

Home Search Collections Journals About Contact us My IOPscience

Harnessing gauge fields for maximally entangled state generation

This content has been downloaded from IOPscience. Please scroll down to see the full text. 2014 EPL 108 20010 (http://iopscience.iop.org/0295-5075/108/2/20010)

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 147.173.98.158 This content was downloaded on 20/03/2015 at 15:50

Please note that terms and conditions apply.



EPL, **108** (2014) 20010 doi: 10.1209/0295-5075/108/20010

## Harnessing gauge fields for maximally entangled state generation

S. A. REYES<sup>1</sup>, L. MORALES-MOLINA<sup>1</sup>, M. ORSZAG<sup>1</sup> and D. SPEHNER<sup>2,3</sup>

<