop}}^\natural + \Upsilon, \rho\big] + \sum_{\omega\in\sigma([\cdot,S])} \sum_{i\in I} \Bigg(\sum_{x,y\in\mathbb{Z}^d} c_i(y-x,\omega) W_{i,\omega} \otimes |x\rangle\langle x| \\ \rho W_{i,\omega}^* \otimes |y\rangle\langle y| - \frac{c_i(0,\omega)}{2} \big\{ W_{i,\omega}^* W_{i,\omega} \otimes 1_{\ell^2(\mathbb{Z}^d)}, \rho \big\} \Bigg) \tag{33}$$

where

$$\Upsilon = \sum_{\omega\in\sigma([\cdot,S])} \sum_{i\in I} \Im \left\{ \int_0^\infty \mathrm{d}t\, f_i(0,t) \mathrm{e}^{-\mathrm{i}t\omega} \right\} W_{i,\omega}^* W_{i,\omega} \otimes 1_{\ell^2(\mathbb{Z}^d)} \tag{34}$$

and $c_i(x,\omega)$ *is the time Fourier transform of* $f_i(x,t)$,

$$c_i(x,\omega) = \int_{\mathbb{R}} \mathrm{d}t\, f_i(x,t) \mathrm{e}^{-\mathrm{i}t\omega}. \tag{35}$$

If $\alpha > 2$, $\mathcal{L}^\natural$ *is given by the same expression as in* (33) *but without the term* $-\mathrm{i}[H_{\mathrm{hop}}^\natural, \rho]$.

To see that $\mathcal{L}^\natural$ in (33) has the Lindblad form (and thus that $(\mathrm{e}^{\tau\mathcal{L}^\natural})_{\tau\geq 0}$ is a QDS), we first rewrite

$$\mathcal{L}^\natural(\rho) = -\mathrm{i}\big[H_{\mathrm{hop}}^\natural + \Upsilon, \rho\big] + \mathfrak{A}(\rho) - \frac{1}{2}\big\{\mathfrak{A}^\star(1), \rho\big\} \tag{36}$$

where the map $\mathfrak{A}$ abbreviates the term involving a sum over x and y in the right-hand side of (33) and $\mathfrak{A}^\star$ is its adjoint with respect to the trace, i.e., $\mathrm{tr}(\mathfrak{A}(\rho)A) = \mathrm{tr}(\rho\mathfrak{A}^*(A))$. We note that $c_i(x,\omega)$ is of positive type in the x-variable (this follows from the fact that $f_i(x,t)$ is a correlation function, see (24)) and therefore it is the Fourier transform of a positive measure $\widehat{c_i}(\mathrm{d}q,\omega)$ on $\mathbb{T}^d$:

$$c_i(x,\omega) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \widehat{c_i}(\mathrm{d}q,\omega)\, \mathrm{e}^{-\mathrm{i}q\cdot x}. \tag{37}$$

This shows that $\mathfrak{A}$ has the Kraus form

$$\mathfrak{A}(\rho) = \sum_{\omega\in\sigma([\cdot,S])} \sum_{i\in I} \int \widehat{c}_i(\mathrm{d}q,\omega) V_{i,\omega}(q)\rho V_{i,\omega}(q)^* \tag{38}$$

with $V_{i,\omega} = (2\pi)^{-d/2} W_{i,\omega} \otimes \mathrm{e}^{\mathrm{i}q\cdot X}$. Thus $\mathfrak{A}$ is a completely positive map [31]. Moreover, $\mathfrak{A}$ is bounded on $\mathcal{B}_1(\mathcal{H}_P)$ because for any $\rho \in \mathcal{B}(\mathcal{H}_P)$, $\rho \geq 0$,

$$\big\|\mathfrak{A}(\rho)\big\|_1 = \mathrm{tr}\big(\mathfrak{A}(\rho)\big) \leq \sum_{\omega\in\sigma([\cdot,S])} \sum_{i\in I} c_i(0,\omega) \|W_{i,\omega}\|^2 \, \|\rho\|_1 \tag{39}$$

and $c_i(0,\omega)$ is finite by the integrability of the correlation function $f_i(0,t)$ (assumption (32)). Since also $\mathfrak{A}^*(1)$ and $H_{\mathrm{hop}} + \Upsilon$ are bounded operators on $\mathcal{H}_P$, it follows that $\mathcal{L}^\natural$ is bounded on $\mathcal{B}_1(\mathcal{H}_P)$. This boundedness and the complete positivity of $\mathfrak{A}$ imply that the operator $\mathcal{L}^\natural$ in (36) generates a norm-continuous QDS [1].

Another way of phrasing Theorem 1 is to say that the rescaled density matrix $\rho_{\mathrm{sl}}(\tau)$ satisfies the *Bloch-Boltzmann* master equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_{\mathrm{sl}}(\tau) = \mathcal{L}^\natural\big(\rho_{\mathrm{sl}}(\tau)\big).$$

Note that $\mathcal{L}^\natural$ commutes with $\mathrm{i}[S,\cdot]$, a generic fact for generators obtained via a weak coupling limit [3, 7]. The self-adjoint operator Υ in (34) acts trivially on $\ell^2(\mathbb{Z}^d)$ and commutes with S; it represents the energy shifts of the particle due to its coupling with the bosons (Lamb shifts). In the following Sects. 3.2 and 3.3, we unwrap the form of the generator $\mathcal{L}^\natural$ in two different situations and discuss the physical phenomena described by the corresponding master equation.

The major technical assumption of Theorem 1 is the integrability condition (32) on the boson correlation functions. This assumption should be compared to the analogous condition for confined systems [3, 7, 8], i.e., the integrability of the correlation function (26) for $x = 0$. An explicit computation and the use of a stationary phase argument performed in the Appendix yields:

Proposition 2 *Let us assume that the form factor $g_{0,i}$ has a support contained in the open ball $\{q \in \mathbb{T}^d; |q| < \pi\}$, that $|g_{0,i}|(q)$ and $\zeta_i(q)$ depend only on the modulus $|q|$ of q, and that the bosons of bath i have a linear dispersion relation $\nu_i(q) = |q|$ on the support of $g_{0,i}$. Furthermore, let the functions $\psi_{i,+}(|q|) = |g_{0,i}(q)|^2\zeta_i(q)$ and $\psi_{i,-}(|q|) = |g_{0,i}(q)|^2(1+\zeta_i(q))$ belong to $C^2(]0,\pi])$ and*

$$|q|^{\min\{d-3,\frac{d-1}{2}\}}\psi_{i,\pm}\big(|q|\big), \qquad |q|^{d-2}\psi'_{i,\pm}\big(|q|\big), \qquad |q|^{d-1}\psi''_{i,\pm}\big(|q|\big) \tag{40}$$

be integrable on $[0,\pi]$. Then assumption (32) on the correlation function $f_i(x,t)$ is satisfied in dimension $d \geq 2$.

If the bosons are initially at thermal equilibrium, in such a way that $\zeta_i(q) = (\mathrm{e}^{\beta|q|} - 1)^{-1}$, then the assumptions on $\psi_{i,\pm}$ in Proposition 2 are satisfied if $g_{0,i}(|q|) \in C^2(]0,\pi])$ and

$$|q|^{\min\{d-4,\frac{d-3}{2}\}}|g_{0,i}|^2, \quad |q|^{d-3}\frac{\mathrm{d}}{\mathrm{d}|q|}|g_{0,i}|^2, \quad \text{and} \quad |q|^{d-2}\frac{\mathrm{d}^2}{\mathrm{d}|q|^2}|g_{0,i}|^2 \tag{41}$$

are integrable on $[0,\pi]$.

Remark 1 For most natural models, assumption (32) fails in dimension $d = 1$; see the Appendix for an explicit computation.

Remark 2 For physical applications it would be of interest to estimate the error terms in the convergence to the scaling limit, that is, to have an explicit bound on $\|\rho_P(\lambda^{-2}\tau) - e^{-i\lambda^{-2}\tau[S,\cdot]}\rho_{sl}(\tau)\|_1$. Due to the repeated use of the dominated convergence theorem, our proof of Theorem 1 does not exhibit such a bound. One could in principle obtain the error terms by assuming some explicit decay of the correlation functions $f_i(x,t)$ as in [33] (see Appendix B in this reference).

Remark 3 The choice of the scaling exponent α in the factor λ^α in front of the hopping Hamiltonian H_{hop} in (16) is more dictated by mathematical than by physical motivations. For $\alpha > 2$ the hopping Hamiltonian H_{hop} is absent in the dynamics in the scaling limit and one has a trivial coupling between the hopping and the internal degrees of freedom in the Lindbladian $\mathcal{L}^\natural$. For $\alpha < 2$, as explained in the introduction, the convergence in the weak coupling limit $\lambda \to 0$ is mathematically more involved. It is not clear whether in this case one can still distill a limiting Lindblad operator.

As already indicated in the introduction, in the case $\alpha = 2$ our scaling limit incorporates both features of the singular and weak coupling limits: the translational degrees of freedom are treated within the singular coupling limit and the internal degree of freedom is treated within the weak coupling limit. The fact that the Hamiltonian

$$H_\lambda = \lambda^2 H_{\text{hop}} + H_B + \lambda H_{\text{int}} \tag{42}$$

corresponds in the limit $\lambda \to 0$ to the singular coupling limit was pointed out by Palmer [11]: the dynamics generated by (42) at time t can be mapped into the dynamics generated by the Hamiltonian

$$H'_\lambda = H_{\text{hop}} + H'_B + H_{\text{int}}^{\lambda\,\prime} \tag{43}$$

at the (unrescaled) time τ and with an (unrescaled) hopping term H_{hop}, where H'_B is a free boson Hamiltonian as in (10). The new interaction Hamiltonian $H_{\text{int}}^{\lambda\,\prime}$ is not multiplied by λ and is given by the original interaction Hamiltonian H_{int}, as given by (13), but with a rescaled form factor $g_{\lambda^2 x}(\lambda^2 q)$ instead of $g_x(q)$. The two dynamics generated by (42) and (43) are exactly the same, in the sense that e^{itH_λ} and $e^{i\tau H'_\lambda}$ coincides up to a conjugation by the unitary operator transforming the field operator $\Phi(\varphi)$ of a boson with wavefunction $\varphi \in L^2(\mathbb{T}^d)$ in the initial problem onto the field operator $\Phi(\varphi_\lambda)$ of a boson in the new problem, with $\varphi_\lambda(q) = \lambda\varphi(\lambda^2 q)$. In the limit $\lambda \to 0$, the rescaled form factor becomes singular.

Let us mention that a model of a quantum system coupled to a free fermion bath has been studied in [22] in the singular coupling limit, using the approach of Refs. [7, 8].

3.2 A Jump Process in Momentum Space

Let us choose $H_{\text{hop}} = -\Delta$ according to (4), $\alpha = 2$, and assume that all baths are initially in thermal equilibrium with the same temperature $\beta_i^{-1} = \beta^{-1}$. We may then just as well consider only one bath initially in the Gibbs state $\rho_B^\Lambda = \rho_\beta^\Lambda$.

This setup is interesting if one sets out to study diffusion and decoherence of the Brownian particle, see [20]. The resulting master equation can be written as

$$\frac{\mathrm{d}\rho_{\mathrm{sl}}(\tau)}{\mathrm{d}\tau} = -\mathrm{i}\big[-\Delta + \Upsilon, \rho_{\mathrm{sl}}(\tau)\big] + \sum_{\omega\in\sigma([S,\cdot])} \int_{\mathbb{T}^d} \frac{\mathrm{d}q}{(2\pi)^d} \widehat{c}(q,\omega) \Big(W_\omega \otimes T_q\, \rho_{\mathrm{sl}}(\tau) W_\omega^* \otimes T_q^* - \frac{1}{2}\big\{ W_\omega^* W_\omega \otimes 1_{\ell^2(\mathbb{Z}^d)}\,,\ \rho_{\mathrm{sl}}(\tau)\big\}\Big) \tag{44}$$

where $T_q = \mathrm{e}^{\mathrm{i}q\cdot X}$ is the unitary momentum translation operator on $\ell^2(\mathbb{Z}^d)$ and we have used the notation $\widehat{c}_i(\mathrm{d}q,\omega) = \widehat{c}(q,\omega)\,\mathrm{d}q$ with $\widehat{c}(q,\omega)$ the positive distribution which for $\omega \neq 0$ is given by (see (26), (35), and (37))

$$\widehat{c}(q,\omega) = 2\pi \big|g_0(q)\big|^2 \Big(\zeta(-q)\delta\big(\nu(-q)-\omega\big) + \big(1+\zeta(q)\big)\delta\big(\nu(q)+\omega\big)\Big). \tag{45}$$

Here $\zeta(q) = (\mathrm{e}^{\beta\nu(q)} - 1)^{-1}$ is the Bose-Einstein distribution, and we have assumed that $\widehat{c}_i(\mathrm{d}q,\omega)$ defines a bona fide measure, which requires some mild additional conditions on ν. The first term in the rate (45) corresponds to absorption processes of a boson with momentum $-q$ and frequency $\nu(-q) = \omega$; it is proportional to the mean number $\zeta(-q)$ of bosons with momentum $-q$. The second term corresponds to spontaneous and stimulated emission of a boson with momentum q and frequency $\nu(q) = -\omega$ and is proportional to $1+\zeta(q)$ (the term 1 comes from spontaneous emission). The delta distributions in (45) accounts for energy conservation (see Fig. 1). Note that we have the detailed balance condition $\widehat{c}(q,\omega) = \mathrm{e}^{-\beta\omega}\widehat{c}(-q,-\omega)$.

To appreciate Eq. (44), it is worthwhile to see what it implies for the evolution of diagonal elements of the density matrix in the eigenbasis of S and the momentum basis for the translational degrees of freedom. Let us define the momentum density (assuming that the sum on the right-hand side is absolutely convergent)

$$\rho_{\mathrm{sl}}(\tau;k,s) = \sum_{x,x'\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(x-x')\cdot k} \langle x,s|\rho_{\mathrm{sl}}(\tau)|x',s\rangle \tag{46}$$

with $|x,s\rangle = |x\rangle \otimes |s\rangle$. Note that $\sum_{s=1}^N \int_{\mathbb{T}^d} \mathrm{d}k\, \rho_{\mathrm{sl}}(\tau;k,s)/(2\pi)^d = 1$ for any $\tau \geq 0$. For simplicity, let us assume that the spectrum of S is non-degenerate, i.e., all eigenvalues of S are simple. Then (44) gives

$$\frac{\partial}{\partial\tau}\rho_{\mathrm{sl}}\big(\tau;k',s'\big) = \sum_{s=1}^N \int_{\mathbb{T}^d} \mathrm{d}k \Big(\gamma\big(k',s'\big|k,s\big)\rho_{\mathrm{sl}}(\tau;k,s) - \gamma\big(k,s\big|k',s'\big)\rho_{\mathrm{sl}}\big(\tau;k',s'\big)\Big) \tag{47}$$

where

$$\gamma\big(k',s'\big|k,s\big) = (2\pi)^{-d}\widehat{c}\big(k-k', E_{s'}-E_s\big)\big|\langle s'|W|s\rangle\big|^2. \tag{48}$$

In formula (47) one recognizes the structure of a forward Markov generator with (singular) transition rates $\gamma(k',s'|k,s)$, acting on densities of absolutely continuous probability measures (hence on L^1-functions) on $\mathbb{T}^d \times \{1,\dots,N\}$. Therefore, the master equation (44) describes the stochastic evolution of a particle with momentum k and internal state s, which may jump from the state (k,s) to (k',s') by emitting or absorbing a boson of momentum q and energy $\nu(q)$, as represented in Fig. 1. According to (45) and (48), only jumps satisfying

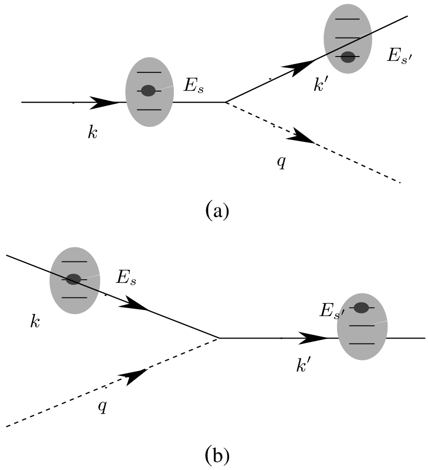

Fig. 1 The processes contributing to the gain term (first term on the right-hand side of (47)). Emission corresponds to $E_s > E_{s'}$ and absorption to $E_s < E_{s'}$. (**a**) The particle makes a transition $(k, s) \to (k', s')$ and emits a boson of momentum q and energy $\nu(q)$ with $k = k' + q$ and $E_s = \nu(q) + E_{s'}$. (**b**) The particle makes a transition $(k, s) \to (k', s')$ and absorbs a boson of momentum q with $k + q = k'$ and $E_s + \nu(q) = E_{s'}$

energy and momentum conservation $\nu(q) = |E_{s'} - E_s|$ and $q = \operatorname{sign}(E_{s'} - E_s)(k' - k)$ are allowed (here sign is the sign function). Note that, in the limit $\lambda \to 0$, the energy of the particle coincides with its internal energy E_s since the hopping energy was assumed to be of the order of λ^α with $\alpha \geq 2$. This explains why the detailed balance condition

$$\gamma\big(k', s'\big|k, s\big) = \mathrm{e}^{\beta(E_s - E_{s'})}\gamma\big(k, s\big|k', s'\big) \tag{49}$$

does not involve the kinetic energy but only the internal energy levels E_s and $E_{s'}$.

Let us recast the master equation (47) in a more explicit form, making some concrete choices. We assume that $N = 2$, label the two spin states as $s \in \{-, +\}$, and choose the two internal energies $E_\pm = \pm\epsilon/2$. Furthermore, we suppose that on $\{|q| \leq \epsilon\}$, the form factor $g_0(q)$ depends only on $|q|$ and one has a linear dispersion relation $\nu(q) = |q|$. Then

$$\frac{\partial}{\partial\tau}\rho_{\mathrm{sl}}\big(\tau; k', \pm\big) = c\int_{S^{d-1}(\epsilon)} \mathrm{d}q\,\big(\mathrm{e}^{\mp\beta\epsilon/2}\rho_{\mathrm{sl}}\big(\tau; k' \mp q, \mp\big) - \mathrm{e}^{\pm\beta\epsilon/2}\rho_{\mathrm{sl}}\big(\tau; k', \pm\big)\big) \tag{50}$$

where $S^{d-1}(\epsilon)$ is the hypersphere with radius ϵ, $\mathrm{d}q$ is its surface measure, and the prefactor c is equal to $(2\pi)^{1-d}|\langle+|W|-\rangle|^2|g_0(|q| = \epsilon)|^2\mathrm{e}^{\beta\epsilon/2}/(\mathrm{e}^{\beta\epsilon} - 1)$.

In the absence of internal Hamiltonian (i.e., for $S = 0$), energy and momentum conservation and the fact that $\nu(q) \neq 0$ for $q \neq 0$ imply that, up to second order in λ, the particle can only emit or absorb zero-momentum bosons and thus cannot change its momentum. Hence the coupling with the bath has no effect on the translational degrees of freedom in the scaling limit (29). This features would not change if we consider baths at different temperatures, or more complicated hopping Hamiltonians.

Given some mild technical conditions, one can show that a particle described by the Lindblad equation (44) diffuses in position space: more precisely, one shows that $\sum_{x\in\mathbb{Z}^d} |x|^2\langle x|\rho_{\mathrm{sl}}(\tau)|x\rangle \propto \tau$ as $\tau \to \infty$, see [20, 21].

3.3 Ratchet

We now apply our model to ratchets. We refer the reader to the review article [32] for more details on this subject. For this application, we choose

$$H_{\mathrm{hop}} = -\sum_{x\in\mathbb{Z}^d} \big(|x\rangle\langle x+e_1| \otimes |s_{\rightarrow}\rangle\langle s_{\leftarrow}| + |x+e_1\rangle\langle x| \otimes |s_{\leftarrow}\rangle\langle s_{\rightarrow}|\big). \tag{51}$$

Here, $|s_{\rightarrow}\rangle$ and $|s_{\leftarrow}\rangle$ are two distinguished eigenstates of S and we singled out a spatial direction by coupling these states to the motion in the direction of the unit vector $e_1 \in \mathbb{Z}^d$. We assume that S has four distinct eigenstates labelled by $s_\uparrow, s_\downarrow, s_\rightarrow$, and $s_\leftarrow$ satisfying

$$S|s_\uparrow\rangle = \epsilon|s_\uparrow\rangle, \quad S|s_\downarrow\rangle = -\epsilon|s_\downarrow\rangle, \quad \text{and} \quad S|s_\rightarrow\rangle = S|s_\leftarrow\rangle = 0. \tag{52}$$

For simplicity, we choose equal eigenvalues of S corresponding to the states $|s_\rightarrow\rangle$ and $|s_\leftarrow\rangle$, so that H_{hop} and S commute. We will exploit the fact that we have two reservoirs. Bosons from the first reservoir couple to the s-variable by

$$W_{i=1} = |s_\uparrow\rangle\langle s_\rightarrow| + |s_\downarrow\rangle\langle s_\leftarrow| + |s_\rightarrow\rangle\langle s_\uparrow| + |s_\leftarrow\rangle\langle s_\downarrow| \tag{53}$$

while bosons of the second reservoir couple as

$$W_{i=2} = |s_\uparrow\rangle\langle s_\leftarrow| + |s_\downarrow\rangle\langle s_\rightarrow| + |s_\leftarrow\rangle\langle s_\uparrow| + |s_\rightarrow\rangle\langle s_\downarrow|. \tag{54}$$

We choose the initial state of the reservoirs to be $\rho_B = \rho^{(1)}_{\beta_1} \otimes \rho^{(2)}_{\beta_2}$, where $\rho^{(i)}_{\beta_i}$ is a Gibbs (thermal) state at temperature $T_i = \beta_i^{-1}$. If the two reservoirs have the same temperature, then the model does not display any current. However, by preparing the reservoirs at different temperatures $T_1 \neq T_2$ one breaks the time-reversal symmetry and a current will in general emerge. To simplify the forthcoming discussion, we choose $g_{0,1} = g_{0,2}$ (the form factors are equal) and

$$T_1 = 0, \qquad T_2 > 0. \tag{55}$$

The boson field induces jumps between the internal states $|s\rangle$ as represented in Fig. 2. By using energy conservation (see (45)), one easily convinces oneself that, with the temperatures chosen as above, the particle can make a transition from $|s_\leftarrow\rangle$ to $|s_\rightarrow\rangle$ by emitting a boson of the first reservoir and absorbing one of the second reservoir, whereas it can not make the reverse transition from $|s_\rightarrow\rangle$ to $|s_\leftarrow\rangle$ (there are no boson with frequency ϵ in the first reservoir at $T_1 = 0$). Since all the jumps between eigenstates of S happen at fixed position, these transitions do not in themselves induce a current. However, the hopping Hamiltonian H_{hop} allows for transitions between the states $|s_\rightarrow\rangle$ and $|s_\leftarrow\rangle$. Hence, a current flows in the e_1-direction. The possibility of extracting work from the system is already visible in Fig. 2, where one sees that the particle can go from $|s_\leftarrow\rangle$ to $|s_\rightarrow\rangle$ via the upper (respectively lower) level only clockwise (anticlockwise). Once one has this property (which is excluded in equilibrium), it is clear that one can devise a scheme to convert this "internal current" into a spatial current.

4 Proofs

4.1 Preliminaries

Our proof of Theorem 1 follows a similar approach as in some previous works of one of the authors, in particular [20, 34]. The starting point is a Dyson expansion of the propagator

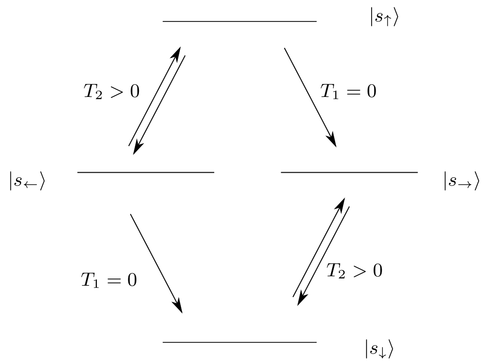

Fig. 2 Possible jumps between internal states of the particle. Note that the jumps due to the coupling with the reservoir at temperature $T_1 = 0$ can take place only from higher to lower energies, whereas the jumps due to coupling with the other reservoir at positive temperature can go either way

$\mathcal{Z}_{t,\lambda}$. We consider the situation in which the particle is coupled to a single bath. Therefore, we do not write the lower index i and the corresponding sums, which do not play any role and obscure the notation. The proof for several baths, i.e., $|I| > 1$, is exactly the same. We also restrict ourselves to the case of a scaling exponent $\alpha = 2$. The case $\alpha > 2$ is simpler and can be treated similarly.

Let us first fix some notation. For products of operators, we use the conventions

$$\overleftarrow{\prod_{j=1,\dots,n}} A_j = A_n \dots A_2 A_1, \qquad \overrightarrow{\prod_{j=1,\dots,n}} A_j = A_1 A_2 \dots A_n. \tag{56}$$

The operators acting on $\mathcal{B}_1(\mathcal{H}_P)$ are denoted by calligraphic fonts, e.g. $\mathcal{T}$ or $\mathcal{L}$, and we use their (operator) norms

$$\|\mathcal{T}\| = \sup_{A \in \mathcal{B}_1(\mathcal{H}_P)} \frac{\|\mathcal{T}(A)\|_1}{\|A\|_1}. \tag{57}$$

We denote by C constants that can depend only on the space dimension d, the internal space dimension N, the parameter κ in the propagation bound (7), and the correlation function $f(x,t)$.

4.2 The Dyson Series

Let $\mathcal{Z}^{\Lambda}_{t,\lambda}$ be the propagator for the reduced dynamics of the particle at finite volume, defined by

$$\mathcal{Z}^{\Lambda}_{t,\lambda}\big(\rho^{\Lambda}_P\big) = \mathrm{tr}_B\big(\mathrm{e}^{-\mathrm{i}tH^{\Lambda}_{\mathrm{tot}}}\big(\rho^{\Lambda}_P \otimes \rho^{\Lambda}_B\big)\mathrm{e}^{\mathrm{i}tH^{\Lambda}_{\mathrm{tot}}}\big), \quad \rho^{\Lambda}_P \in \mathcal{S}\big(\mathcal{H}^{\Lambda}_P\big). \tag{58}$$

We will show below that the Dyson expansion with respect to the interaction Hamiltonian (13) of this propagator converges in norm; this expansion reads

$$\begin{aligned}\mathcal{D}^{\Lambda}_{t,\lambda}\big(\rho^{\Lambda}_P\big) = \rho^{\Lambda}_P + \sum_{n\ge 1}(-\mathrm{i}\lambda)^n \int_{0\le t_1\le\cdots\le t_n\le t} \mathrm{d}t_1\cdots\mathrm{d}t_n \sum_{(x_1,\dots,x_n)\in\Lambda^n} \mathrm{tr}_B\big(\big[V^{\Lambda}_{x_n}(t_n)\otimes\Phi^{\Lambda}_{x_n}(t_n),\dots,\\ \big[V^{\Lambda}_{x_2}(t_2)\otimes\Phi^{\Lambda}_{x_2}(t_2),\big[V^{\Lambda}_{x_1}(t_1)\otimes\Phi^{\Lambda}_{x_1}(t_1),\rho^{\Lambda}_P\otimes\rho^{\Lambda}_B\big]\big]\cdots\big]\big)\end{aligned} \tag{59}$$

where $\Phi^{\Lambda}_x(t) = \Phi^{\Lambda}(\mathrm{e}^{\mathrm{i}t\nu}g_x)$ is the freely-evolved field operator, see (23), and

$$V^{\Lambda}_x(t) = \mathrm{e}^{\mathrm{i}tH^{\Lambda}_P} W \otimes |x\rangle\langle x|\mathrm{e}^{-\mathrm{i}tH^{\Lambda}_P}. \tag{60}$$

For any $x \in \mathbb{Z}^d$, $t \geq 0$, and $T \in \mathcal{B}(\mathcal{H}_P^\Lambda)$, let us set

$$\begin{aligned} \mathcal{I}^\Lambda(x,t,l)(T) &= \begin{cases} V_x^\Lambda(t)T & \text{if } l = L \\ -TV_x^\Lambda(t) & \text{if } l = R. \end{cases} \\ f^\Lambda(x,t,l) &= \begin{cases} f^\Lambda(x,t) & \text{if } l = L \\ \overline{f^\Lambda(x,t)} & \text{if } l = R \end{cases} \end{aligned} \tag{61}$$

where $f^\Lambda(x,t)$ is the correlation function (25), which satisfies $\overline{f^\Lambda(x,t)} = f^\Lambda(-x,-t)$. By using the quasi-freeness assumption (B2), one gets

$$\mathcal{D}_{t,\lambda}^\Lambda = 1 + \sum_{n\geq 1} \sum_{\text{pairings } \underline{\pi}} \sum_{(\underline{x},\underline{l})\in\Lambda^{2n}\times\{L,R\}^{2n}} \int_{0\leq t_1\leq\cdots\leq t_{2n}\leq t} \mathrm{d}\underline{t}\, \mathcal{V}_{\lambda,n}^\Lambda(\underline{\pi},\underline{t},\underline{x},\underline{l}) \tag{62}$$

where $\mathrm{d}\underline{t}$ stands for $\mathrm{d}t_1\cdots\mathrm{d}t_{2n}$, the second sum runs over all pairings $\underline{\pi} = \{(\iota_1,\sigma_1),\ldots,(\iota_n,\sigma_n)\}$ of $(1,\ldots,2n)$, and

$$\mathcal{V}_{\lambda,n}^\Lambda(\underline{\pi},\underline{t},\underline{x},\underline{l}) = \left(-\lambda^2\right)^n \overleftarrow{\prod_{j=1,\ldots,2n}} \mathcal{I}^\Lambda(x_j,t_j,l_j) \prod_{m=1}^n f^\Lambda(x_{\sigma_m}-x_{\iota_m},t_{\sigma_m}-t_{\iota_m},l_{\iota_m}) \tag{63}$$

if $x_1,\ldots,x_{2n} \in \Lambda$, and $\mathcal{V}_{\lambda,n}^\Lambda(\underline{\pi},\underline{t},\underline{x},\underline{l}) = 0$ otherwise. We do not write explicitly the dependence of H_P^Λ, V_x^Λ, and $\mathcal{I}^\Lambda$ on the coupling constant λ to simplify notation, but we keep it in $\mathcal{V}_{\lambda,n}^\Lambda$ and $\mathcal{D}_{t,\lambda}^\Lambda$ because we will later consider the limit $\lambda \to 0$ of these quantities.

Already at this point, we can establish the norm-convergence of the series (62). Indeed, $\|\mathcal{I}^\Lambda(x,t,l)\| \leq \|W\|$ and $|f^\Lambda(x,t,l)| \leq f^\Lambda(0,0)$, thus the nth term in the series (62) has a norm bounded by

$$\left(2|\Lambda|\lambda\|W\|\right)^{2n} \frac{t^{2n}}{(2n)!} \left(f^\Lambda(0,0)\right)^n \left[2^{-n}\binom{2n}{n} n!\right] = \frac{((|\Lambda|\lambda\|W\|t)^2 2 f^\Lambda(0,0))^n}{n!} \tag{64}$$

where the term between the square brackets $[\cdot]$ is the number of pairings $\underline{\pi}$ of $(1,\ldots,2n)$. Hence the Dyson series (62) at finite volume converges in norm. One can prove that its sum is equal to the propagator in the interaction picture,

$$\mathcal{D}_{t,\lambda}^\Lambda = \mathrm{e}^{\mathrm{i}t[H_P^\Lambda,\cdot]} \mathcal{Z}_{t,\lambda}^\Lambda. \tag{65}$$

We will consider the Dyson series at infinite volume and we simply drop the superscript Λ on $H_P, \mathcal{I}, \mathcal{V}$, and f to denote the corresponding objects for $\Lambda = \mathbb{Z}^d$. We argue that in the infinite volume limit $\Lambda \uparrow \mathbb{Z}^d$,

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}t[H_P^\Lambda,\cdot]} \to \mathrm{e}^{-\mathrm{i}t[H_P,\cdot]}, \qquad \mathcal{I}^\Lambda(x,t,l) \to \mathcal{I}(x,t,l), \\ \text{and} \quad \mathcal{V}_{\lambda,n}^\Lambda(\underline{\pi},\underline{t},\underline{x},\underline{l}) \to \mathcal{V}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l}) \end{aligned} \tag{66}$$

strongly on $\mathcal{B}_1(\mathcal{H}_P)$. Indeed, recall that given some bounded operators A^Λ on a Hilbert space such that $A^\Lambda \to 0$ and $(A^\Lambda)^* \to 0$ strongly, then $\|A^\Lambda T\|_1 \to 0$ and $\|TA^\Lambda\|_1 \to 0$ for any $T \in \mathcal{B}_1(\mathcal{H}_P)$. The first limit in (66) follows from this property, the inequality $\|A^\Lambda TU\|_1 \leq \|A^\Lambda T\|_1$ for U unitary, and the strong convergence of $\mathrm{e}^{-\mathrm{i}tH_P^\Lambda}$ on $\mathcal{H}_P$, see (22). As $V_x^\Lambda(t) \to V_x(t)$ strongly on $\mathcal{H}_P$ (again by (22)), the second limit follows from the same property.

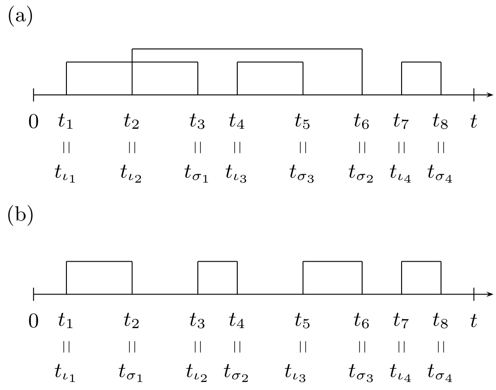

Fig. 3 (**a**) Crossed diagram, (**b**) ladder diagram

Since $\mathcal{V}^{\Lambda}_{\lambda,n}(\underline{\pi}, \underline{t}, \underline{x}, \underline{l})$ is a finite product of the $\mathcal{I}$-operators and correlation functions f^{Λ}, the third limit in (66) then follows from the second one and from the pointwise convergence $f^{\Lambda} \to f$ (see Sect. 2.5). This implies that term-by-term, the series $\mathcal{D}^{\Lambda}_{t,\lambda}$ converges strongly to the infinite-volume Dyson expansion

$$\mathcal{D}_{t,\lambda} = \sum_{n} \sum_{\underline{\pi}} \sum_{\underline{x},\underline{l}} \int_{Z_{2n}(t)} \mathrm{d}\underline{t}\, \mathcal{V}_{\lambda,n}(\underline{\pi}, \underline{t}, \underline{x}, \underline{l}). \tag{67}$$

The convergence of this series has not been addressed yet, so for the moment we consider it as a formal series. To prove Proposition 1, we will show that this convergence is in some way uniform in Λ and apply the dominated convergence theorem. We have made in (67) the following abbreviations, which will also be in place in the remaining of the paper:

(a) we sum over $n = 0, 1, \ldots,$ where it is understood that the term corresponding to $n = 0$ is equal to 1;
(b) the sum over $\underline{\pi}$ ranges over all pairings of $(1, 2, \ldots, 2n)$;
(c) $Z_n(t)$ denotes the simplex $\{\underline{t} = (t_1, \ldots, t_n) \in [0, t]^n;\ t_1 \le t_2 \le \cdots \le t_n\}$;
(d) the sum over $\underline{x}$ and $\underline{l}$ range over $\mathbb{Z}^{2nd}$ and $\{L, R\}^{2n}$, respectively.

4.2.1 Graphical Representation

It is convenient to represent $\mathcal{V}_{\lambda,n}$ or $\mathcal{V}^{\Lambda}_{\lambda,n}$ by a diagram in which all times $t_1 \le t_2 \le \cdots \le t_{2n}$ are ordered on the real line and pairings are represented by bridges linking two distinct times. Two examples of diagrams are represented in Fig. 3. Replacing the times by their indices $1, 2, \ldots, 2n$, the diagrams with $2n$ points are in one-to-one correspondence with the pairings of $(1, 2, \ldots, 2n)$. A diagram containing two pairs $(\iota, \sigma) \in \underline{\pi}$ and $(\iota', \sigma') \in \underline{\pi}$ such that $\iota < \iota'$ and $\iota' < \sigma$ is called a *crossing diagram*.[4] A non-crossing diagram will be called a *ladder diagram*; it corresponds to the pairing $\underline{\pi}_{\text{ladder}} = \{(1, 2), (3, 4), \ldots, (2n - 1, 2n)\}$ respecting the order, see Fig. 3.

The strategy of the proof of Theorem 1 consists in showing that in the scaling limit $\lambda \to 0$, $t = \lambda^{-2}\tau \to \infty$,

[4]Traditionally, some of these diagrams are called "nested", but we call all of them "crossing diagrams" as we represent them on a single time axis, see Fig. 3.

(i) all crossing diagrams in the infinite-volume Dyson series (67) converge to zero;
(ii) the ladder diagram in (67) of order $2n$ converges to the corresponding diagram of the Dyson expansion of $\mathrm{e}^{\mathrm{i}\tau[H^{\natural}_{\mathrm{hop}},\cdot]}\mathrm{e}^{\tau\mathcal{L}^{\natural}}$, where $\mathcal{L}^{\natural}$ is the Lindblad generator given in (33).

4.3 Plan of the Proof

Below, we give the main steps of the proof of Theorem 1.

4.3.1 Topology

We first introduce a notion of convergence that is particularly useful for the problem. Let $(\mathcal{T}_\lambda)_\lambda$ be a family of operators on $\mathcal{B}_1(\mathcal{H}_P)$. We associate to $\mathcal{T}_\lambda$ a kernel with values in operators on the vector space $\mathcal{B}(\mathbb{C}^N)$ of $N\times N$ matrices, defined as follows:

$$(\mathcal{T}_\lambda)_{x_0,y_0;x,y}(M)=\langle x|\mathcal{T}_\lambda\big(M\otimes|x_0\rangle\langle y_0|\big)|y\rangle,\quad M\in\mathcal{B}\big(\mathbb{C}^N\big). \tag{68}$$

We write $\text{pt-lim}_{\lambda\to 0}\mathcal{T}_\lambda=0$ whenever

$$\lim_{\lambda\to 0}\sum_{x,y\in\mathbb{Z}^d}\big\|(\mathcal{T}_\lambda)_{x_0,y_0;x,y}\big\|=0\quad\text{for any } x_0,y_0\in\mathbb{Z}^d,$$

where the norm inside the sum is the matrix norm.

Lemma 1 *Let $\mathcal{T}_\lambda$ be uniformly bounded operators on $\mathcal{B}_1(\mathcal{H}_P)$. If one has* $\text{pt-lim}_{\lambda\to 0}\mathcal{T}_\lambda=0$ *then $\mathcal{T}_\lambda\to 0$ strongly, that is,* $\lim_{\lambda\to 0}\|\mathcal{T}_\lambda(T)\|_1\to 0$ *for any $T\in\mathcal{B}_1(\mathcal{H}_P)$.*

Proof Note first that for any $T\in\mathcal{B}_1(\mathcal{H}_P)$,

$$\begin{aligned}\|T\|_1&=\sup_{A\in\mathcal{B}(\mathcal{H}_P)}\frac{|\operatorname{tr}(TA)|}{\|A\|}\le\sup_{A\in\mathcal{B}(\mathcal{H}_P)}\sum_{x,y\in\mathbb{Z}^d}\operatorname{tr}_{\mathbb{C}^N}\big(\big|\langle x|T|y\rangle\big|\big)\frac{\|\langle y|A|x\rangle\|}{\|A\|}\\&\le N\sum_{x,y\in\mathbb{Z}^d}\big\|\langle x|T|y\rangle\big\|\end{aligned}\tag{69}$$

where the last inequality follows because $\operatorname{tr}(|M|)\le N\|M\|$ for any finite matrix $M\in\mathcal{B}(\mathbb{C}^N)$. Let T have finite support in the sense that $\langle x_0|T|y_0\rangle$ is nonzero only for a finite number of $x_0,y_0\in\mathbb{Z}^d$. Then, by (69), $\text{pt-lim}_{\lambda\to 0}\mathcal{T}_\lambda=0$ implies $\|\mathcal{T}_\lambda(T)\|_1\to 0$. Next, one checks that operators T with finite support are dense in $\mathcal{B}_1(\mathcal{H}_P)$. Actually, let $P_\Lambda=\sum_{x\in\Lambda}|x\rangle\langle x|\otimes 1_{\mathbb{C}^N}$ be the finite-rank projector on $\operatorname{span}\{|x\rangle;\ x\in\Lambda\}\otimes\mathbb{C}^N$, with $\Lambda=\mathbb{Z}^d/(2L\mathbb{Z})^d$ as before. Without loss of generality, we may assume that $T\ge 0$. If $\{|\psi_j\rangle\}$ is an orthonormal basis of $\mathcal{H}_P$ diagonalizing T and $p_j\ge 0$ are the eigenvalues of T, then

$$\begin{aligned}\|P_\Lambda TP_\Lambda-T\|_1&\le\sum_j p_j\big(\big\|P_\Lambda|\psi_j\rangle\langle\psi_j|(P_\Lambda-1)\big\|_1+\big\|(P_\Lambda-1)|\psi_j\rangle\langle\psi_j|\big\|_1\big)\\&\le 2\sum_j p_j\big\|(P_\Lambda-1)|\psi_j\rangle\big\|\to 0\end{aligned}$$

as $L\to\infty$ by dominated convergence. Hence, for any $T\in\mathcal{B}_1(\mathcal{H}_P)$ and $\varepsilon>0$, one can choose a decomposition $T=T_0+T_1$ such that T_0 has finite support and $\|T_1\|_1\le\varepsilon$. Then $\|\mathcal{T}_\lambda(T)\|_1\le\|\mathcal{T}_\lambda(T_0)\|_1+C\varepsilon$ by the uniform boundedness of $(\mathcal{T}_\lambda)_\lambda$. The claim follows. □

A consequence of Lemma 1 is that in order to prove the strong convergence of $\mathcal{D}_{\lambda^{-2}\tau,\lambda}$ to $\mathcal{D}_\tau$ in the scaling limit, it is enough to show that

$$\operatorname*{pt-lim}_{\lambda\to 0}\{\mathcal{D}_{\lambda^{-2}\tau,\lambda} - \mathcal{D}_\tau\} = 0. \tag{70}$$

In fact, one has $\|\mathcal{D}_{t,\lambda}\| = 1$ for any t by the following standard argument. One first notes that the finite volume propagator $\mathcal{Z}^{\Lambda}_{t,\lambda}$ defined in (58) preserves positivity and trace and hence so does the sum $\mathcal{D}^{\Lambda}_{t,\lambda}$ of its Dyson expansion (related to $\mathcal{Z}^{\Lambda}_{t,\lambda}$ by (65)) as well as its strong limit $\mathcal{D}_{t,\lambda}$ as $\Lambda \uparrow \mathbb{Z}^d$ (we assume here that Proposition 1 has been already established). Then the dual of $\mathcal{D}_{t,\lambda}$ under the trace, $\mathcal{D}^*_{t,\lambda}$, which acts on $\mathcal{B}(\mathcal{H}_P)$, preserves positivity and satisfies $\mathcal{D}^*_{t,\lambda}(1) = 1$. It follows that $\|\mathcal{D}^*_{t,\lambda}\| = 1$ (see e.g. [27], Corollary 3.2.6) and thus $\|\mathcal{D}_{t,\lambda}\| = 1$.

4.3.2 Assumptions

In the remainder of this paper, we always assume (A1)–(A2) and (B1)–(B4) to be valid without further mentioning it (note that once the Dyson series for $\mathcal{Z}_{t,\lambda}$ is accepted as the basic object of study, one does no need those assumptions anymore). To prove Theorem 1, we rely on:

(a) The propagation bound (7), but for $\Lambda = \mathbb{Z}^d$: by (22) and as the right-hand side of (7) is independent of Λ, this bound remains valid for $\Lambda = \mathbb{Z}^d$ (alternatively, one can check this directly by the Combes-Thomas estimate, using the fact that H_{hop} has a finite range).
(b) Assumption (32) on the infinite volume correlation function $f(x,t)$ or, in most intermediate steps, the weaker requirement that $f(x,\cdot)$ is integrable for any $x \in \mathbb{Z}^d$.

4.3.3 Step I

We first prove that the Dyson series $\mathcal{D}_{\lambda^{-2}\tau,\lambda}$, considered as a series in n, $\underline{x}$, and $\underline{l}$, converges absolutely and uniformly in λ, in the sense that

$$\sum_n \sum_{\underline{x},\underline{l}} \sum_{\underline{\pi}} \sum_{x,y} \sup_{\lambda>0} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t}\, \left\| \left(\mathcal{V}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l})\right)_{x_0,y_0;x,y} \right\| < \infty \tag{71}$$

for any $x_0, y_0 \in \mathbb{Z}^d$ (see Proposition 3 below). This bound is the crucial point in the proof, since it allows us to estimate the perturbation series term by term. It relies heavily on the assumption (32).

By a similar bound on the finite-volume Dyson series $\mathcal{D}^{\Lambda}_{t,\lambda}$, the fact that this series converges term by term as $\Lambda \uparrow \mathbb{Z}^d$, Lemma 1, and (66), we obtain Proposition 1 with

$$\mathcal{Z}_{t,\lambda} = \mathrm{e}^{-\mathrm{i}t[H_P,\cdot]}\mathcal{D}_{t,\lambda}. \tag{72}$$

Here the integrability of $f^{\Lambda}(x,\cdot)$ is not needed, the only requirement is its pointwise convergence as $\Lambda \uparrow \mathbb{Z}^d$ (which follows from (B4)).

These results are accomplished in Sects. 4.4 and 4.5.

4.3.4 Step II

We show that every single crossing diagram vanishes in the scaling limit, in the sense that, for any $n \geq 1$ and $x_0, y_0, x, y \in \mathbb{Z}^d$,

$$\lim_{\lambda\to 0} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t}\, \left\| \mathcal{V}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l})_{x_0,y_0;x,y} \right\| = 0 \quad \text{whenever } \underline{\pi} \neq \underline{\pi}_{\text{ladder}}. \tag{73}$$

This step is essentially taken over from the original work [7], but we review it in Sect. 4.7. To feel why this holds true, note that when $t = \lambda^{-2}\tau \gg 1$, the time integration domain of a crossing diagram of order $2n$ is much smaller than that of the ladder diagram of the same order. This is due to the restriction $t_\iota \leq t_{\iota'} \leq t_\sigma$ associated to the nested pairs (ι, σ) and (ι', σ'), see Fig. 3.

By dominated convergence and (71), this implies that the contribution of crossing diagrams in the Dyson series $\mathcal{D}_{\lambda^{-2}t,\lambda}$ vanishes in the limit $\lambda \to 0$ in the topology introduced in Sect. 4.3.1, i.e.,

$$\sum_n \sum_{\underline{x},\underline{l}} \sum_{x,y} \sum_{\underline{\pi} \neq \underline{\pi}_{\text{ladder}}} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t} \left\| \left(\mathcal{V}_{\lambda,n}(\underline{\pi}, \underline{t}, \underline{x}, \underline{l}) \right)_{x_0,y_0;x,y} \right\| \underset{\lambda \to 0}{\to} 0 \tag{74}$$

for any $x_0, y_0 \in \mathbb{Z}^d$.

4.3.5 *Step III*

It remains to evaluate the contribution of ladder diagrams $\underline{\pi}_{\text{ladder}}$. Let us denote by $\mathcal{K}(\underline{\tau}, \underline{x}, \underline{l})$ the bounded operators, defined in Proposition 6 below, which yield the limiting QDS $(\mathrm{e}^{\tau \mathcal{L}^{\natural}})_{\tau \geq 0}$ upon summing over n, $\underline{x}$, $\underline{l}$, and $\underline{\tau}$:

$$\sum_n \int_{Z_n(\tau)} \mathrm{d}\underline{\tau} \sum_{\underline{x},\underline{l}} \mathcal{K}_n(\underline{\tau}, \underline{x}, \underline{l}) = \mathrm{e}^{\tau \mathcal{L}^{\natural}}, \tag{75}$$

where the sums and integrals are absolutely convergent in norm. We will show in Sects. 4.8 and 4.9 that

$$\lim_{\lambda \to 0} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t}\, \mathcal{V}_{\lambda,n}(\underline{\pi}_{\text{ladder}}, \underline{t}, \underline{x}, \underline{l}) = \int_{Z_n(\tau)} \mathrm{d}\underline{\tau}\, \mathcal{K}_n(\underline{\tau}, \underline{x}, \underline{l}) \tag{76}$$

in norm on $\mathcal{B}(\mathcal{B}_1(\mathcal{H}_P))$, which of course implies

$$\int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t} \left(\mathcal{V}_{\lambda,n}(\underline{\pi}_{\text{ladder}}, \underline{t}, \underline{x}, \underline{l}) \right)_{x_0,y_0;x,y} \to \int_{Z_{2n}(\tau)} \mathrm{d}\underline{\tau} \left(\mathcal{K}_n(\underline{\tau}, \underline{x}, \underline{l}) \right)_{x_0,y_0;x,y}. \tag{77}$$

Dominated convergence allows us to conclude from (71) and (77) that (70) holds true.

4.4 Estimating Each Term of the Dyson Series

Lemma 2 *Fix $n, \underline{\pi}, x_0, y_0$, and $\underline{t} \in Z_{2n}(t)$. Then*

$$\left\| \left(\mathcal{V}_{\lambda,n}(\underline{\pi}, \underline{t}, \underline{x}, \underline{l}) \right)_{x_0,y_0;x,y} \right\| \leq (C\lambda)^{2n} \mathrm{e}^{4\lambda^2 t} \prod_{m=1}^{n} h_n(t_{\sigma_m} - t_{\iota_m}) R^{(n)}_{x_0,y_0;x,y}(\underline{x}, \underline{l}) \tag{78}$$

where

$$h_n = \sup_{x \in \mathbb{Z}^d} \left| f(x, \cdot) \right| \mathrm{e}^{-\frac{|x|}{2n}} \tag{79}$$

and $R^{(n)}_{x_0,y_0;x,y}(\underline{x}, \underline{l})$ is independent of λ and Λ and such that

$$\sup_{n \geq 0} \left\{ \sum_{\underline{x},\underline{l}} \sum_{x,y} R^{(n)}_{x_0,y_0;x,y}(\underline{x}, \underline{l}) \right\} < \infty. \tag{80}$$

The bound (78) also holds true if one replaces $\mathcal{V}_n$ by $\mathcal{V}_n^{\Lambda}$ and h_n by $\max_{x \in \Lambda} |f^{\Lambda}(x, \cdot)|$.

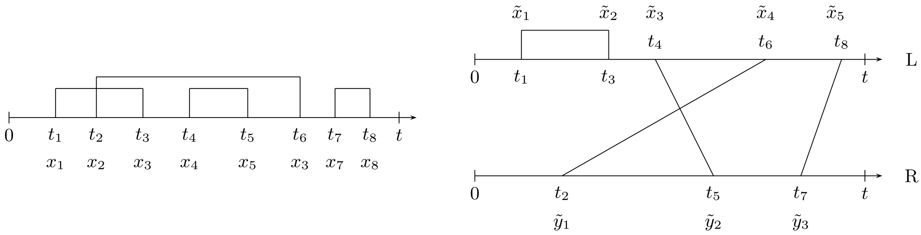

Fig. 4 Equivalent representations of an $n = 8$-points diagram: on the *right diagram*, the times t_j with $l_j = L$ are put on the *upper axis* and the times t_k with $l_k = R$ on the *lower axis*. Here $|l| = 5$, $(j_1, j_2, j_3, j_4, j_5) = (1, 3, 4, 6, 8)$, and $(k_1, k_2, k_3) = (2, 5, 7)$

Proof We first give an explicit formula for $\mathcal{V}_{\lambda,n}(\underline{\pi}, \underline{t}, \underline{x}, \underline{l})(M \otimes |x_0\rangle\langle y_0|)$ in terms of the operators $V_x(t)$, with $M \in \mathcal{B}(\mathbb{C}^N)$ and $x_0, y_0 \in \mathbb{Z}^d$. For a fixed $\underline{l} \in \{L, R\}^{2n}$, let $|l|$ denotes the number of indices j such that $l_j = L$, $j = 1, \ldots, 2n$. We change the labelling of the indices and coordinates by defining (see Fig. 4)

$$\{j_1 < \cdots < j_{|l|}\} = \big\{j \in \{1, \ldots, 2n\};\ l_j = L\big\}$$
$$\{k_1 < \cdots < k_{2n-|l|}\} = \big\{k \in \{1, \ldots, 2n\};\ l_k = R\big\}$$

and

$$\tilde{x}_0 = x_0, \qquad \tilde{x}_1 = x_{j_1}, \quad \ldots, \quad \tilde{x}_{|l|} = x_{j_{|l|}},$$
$$\tilde{y}_0 = y_0, \qquad \tilde{y}_1 = x_{k_1}, \quad \ldots, \quad \tilde{y}_{2n-|l|} = x_{k_{2n-|l|}}.$$

It follows from (61) and (63) that for any matrix $M \in \mathcal{B}(\mathbb{C}^N)$,

$$\begin{aligned}
&\mathcal{V}_{\lambda,n}(\underline{\pi}, \underline{t}, \underline{x}, \underline{l})\big(M \otimes |x_0\rangle\langle y_0|\big) \\
&\quad = \lambda^{2n}(-1)^{n+|l|} V_{\tilde{x}_{|l|}}(t_{j_{|l|}}) \cdots V_{\tilde{x}_1}(t_{j_1}) M \otimes |x_0\rangle\langle y_0| \\
&\qquad \times V_{\tilde{y}_1}(t_{k_1}) \cdots V_{\tilde{y}_{2n-|l|}}(t_{k_{2n-|l|}}) \prod_{m=1}^{n} f(x_{\sigma_m} - x_{\iota_m}, t_{\sigma_m} - t_{\iota_m}, l_{\iota_m}).
\end{aligned} \tag{81}$$

Note that a similar formula holds at finite volume for $\mathcal{V}^{\Lambda}_{\lambda,n}(\underline{\pi}, \underline{t}, \underline{x}, \underline{l})$.

We must bound the norm of the right-hand side of (81). Let us denote by $G(t; x, y) = \langle x|\mathrm{e}^{-\mathrm{i}tH_P}|y\rangle$ the time-dependent Green function associated to the free motion of the particle at infinite volume. Using (60) and setting $t_{j_0} = t_{k_0} = 0$, one gets

$$\begin{aligned}
&\langle x|V_{\tilde{x}_{|l|}}(t_{j_{|l|}}) \cdots V_{\tilde{x}_1}(t_{j_1})|x_0\rangle \\
&\quad = G(-t_{j_{|l|}}; x, \tilde{x}_{|l|}) \overleftarrow{\prod_{p=1,\ldots,|l|}} WG(t_{j_p} - t_{j_{p-1}}; \tilde{x}_p, \tilde{x}_{p-1})
\end{aligned} \tag{82}$$

and

$$\begin{aligned}
&\langle y_0|V_{\tilde{y}_1}(t_{k_1}) \cdots V_{\tilde{y}_{2n-|l|}}(t_{k_{2n-|l|}})|y\rangle \\
&\quad = \overrightarrow{\prod_{q=1,\ldots,2n-|l|}} G(t_{k_{q-1}} - t_{k_q}; \tilde{y}_{q-1}, \tilde{y}_q) WG(t_{k_{2n-|l|}}; \tilde{y}_{2n-|l|}, y).
\end{aligned} \tag{83}$$

Thanks to the propagation bound (7), we have

$$
\begin{aligned}
&\left\| \left(\mathcal{V}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l})\right)_{x_0,y_0;x,y} \right\| \\
&\quad \le \left(\|W\|\lambda\right)^{2n} \mathrm{e}^{4\lambda^2 t} \mathrm{e}^{-|x-\tilde{x}_{|l|}|-|y-\tilde{y}_{2n-|l|}|} \prod_{p=1}^{|l|} \mathrm{e}^{-|\tilde{x}_p-\tilde{x}_{p-1}|} \prod_{q=1}^{2n-|l|} \mathrm{e}^{-|\tilde{y}_q-\tilde{y}_{q-1}|} \\
&\qquad \times \prod_{m=1}^{n} h_n(t_{\sigma_m}-t_{\iota_m})\, \mathrm{e}^{\frac{|x_{\sigma_m}-x_{\iota_m}|}{2n}}.
\end{aligned}
\tag{84}
$$

Next, observe that

$$
\sum_{p=1}^{|l|} |\tilde{x}_p-\tilde{x}_{p-1}| + \sum_{q=1}^{2n-|l|} |\tilde{y}_q-\tilde{y}_{q-1}| \ge \frac{1}{n}\sum_{m=1}^{n} |x_{\sigma_m}-x_{\iota_m}| - |\tilde{x}_0-\tilde{y}_0|. \tag{85}
$$

Actually, (85) is a consequence of the inequality

$$
\sum_{m=0}^{2n+1} |z_{m+1}-z_m| \ge \max\left\{|z_\sigma - z_\iota|;\ \iota,\sigma = 0,\dots,2n+1\right\}
$$

applied to $(z_0,\dots,z_{2n+1}) = (\tilde{x}_{|l|},\dots,\tilde{x}_0,\tilde{y}_0,\dots,\tilde{y}_{2n-|l|}) \in \mathbb{Z}^{(2n+2)d}$. Replacing (85) into (84), one gets the result with

$$
\begin{aligned}
R_{x_0,y_0;x,y}(\underline{x},\underline{l}) &= \left(2\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\frac{1}{2}|z|}\right)^{-2n} \mathrm{e}^{-|x-\tilde{x}_{|l|}|-|y-\tilde{y}_{2n-|l|}|} \mathrm{e}^{\frac{1}{2}|x_0-y_0|} \\
&\quad \times \prod_{p=1}^{|l|} \mathrm{e}^{-\frac{1}{2}|\tilde{x}_p-\tilde{x}_{p-1}|} \prod_{q=1}^{2n-|l|} \mathrm{e}^{-\frac{1}{2}|\tilde{y}_q-\tilde{y}_{q-1}|}.
\end{aligned}
\tag{86}
$$

The proof for the finite lattice is the same, since we only used the propagation estimate. □

4.5 Uniform Convergence of the Dyson Series

Proposition 3 *Let $x_0, y_0 \in \mathbb{Z}^d$. For fixed λ and t, one has*

$$
\sum_n \sum_{\underline{x},\underline{l}} \sum_{\underline{\pi}} \sum_{x,y} \int_{Z_{2n}(t)} \mathrm{d}\underline{t} \sup_{\Lambda\subset\mathbb{Z}^d} \left\| \left(\mathcal{V}^{\Lambda}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l})\right)_{x_0,y_0;x,y} \right\| < \infty \tag{87}
$$

where the supremum is taken over $\Lambda =]-L, L]^d \cap \mathbb{Z}^d$ for all finite $L > 0$. Similarly, assume that the correlation function $f(x,t)$ satisfies (32) and fix $\tau \ge 0$, then,

$$
\sum_n \sum_{\underline{x},\underline{l}} \sum_{x,y} \sup_{\lambda>0} \left\{ \sum_{\underline{\pi}} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t} \left\| \left(\mathcal{V}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l})\right)_{x_0,y_0;x,y} \right\| \right\} < \infty. \tag{88}
$$

Proof We first show the second claim. The proof of the first one follows similar lines. We bound (88) with the help of Lemma 2 by

$$
C_{x_0,y_0} \mathrm{e}^{4\tau} \sum_n \sup_{\lambda>0} \left\{ (C\lambda)^{2n} \sum_{\underline{\pi}} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t} \prod_{m=1}^{n} h_n(t_{\sigma_m}-t_{\iota_m}) \right\} \tag{89}
$$

where $C_{x_0,y_0} < \infty$ denotes the supremum in (80). The sum over all pairings $\underline{\pi}$ and the time integrals are conveniently rewritten with the help of

Lemma 3 *For any locally integrable function* $g : \mathbb{R}^{2n} \to \mathbb{R}$ *and* $0 \le t_0 < t$,

$$\sum_{\underline{\pi}} \int_{t_0 \le t_1 \le \cdots \le t_{2n} \le t} \mathrm{d}\underline{t}\, g(t_{\iota_1}, t_{\sigma_1}; \cdots ; t_{\iota_n}, t_{\sigma_n})$$

$$= \int_{t_0 \le u_1 \le \cdots \le u_n \le t} \mathrm{d}u_1 \cdots \mathrm{d}u_n \int_{u_m \le u'_m \le t;\, m=1,\ldots,n} \mathrm{d}u'_1 \cdots \mathrm{d}u'_n\, g\big(u_1, u'_1; \cdots ; u_n, u'_n\big).$$

We leave to the reader the proof of this lemma, which is based on change of variables.

Applying Lemma 3 with $g(u_1, u'_1; \cdots ; u_n, u'_n) = \prod_m h_n(u'_m - u_m)$ as in (89), bounding the integrals over the u_m and u'_m by $\|h_n\|_1^n = (\int_0^\infty \mathrm{d}t\, h_n(t))^n$ times the volume of the n-dimensional simplex $Z_n(\lambda^{-2}\tau)$, we conclude that (89) is smaller than

$$C_{x_0,y_0} \mathrm{e}^{4\tau} \sum_{n \ge 0} \frac{1}{n!} \big(C^2 \|h_n\|_1\, \tau\big)^n. \tag{90}$$

By Stirling formula $n! \sim (2\pi n)^{1/2} (n/e)^n$ as $n \to \infty$, one finds that the convergence of the series in (90) is ensured by assumption (32), that is, by the condition $\|h_n\|_1/n \to 0$ as $n \to \infty$. Thus the second claim of Proposition 3 is proven.

To show the first claim, we replace h_n by $\max_{x \in \Lambda} |f^\Lambda(x, \cdot)|$ (see Lemma 2) in (89). Here we do not need to assume that this function has a finite L^1-norm, we bound it by $f^\Lambda(0,0)$, see (25). The quantity in (87) is thus smaller than

$$C_{x_0,y_0} \mathrm{e}^{4\lambda^2 t} \sum_{n \ge 0} \frac{1}{n!} \left(\frac{C^2 \lambda^2 t^2 \sup_\Lambda f^\Lambda(0,0)}{2} \right)^n < \infty. \tag{91}$$

Note that $\sup_\Lambda f^\Lambda(0,0)$ is finite since $f^\Lambda(0,0)$ converges as $\Lambda \uparrow \mathbb{Z}^d$. This concludes the proof of Proposition 3. □

4.6 Proof of Proposition 1

One has

$$\big\| \big((\mathcal{V}^\Lambda_{\lambda,n} - \mathcal{V}_{\lambda,n})(\underline{\pi}, \underline{t}, \underline{x}, \underline{l}) \big)_{x_0,y_0;x,y} \big\| \longrightarrow 0 \quad \text{as } \Lambda \nearrow \mathbb{Z}^d. \tag{92}$$

Actually, let us set $\mathcal{A}^\Lambda = (\mathcal{V}^\Lambda_{\lambda,n} - \mathcal{V}_{\lambda,n})(\underline{\pi}, \underline{t}, \underline{x}, \underline{l})$. It is sufficient to prove that $\|(\mathcal{A}^\Lambda)_{x_0,y_0;x,y}(M)\| \le \|(\mathcal{A}^\Lambda)_{x_0,y_0;x,y}(M)\|_1 \to 0$ for any $M \in \mathcal{B}(\mathbb{C}^N)$ (recall that $(\mathcal{A}^\Lambda)_{x_0,y_0;x,y}$ is a finite matrix), so that the convergence (92) follows directly from the last claim in (66). By Proposition 3, (92), and dominated convergence, for any fixed λ and t we have

$$\underset{\Lambda \nearrow \mathbb{Z}^d}{\text{pt-lim}} \big\{ \mathcal{D}^\Lambda_{t,\lambda} - \mathcal{D}_{t,\lambda} \big\} = 0. \tag{93}$$

Since $\mathcal{Z}^\Lambda_{t,\lambda}$ preserves the trace and is completely positive, the same holds true for $\mathcal{D}^\Lambda_{t,\lambda}$ in (65). This implies that $\|\mathcal{D}^\Lambda_{t,\lambda}\| = 1$ for any Λ (see the discussion after (70)). Applying Lemma 1, we deduce from (93) that

$$\mathcal{D}^\Lambda_{t,\lambda}(T) \to \mathcal{D}_{t,\lambda}(T) \tag{94}$$

for any $T \in \mathcal{B}_1(\mathcal{H}_P)$ with finite support on the lattice $\mathbb{Z}^d$. Hence $\mathcal{D}_{t,\lambda}$ can be extended to a bounded, trace-preserving and completely positive operator on $\mathcal{B}_1(\mathcal{H}_P)$. From this, we straightforwardly deduce that, for a convergent sequence $\rho_P^\Lambda \to \rho_P$ in $\mathcal{B}_1(\mathcal{H}_P)$,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathcal{D}_{t,\lambda}^\Lambda\big(\rho_P^\Lambda\big) = \mathcal{D}_{t,\lambda}(\rho_P). \tag{95}$$

Since $\mathrm{e}^{-\mathrm{i}t[H_P^\Lambda,\cdot]} \to \mathrm{e}^{-\mathrm{i}t[H_P,\cdot]}$ strongly, (95) also holds if we replace $\mathcal{D}_{t,\lambda}^\Lambda$ by $\mathcal{Z}_{t,\lambda}^\Lambda = \mathrm{e}^{-\mathrm{i}t[H_P^\Lambda,\cdot]}\mathcal{D}_{t,\lambda}^\Lambda$ and $\mathcal{D}_{t,\lambda}$ by $\mathcal{Z}_{t,\lambda}$ which is given by (72). Therefore, the limit in Proposition 1 exists and $\mathcal{Z}_{t,\lambda}$ is trace-preserving and completely positive on $\mathcal{B}_1(\mathcal{H}_P)$. □

4.7 Crossing Diagrams Vanish in the van Hove Limit

In this subsection and the following ones, the correlation function $f(x,t)$ does not need to satisfy the assumption (32) and we only require that

$$\big\| f(x,\cdot)\big\|_1 = \int_0^\infty \mathrm{d}t\, \big|f(x,t)\big| < \infty \quad \text{for any } x \in \mathbb{Z}^d. \tag{96}$$

As announced in Step II of the plan of the proof, we show:

Proposition 4 *Assume the integrability condition* (96). *If* $\underline{\pi}$ *is a crossing diagram, i.e.,* $\underline{\pi} \neq \underline{\pi}_{\text{ladder}}$, *then for any fixed* n, τ, $\underline{x}$, $\underline{l}$, *and* x_0, y_0, x, y,

$$\lim_{\lambda\to 0} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t}\, \big\|\mathcal{V}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l})_{x_0,y_0;x,y}\big\| = 0. \tag{97}$$

Proof It is a simple adaptation of the arguments used in [36], Sect. 6.3. Let $\underline{\pi} = \{(\iota_1,\sigma_1),\dots,(\iota_n,\sigma_n)\}$ be a crossing diagram. This means that one can find two pairs $(\iota_\mu,\sigma_\mu) \in \underline{\pi}$ and $(\iota_\nu,\sigma_\nu) \in \underline{\pi}$ such that $\iota_\mu < \iota_\nu$ (i.e., $\mu < \nu$) and $\iota_\nu < \sigma_\mu$. According to (84), we need to show that

$$J_\lambda(\underline{\pi},\tau) = \lambda^{2n} \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t} \prod_{m=1}^n \big|f_m(t_{\sigma_m} - t_{\iota_m})\big| \longrightarrow 0 \tag{98}$$

as $\lambda \to 0$, where we abbreviated $f(x_{\sigma_m} - x_{\iota_m}, t_{\sigma_m} - t_{\iota_m})$ by $f_m(t_{\sigma_m} - t_{\iota_m})$ (recall that here $\underline{x}$ is fixed). One has

$$\begin{aligned} J_\lambda(\underline{\pi},\tau) \le \prod_{m\neq\mu,\nu}^n \int_{0\le t_{\iota_m}\le t_{\sigma_m}\le\lambda^{-2}\tau} \mathrm{d}t_{\iota_m}\,\mathrm{d}t_{\sigma_m}\,\lambda^2\big|f_m(t_{\sigma_m}-t_{\iota_m})\big| \\ \times \int_{0\le t_{\iota_\mu}\le t_{\iota_\nu}\le t_{\sigma_\mu}\le\lambda^{-2}\tau,\, t_{\iota_\nu}\le t_{\sigma_\nu}} \mathrm{d}t_{\iota_\mu}\,\mathrm{d}t_{\sigma_\mu}\,\mathrm{d}t_{\iota_\nu}\,\mathrm{d}t_{\sigma_\nu}\,\lambda^4\big|f_\mu(t_{\sigma_\mu}-t_{\iota_\mu})\big|\big|f_\nu(t_{\sigma_\nu}-t_{\iota_\nu})\big|. \end{aligned}$$

The product of integrals in the first line is smaller than $(\sup_m \|f_m\|_1\tau)^{n-2}$. To deal with the integral in the second line, we first bound the integral over t_{σ_ν} by $\lambda^4\|f_\nu\|_1 f_\mu(t_{\sigma_\mu} - t_{\iota_\mu})$ and then substitute $v = \lambda^2 t_{\sigma_\mu}$, $w = \lambda^2(t_{\sigma_\mu} - t_{\iota_\nu})$, and $t' = t_{\sigma_\mu} - t_{\iota_\mu}$. This gives the bound

$$\|f_\nu\|_1 \int_0^\tau \mathrm{d}v \int_0^v \mathrm{d}w \int_{\lambda^{-2}w}^{\lambda^{-2}v} \mathrm{d}t'\,\big|f_\mu\big(t'\big)\big|. \tag{99}$$

For any $(v, w) \in [0, \tau]^2$ such that $0 < w < v$, $\int_{\lambda^{-2}w}^{\lambda^{-2}v} \mathrm{d}t' \, |f_\mu(t')|$ converges to zero as $\lambda \to 0$ (because $\|f_\mu\|_1 < \infty$). This integral is also bounded by $\|f_\mu\|_1$, therefore (99) converges to zero by dominated convergence. □

4.8 Contribution of the Ladder Diagrams

In this subsection, we determine the contribution of the ladder diagrams and accomplish Step III of the proof. We first introduce the following operators on $\mathcal{B}_1(\mathcal{H}_P)$:

$$\mathcal{U}_\lambda(t) = \mathrm{e}^{-\mathrm{i}t[H_P,\cdot]} = \mathrm{e}^{-it[\lambda^2 H_{\mathrm{hop}}+S,\cdot]}, \qquad \mathcal{U}_0(t) = \mathrm{e}^{-\mathrm{i}t[S,\cdot]}. \tag{100}$$

Define the family of operators

$$\mathcal{A}\big(x', l'; x, l\big) = -\int_0^\infty \mathrm{d}t \; f\big(x' - x, t, l\big)\mathcal{U}_0(-t)\mathcal{I}\big(x', 0, l'\big)\mathcal{U}_0(t)\mathcal{I}(x, 0, l). \tag{101}$$

The integral is convergent by the integrability condition (96). Let

$$\mathcal{W}_{\lambda,n}(\underline{\tau}, \underline{x}, \underline{l}) = \overleftarrow{\prod_{j=1,\dots,n}} \mathcal{U}_\lambda\big(-\lambda^{-2}\tau_j\big)\mathcal{A}(x_{2j}, l_{2j}; x_{2j-1}, l_{2j-1})\mathcal{U}_\lambda\big(\lambda^{-2}\tau_j\big) \tag{102}$$

for any $\underline{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}_+^n$. We have

Proposition 5 *Assume the integrability condition* (96). *For any fixed* $\underline{x}$, $\underline{t}$ *and* τ,

$$\lim_{\lambda\to 0}\left\| \int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t}\; \mathcal{V}_{\lambda,n}(\underline{\pi}_{\mathrm{ladder}}, \underline{t}, \underline{x}, \underline{l}) - \int_{Z_n(\tau)} \mathrm{d}\underline{\tau}\; \mathcal{W}_{\lambda,n}(\underline{\tau}, \underline{x}, \underline{l}) \right\| = 0. \tag{103}$$

Proof Let us set, for any $\delta > 0$,

$$\mathcal{A}_{\lambda,\delta}\big(x', l'; x, l\big) = -\int_0^{\lambda^{-2}\delta} \mathrm{d}t \; f\big(x' - x, t, l\big)\mathcal{U}_\lambda(-t)\mathcal{I}\big(x', 0, l'\big)\mathcal{U}_\lambda(t)\mathcal{I}(x, 0, l). \tag{104}$$

Then

$$\lim_{\lambda\to 0} \mathcal{A}_{\lambda,\delta}\big(x', l'; x, l\big) = \mathcal{A}\big(x', l'; x, l\big) \quad \text{in norm} \tag{105}$$

and $\|\mathcal{A}_{\lambda,\delta}(x', l'; x, l)\| \le \|f(x'-x, \cdot)\|_1 \|W\|^2$ for any $x, x' \in \mathbb{Z}^d$, and $l, l' \in \{L, R\}$. This follows from the dominated convergence theorem, using (i) the integrability of $|f(x' - x, \cdot)|$, (ii) the norm convergence $\lim_{\lambda\to 0}\mathcal{U}_\lambda(t) = \mathcal{U}_0(t)$ (which follows directly from the boundedness of H_{hop}), and (iii) the bounds $\|\mathcal{I}(x, 0, l)\| \le \|W\|$ and $\|\mathcal{U}_\lambda(t)\| = 1$. Using $\mathcal{I}(x, t, l) = \mathcal{U}_\lambda(-t)\mathcal{I}(x, 0, l)\mathcal{U}_\lambda(t)$ and setting $s_j = t_{2j} - t_{2j-1}$, we rewrite the (infinite volume version of) (63) as

$$\begin{aligned}\mathcal{V}_{\lambda,n}(\underline{\pi}_{\mathrm{ladder}}, \underline{t}, \underline{x}, \underline{l}) = \big(-\lambda^2\big)^n \overleftarrow{\prod_{j=1,\dots,n}} & f(x_{2j} - x_{2j-1}, s_j, l_{2j-1})\mathcal{U}_\lambda(-t_{2j-1})\mathcal{U}_\lambda(-s_j) \\ & \times \mathcal{I}(x_{2j}, 0, l_{2j})\mathcal{U}_\lambda(s_j)\mathcal{I}(x_{2j-1}, 0, l_{2j-1})\mathcal{U}_\lambda(t_{2j-1}).\end{aligned} \tag{106}$$

We now perform the variable substitutions $\tau_j = \lambda^2 t_{2j-1}$, $s_j = t_{2j} - t_{2j-1}$ for $j = 1, \dots, n$, to get

$$\int_{Z_{2n}(\lambda^{-2}\tau)} \mathrm{d}\underline{t}\, \mathcal{V}_{\lambda,n}(\underline{\pi}_{\mathrm{ladder}}, \underline{t}, \underline{x}, \underline{l}) - \int_{Z_n(\tau)} \mathrm{d}\underline{\tau}\, \mathcal{W}_{\lambda,n}(\underline{\tau}, \underline{x}, \underline{l})$$
$$= \int_{Z_n(\tau)} \mathrm{d}\underline{\tau} \left\{ \overleftarrow{\prod_{j=1,\dots,n}} \mathcal{U}_\lambda\left(-\frac{\tau_j}{\lambda^2}\right) \mathcal{A}_{\lambda,\tau_{j+1}-\tau_j}(x_{2j}, l_{2j}; x_{2j-1}, l_{2j-1}) \mathcal{U}_\lambda\left(\frac{\tau_j}{\lambda^2}\right) \right.$$
$$\left. - \overleftarrow{\prod_{j=1,\dots,n}} \mathcal{U}_\lambda\left(-\frac{\tau_j}{\lambda^2}\right) \mathcal{A}(x_{2j}, l_{2j}; x_{2j-1}, l_{2j-1}) \mathcal{U}_\lambda\left(\frac{\tau_j}{\lambda^2}\right) \right\}. \tag{107}$$

For any $0 \le \tau_1 < \tau_2 < \cdots < \tau_n$, the integrand inside the curly brackets converges in norm to zero because of (105). This integrand is bounded by $2(\max_{j=1,\dots,n} \| f(x_{2j} - x_{2j-1}, \cdot)\|_1 \|W\|^2)^n$. Hence an application of the dominated convergence theorem yields the result. □

4.9 Spectral Averaging

To end the proof of Theorem 1, we use some standard techniques of "dynamical spectral averaging", originally used in Ref. [8] in the same context.

Lemma 4 *Let A and B be bounded operators on a Banach space $\mathcal{Y}$ such that $(\mathrm{e}^{tB})_{t\in\mathbb{R}}$ is a one-parameter group of isometries on $\mathcal{Y}$ and the norm limit*

$$A^\natural = \lim_{t\to\infty} t^{-1} \int_0^t \mathrm{d}u\, \mathrm{e}^{-uB} A \mathrm{e}^{uB} \tag{108}$$

exists. Let $D(\cdot), E(\cdot)$ be in $\mathcal{C}^1(\mathbb{R}, \mathcal{B}(\mathcal{Y}))$ (continuously differentiable $\mathcal{B}(\mathcal{Y})$-valued functions). Then, for any $\tau > 0$,

(1) $\lim_{\varepsilon\to 0} \int_0^\tau \mathrm{d}\tau_1\, D(\tau_1) \mathrm{e}^{-(\tau_1/\varepsilon)B} A \mathrm{e}^{(\tau_1/\varepsilon)B} E(\tau_1) = \int_0^\tau \mathrm{d}\tau_1\, D(\tau_1) A^\natural E(\tau_1)$
(2) $\lim_{\varepsilon\to 0} \mathrm{e}^{-(\tau/\varepsilon)B} \mathrm{e}^{(\tau/\varepsilon)(B+\varepsilon A)} = \lim_{\varepsilon\to 0} \mathrm{e}^{(\tau/\varepsilon)(B+\varepsilon A)} \mathrm{e}^{-(\tau/\epsilon)B} = \mathrm{e}^{\tau A^\natural}$.

Proof To show the claim (1), we put $E(\tau) = 1$ (the general result follows by an obvious extension of the proof). Let us write $D'(\tau) = \frac{d}{d\tau} D(\tau)$, then

$$\int_0^\tau \mathrm{d}\tau_1\, D(\tau_1) \mathrm{e}^{-(\tau_1/\varepsilon)B} A \mathrm{e}^{(\tau_1/\varepsilon)B}$$
$$= \int_0^\tau \mathrm{d}\tau_1 \left(D(0) + \int_0^{\tau_1} \mathrm{d}\tau_2\, D'(\tau_2) \right) \mathrm{e}^{-(\tau_1/\varepsilon)B} A \mathrm{e}^{(\tau_1/\varepsilon)B}$$
$$= \int_0^\tau \mathrm{d}\tau_1\, D(0) \mathrm{e}^{-(\tau_1/\varepsilon)B} A \mathrm{e}^{(\tau_1/\varepsilon)B} + \int_0^\tau \mathrm{d}\tau_2\, D'(\tau_2) \int_{\tau_2}^\tau \mathrm{d}\tau_1\, \mathrm{e}^{-(\tau_1/\varepsilon)B} A \mathrm{e}^{(\tau_1/\varepsilon)B}$$
$$\underset{\varepsilon\to 0}{\to} \left(\tau D(0) + \int_0^\tau \mathrm{d}\tau_2\, (\tau - \tau_2) D'(\tau_2) \right) A^\natural = \int_0^\tau \mathrm{d}\tau_1\, D(\tau_1) A^\natural. \tag{109}$$

To get the last line, we used (108) to estimate the integrals over τ_1 for all $\tau_2 < \tau$, together with the dominated convergence theorem (since $D'(\cdot)$ is norm continuous and e^{tB} is an isometry).

The claim (2) can be proven from (1) by expanding the two first members as Dyson series in A (alternatively, see [37], Theorem 5.11, and the review by Dereziński and Fruboes in [19] for the same result under weaker conditions). □

Let us now apply Lemma 4 to the case at hand. We choose $B = \mathrm{i}[S, \cdot]$, $\varepsilon = \lambda^2$ and $\mathcal{Y} = \mathcal{B}_1(\mathcal{H}_P)$. Then $(\mathrm{e}^{tB})_{t\in\mathbb{R}}$ is a group of isometries on $\mathcal{B}_1(\mathcal{H}_P)$. Since S has a finite number of eigenvalues, the existence of the norm limit (108) is automatic. Moreover, for any $\mathcal{A} \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}_P))$ one has

$$\mathcal{A}^\natural = \sum_{\omega\in\sigma([S,\cdot])} \mathcal{P}_\omega \mathcal{A} \mathcal{P}_\omega \tag{110}$$

where $\mathcal{P}_\omega$ are the spectral projectors of $[S, \cdot]$, i.e.,

$$\mathcal{P}_\omega(T) = \sum_{s,s'=1,\dots,N} \delta_{E_s - E_{s'},\omega} |s\rangle\langle s'| \langle s|T|s'\rangle, \qquad T \in \mathcal{B}_1(\mathcal{H}_P). \tag{111}$$

We first prove

Proposition 6 *Let us define*

$$\mathcal{K}_n(\underline{\tau}, \underline{x}, \underline{l}) = \overleftarrow{\prod_{j=1,\dots,n}} \mathcal{U}_\natural(-\tau_j)\big[\mathcal{A}(x_{2j}, l_{2j}; x_{2j-1}, l_{2j-1})\big]^\natural \mathcal{U}_\natural(\tau_j) \tag{112}$$

with $\mathcal{U}_\natural(\tau) = \mathrm{e}^{-\mathrm{i}\tau[H^\natural_{\mathrm{hop}},\cdot]}$, *where* $H^\natural_{\mathrm{hop}}$ *is given by* (31). *Then for the norm topology on* $\mathcal{B}(\mathcal{B}_1(\mathcal{H}_P))$,

$$\lim_{\lambda\to 0} \int_{Z_n(\tau)} \mathrm{d}\underline{\tau}\, \mathcal{W}_{\lambda,n}(\underline{\tau}, \underline{x}, \underline{l}) = \int_{Z_n(\tau)} \mathrm{d}\underline{\tau}\, \mathcal{K}_n(\underline{\tau}, \underline{x}, \underline{l}). \tag{113}$$

Proof An explicit calculation yields

$$[H_{\mathrm{hop}}, \cdot]^\natural = \big[H^\natural_{\mathrm{hop}}, \cdot\big]. \tag{114}$$

Choosing $A = \mathrm{i}[H_{\mathrm{hop}}.\cdot]$, the claim (2) of Lemma 4 yields

$$\lim_{\lambda\to 0} \big\|\mathcal{U}_0\big(-\tau/\lambda^2\big)\mathcal{U}_\lambda\big(\tau/\lambda^2\big) - \mathcal{U}_\natural(\tau)\big\| = \lim_{\lambda\to 0} \big\|\mathcal{U}_\lambda\big(\tau/\lambda^2\big)\mathcal{U}_0\big(-\tau/\lambda^2\big) - \mathcal{U}_\natural(\tau)\big\| = 0. \tag{115}$$

We use the abbreviation $\mathcal{A}_j = \mathcal{A}(x_{2j}, l_{2j}; x_{2j-1}, l_{2j-1})$. Since $\mathcal{U}_0(t)$ is an isometry, this gives

$$\lim_{\lambda\to 0} \big\|\mathcal{U}_\lambda\big(-\tau/\lambda^2\big)\mathcal{A}_j\mathcal{U}_\lambda\big(\tau/\lambda^2\big) - \mathcal{U}_\natural(-\tau)\mathcal{U}_0\big(-\tau/\lambda^2\big)\mathcal{A}_j\mathcal{U}_0\big(\tau/\lambda^2\big)\mathcal{U}_\natural(\tau)\big\| = 0. \tag{116}$$

To prove the proposition, we consider first the cases $n = 1$ and $n = 2$ and then conclude by induction.

For $n = 1$, one has

$$\lim_{\lambda\to 0} \int_{Z_1(\tau)} \mathrm{d}\underline{\tau}\, \mathcal{W}_{1,\lambda}(\underline{\tau}, \underline{x}, \underline{l}) = \lim_{\lambda\to 0} \int_0^\tau \mathrm{d}\tau_1\, \mathcal{U}_\lambda\big(-\tau_1/\lambda^2\big)\mathcal{A}_1\mathcal{U}_\lambda\big(\tau_1/\lambda^2\big)$$

$$
\begin{aligned}
&= \lim_{\lambda\to 0}\int_0^{\tau} \mathrm{d}\tau_1\, \mathcal{U}_\natural(-\tau_1)\mathcal{U}_0\big(-\tau_1/\lambda^2\big)\mathcal{A}_1\mathcal{U}_0\big(\tau_1/\lambda^2\big)\mathcal{U}_\natural(\tau_1)\\
&= \int_0^{\tau} \mathrm{d}\tau_1\, \mathcal{U}_\natural(-\tau_1)\mathcal{A}_1^\natural\mathcal{U}_\natural(\tau_1)
\end{aligned}
\tag{117}
$$

where we used (116) in the second equality, relying also on dominated convergence and the uniform boundedness of the integrand, and claim (1) of Lemma 4 in the third equality (note that $\mathcal{U}_\natural(\cdot)$ is C^1).

For $n=2$, one has

$$
\begin{aligned}
&\lim_{\lambda\to 0}\int_{Z_2(\tau)} \mathrm{d}\underline{\tau}\, \mathcal{W}_{2,\lambda}(\underline{\tau},\underline{x},\underline{l})\\
&\quad= \lim_{\lambda\to 0}\int_0^{\tau}\mathrm{d}\tau_2\int_0^{\tau_2}\mathrm{d}\tau_1\, \mathcal{U}_\lambda\big(-\tau_2/\lambda^2\big)\mathcal{A}_2\mathcal{U}_\lambda\big(\tau_2/\lambda^2\big)\mathcal{U}_\lambda\big(-\tau_1/\lambda^2\big)\mathcal{A}_1\mathcal{U}_\lambda\big(\tau_1/\lambda^2\big)\\
&\quad= \lim_{\lambda\to 0}\int_0^{\tau}\mathrm{d}\tau_2\, \mathcal{U}_\lambda\big(-\tau_2/\lambda^2\big)\mathcal{A}_2\mathcal{U}_\lambda\big(\tau_2/\lambda^2\big)\int_0^{\tau_2}\mathrm{d}\tau_1\, \mathcal{U}_\natural(-\tau_1)\mathcal{A}_1^\natural\mathcal{U}_\natural(\tau_1)\\
&\quad= \lim_{\lambda\to 0}\int_0^{\tau}\mathrm{d}\tau_2\, \mathcal{U}_\natural(-\tau_2)\mathcal{U}_0\big(-\tau_2/\lambda^2\big)\mathcal{A}_2\mathcal{U}_0\big(\tau_2/\lambda^2\big)\mathcal{U}_\natural(\tau_2)\\
&\qquad\times\int_0^{\tau_2}\mathrm{d}\tau_1\, \mathcal{U}_\natural(-\tau_1)\mathcal{A}_1^\natural\mathcal{U}_\natural(\tau_1)\\
&\quad= \int_0^{\tau}\mathrm{d}\tau_2\, \mathcal{U}_\natural(-\tau_2)\mathcal{A}_2^\natural\mathcal{U}_\natural(\tau_2)\int_0^{\tau_2}\mathrm{d}\tau_1\, \mathcal{U}_\natural(-\tau_1)\mathcal{A}_1^\natural\mathcal{U}_\natural(\tau_1).
\end{aligned}
\tag{118}
$$

The second equality is the case $n=1$, the third is (116), and the fourth follows from the claim (1) of Lemma 4.

The case $n>2$ follows by a similar induction step. □

End of the proof of Theorem 1 Collecting Propositions 4, 5, and 6, we have

$$
\lim_{\lambda\to 0}\sum_{\underline{\pi}}\int_{Z_{2n}(\lambda^{-2}\tau)}\mathrm{d}\underline{t}\big(\mathcal{V}_{\lambda,n}(\underline{\pi},\underline{t},\underline{x},\underline{l})\big)_{x_0,y_0;x,y} = \int_{Z_n(\tau)}\mathrm{d}\underline{\tau}\big(\mathcal{K}_n(\underline{\tau},\underline{x},\underline{l})\big)_{x_0,y_0;x,y}
\tag{119}
$$

for any fixed n, $\underline{x}$, $\underline{l}$, and x, y, x_0, y_0. Thanks to the second statement of Proposition 3 and to the dominated convergence theorem, we obtain in view of (67)

$$
\underset{\lambda\to 0}{\text{pt-lim}}\,\mathcal{D}_{\lambda^{-2}\tau,\lambda} = \sum_n\sum_{\underline{x},\underline{l}}\int_{Z_n(\tau)}\mathrm{d}\underline{\tau}\,\mathcal{K}_n(\underline{\tau},\underline{x},\underline{l}) = \mathrm{e}^{\mathrm{i}\tau[H_{\mathrm{hop}}^\natural,\cdot]}\mathrm{e}^{-\mathrm{i}\tau[H_{\mathrm{hop}}^\natural,\cdot]+\tau\mathcal{A}^\natural}
\tag{120}
$$

where the last equality comes from a Dyson expansion in powers of $\mathcal{A}^\natural$ (the series is convergent in the pt-lim sense by dominated convergence), and

$$
\mathcal{A}^\natural = \sum_{x,x',l,l'}\big[\mathcal{A}\big(x,l;x',l'\big)\big]^\natural.
\tag{121}
$$

One concludes by invoking Lemma 1 (see also the discussion after this lemma) that

$$
\mathcal{D}_{\lambda^{-2}\tau,\lambda}\to \mathrm{e}^{\mathrm{i}\tau[H_{\mathrm{hop}}^\natural,\cdot]}\mathrm{e}^{-\mathrm{i}\tau[H_{\mathrm{hop}}^\natural,\cdot]+\tau\mathcal{A}^\natural}\quad \text{strongly as } \lambda\to 0.
\tag{122}
$$

But $\mathrm{e}^{\mathrm{i}t[S,\cdot]}$ and $\mathrm{e}^{-\mathrm{i}t[H_P,\cdot]}$ are isometries on $\mathcal{B}_1(\mathcal{H}_P)$. Therefore, in view of (29) and (72),

$$\begin{aligned}\rho_{\mathrm{sl}}(\tau) &= \lim_{\lambda\to 0} \mathrm{e}^{\mathrm{i}\lambda^{-2}\tau[S,\cdot]}\mathrm{e}^{-\mathrm{i}\lambda^{-2}\tau[\lambda^2 H_{\mathrm{hop}}+S,\cdot]}\mathcal{D}_{\lambda^{-2}\tau,\lambda}(\rho_P)\\ &= \mathrm{e}^{-\mathrm{i}\tau[H^{\natural}_{\mathrm{hop}},\cdot]+\tau\mathcal{A}^{\natural}}(\rho_P)\end{aligned} \tag{123}$$

in the trace-norm topology, where we have used again Lemma 4 (2).

To establish the agreement with the generator $\mathcal{L}^{\natural}$ given by (33), we check by inspection that, for any $\rho_P \in \mathcal{B}_1(\mathcal{H}_P)$,

$$\begin{aligned}&\big[\mathcal{A}\big(x,l=L;y,l'=R\big)\big]^{\natural}(\rho_P)\\ &\quad= \sum_{\omega\in\sigma([\cdot,S])} c(y-x,\omega)W_\omega\otimes|x\rangle\langle x|\,\rho_P\,W^*_\omega\otimes|y\rangle\langle y|,\end{aligned} \tag{124}$$

$$\begin{aligned}&\big[\mathcal{A}\big(x,l=L;y,l'=L\big)\big]^{\natural}(\rho_P)\\ &\quad= \mathrm{i}\delta_{x,y}\Upsilon\rho_P+\delta_{x,y}\sum_{\omega\in\sigma([\cdot,S])} c(0,\omega)W_\omega W^*_\omega\otimes|x\rangle\langle x|\,\rho_P\end{aligned} \tag{125}$$

and

$$\begin{aligned}\big[\mathcal{A}(x,R;y,L)\big]^{\natural}(\rho_P) &= \big(\big[\mathcal{A}(x,L;y,R)\big]^{\natural}(\rho_P)\big)^*\\ \big[\mathcal{A}(x,R;y,R)\big]^{\natural}(\rho_P) &= \big(\big[\mathcal{A}(x,L;y,L)\big]^{\natural}(\rho_P)\big)^*.\end{aligned} \tag{126}$$

Consequently, we have

$$\mathcal{A}^{\natural}(\rho_P) = i[\Upsilon,\rho_P] + \mathfrak{A}(\rho_P) - \frac{1}{2}\big\{\mathfrak{A}^{\star}(1),\rho_P\big\} \tag{127}$$

with $\mathfrak{A}$ as defined in (36). □

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Appendix: Proof of Proposition 2

We show in this Appendix that the main hypothesis (32) of Theorem 1 concerning the decay of the bath correlation functions $f_i(x,t)$ is satisfied under the following conditions:

(i) $d \geq 2$;
(ii) the support of $g_{0,i}(q)$ belongs to the open ball $B_r = \{q\in\mathbb{T}^d;\ |q|<r\}$ with $0<r\leq\pi$; the form factor $g_{0,i}(q)$ and the momentum occupation numbers $\zeta_i(q)$ depend only on $|q|$ on B_r;
(iii) the bosons have a linear dispersion relation: $\nu_i(q)=|q|$ for $q\in B_r$;
(iv) the non-negative functions

$$\psi_{i,+}\big(|q|\big) = \big|g_{0,i}(q)\big|^2\zeta_i(q), \qquad \psi_{i,-}\big(|q|\big) = \big|g_{0,i}(q)\big|^2\big(1+\zeta_i(q)\big) \tag{128}$$

belong to $C^2(]0,\pi])$ and the three functions of $|q|$ below are in $L^1([0,\pi])$:

$$|q|^{\min\{d-3,\frac{d-1}{2}\}}\psi_{i,\pm}\big(|q|\big), \qquad |q|^{d-2}\psi'_{i,\pm}\big(|q|\big), \qquad |q|^{d-1}\psi''_{i,\pm}\big(|q|\big). \tag{129}$$

For simplicity we omit the index i labelling the baths. Using the notation (128), the correlation function (26) reads

$$\begin{aligned} f(x,t) &= f_+(x,t) + f_-(x,t) \\ &= \int_{\mathbb{T}^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \big(\psi_+(|q|)\mathrm{e}^{\mathrm{i}q\cdot x}\mathrm{e}^{\mathrm{i}t\nu(q)} + \psi_-(|q|)\mathrm{e}^{-\mathrm{i}q\cdot x}\mathrm{e}^{-\mathrm{i}t\nu(q)}\big). \end{aligned} \tag{130}$$

We first show that under conditions (i)–(iv), there exists a constant $C_d > 0$ such that for any $n \in \mathbb{N}^\star$,

$$\begin{aligned} &\int_1^\infty \mathrm{d}t \sup_{x\in\mathbb{Z}^d, |x|\geq t/2} \big|f(x,t)\big| \mathrm{e}^{-\frac{|x|}{n}} \leq C_d \quad \text{if } d \geq 3, \\ &\int_1^\infty \mathrm{d}t \sup_{x\in\mathbb{Z}^d, |x|\geq t/2} \big|f(x,t)\big| \mathrm{e}^{-\frac{|x|}{n}} \leq C_2\sqrt{n} \quad \text{if } d = 2. \end{aligned} \tag{131}$$

Actually, by (130) and (ii)–(iii), $f(x,t)$ can be rewritten as the Fourier transform

$$f(x,t) = \frac{1}{(2\pi)^{\frac{d}{2}}|x|^{\frac{d-2}{2}}} \int_{-r}^{r} \mathrm{d}\omega\, |\omega|^{\frac{d}{2}} \psi_{\mathrm{sign}(\omega)}\big(|\omega|\big) J_{\frac{d-2}{2}}\big(|\omega x|\big)\mathrm{e}^{\mathrm{i}t\omega} \tag{132}$$

where $\mathrm{sign}(\omega) = \pm 1$ for $\pm\omega > 0$ and $J_m(r)$ is the Bessel function of order m, $J_m(r) = (r/2)^m \Gamma(m+\frac{1}{2})^{-1}(2/\sqrt{\pi}) \int_0^1 \mathrm{d}u\, (1-u^2)^{m-\frac{1}{2}} \cos(ru)$ (here Γ is the Gamma function). A standard bound, see e.g. [38], yields $C = \sup_{r\geq 0}\{\sqrt{r}|J_m(r)|\} < \infty$. Hence

$$\sup_{|x|\geq t/2} \big|f(x,t)\big| \leq \frac{C}{2^{\frac{1}{2}}\pi^{\frac{d}{2}} t^{\frac{d-1}{2}}} \int_{-r}^{r} \mathrm{d}\omega\, |\omega|^{\frac{d-1}{2}} \psi_{\mathrm{sign}(\omega)}\big(|\omega|\big). \tag{133}$$

The last integral is convergent thanks to assumption (iv). Then (131) follows from

$$\begin{aligned} &\sup_{n\in\mathbb{N}^\star} \int_1^\infty \mathrm{d}t\, t^{-\frac{d-1}{2}} \mathrm{e}^{-\frac{t}{2n}} < \infty \qquad \text{if } d \geq 3, \\ &\sup_{n\in\mathbb{N}^\star} \frac{1}{\sqrt{n}} \int_1^\infty \mathrm{d}t\, t^{-\frac{d-1}{2}} \mathrm{e}^{-\frac{t}{2n}} < \infty \quad \text{if } d = 2. \end{aligned} \tag{134}$$

We now show that there exists a constant $C > 0$ such that

$$\max_{x\in\mathbb{Z}^d, |x|<t/2} \big|f(x,t)\big| \leq \frac{C}{t^2}. \tag{135}$$

Let us set

$$v(q,x,t) = \nabla\nu(q) + \frac{x}{t} \tag{136}$$

where ∇ is the gradient with respect to q. Note that $v(q,x,t)$ is the gradient of the phase $S(q,x,t) = \nu(q) + q\cdot x/t$ appearing in the oscillatory integral (130). Assuming $|x| < t/2$, one has $|v(q,x,t)| > 1/2$ (since $\nabla\nu(q) = q/|q|$ has norm one), thus this phase has no stationary points. Noting that $\mathrm{e}^{\mathrm{i}tS(q,x,t)} = (\mathrm{i}t)^{-1}(v/|v|^2)\cdot\nabla\mathrm{e}^{\mathrm{i}tS(q,x,t)}$ and integrating twice by part yields

$$\big|f_\pm(x,t)\big| \leq \frac{1}{t^2}\sum_{k,l=1}^{d} \int_{\mathbb{T}^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \left|\partial_k\left(\frac{v_k}{|v|^2}\partial_l\left(\frac{v_l\,\psi_\pm}{|v|^2}\right)\right)\right| \tag{137}$$

where ∂_k is the derivative with respect to the kth component of q. Note that the boundary terms vanish thanks to condition (ii). A simple calculation shows that the integrand in the right-hand side of (137) is bounded by

$$c_1|q|^{-2}\psi_\pm\big(|q|\big) + c_2|q|^{-1}\big|\nabla\psi_\pm\big(|q|\big)\big| + c_3 \max_{k,l=1,\dots,d}\big|\partial_k\partial_l\psi_\pm\big(|q|\big)\big| \tag{138}$$

for some constants c_1, c_2, and $c_3 > 0$. The last function is integrable by assumption (iv) and this proves our claim (135).

Collecting the above results and recalling that $|f(x,t)| \le f(0,0)$, we conclude that

$$\begin{aligned}\frac{1}{n}\int_0^\infty \mathrm{d}t \sup_{x\in\mathbb{Z}^d}\big|f(x,t)\big|\mathrm{e}^{-\frac{|x|}{n}} \le \frac{1}{n}f(0,0) + \frac{1}{n}\int_1^\infty \mathrm{d}t \sup_{x,|x|\ge t/2}\big|f(x,t)\big|\mathrm{e}^{-\frac{|x|}{n}} \\ + \frac{1}{n}\int_1^\infty \mathrm{d}t \max_{x,|x|<t/2}\big|f(x,t)\big|\end{aligned} \tag{139}$$

converges to zero as $n \to \infty$. This proves Proposition 2. □

Let us stress that conditions (i)–(iv) are not optimal for the hypothesis (32) to hold. In particular, the rotation invariance of g_0 and ζ in (ii) and the linear dispersion (iii) have been chosen to simplify the proof and could be omitted at the expense of using the stationary phase method to evaluate the integral over the manifold $\nu(q) = \omega$ in (130), instead of using (132). In contrast, the first condition $d \ge 2$ is crucial: in dimension $d = 1$, (32) is not fulfilled. To see this, let us assume that (ii)–(iii) hold and that the baths are initially at thermal equilibrium, i.e., $\zeta(q) = (\mathrm{e}^{\beta|q|} - 1)^{-1}$. For $d = 1$, (130) gives

$$f(x,t) = \int_{-r}^{r}\frac{\mathrm{d}q}{2\pi}\,\big|g_0(q)\big|^2\frac{\mathrm{e}^{\mathrm{i}q(t+x)} + \mathrm{e}^{\mathrm{i}q(t-x)}}{|\mathrm{e}^{\beta q} - 1|}. \tag{140}$$

It is clear that $\sup_{x\in\mathbb{Z}}|f(x,t)|$ does not decay to zero at large times t. As a result, the integral in (32) diverges.

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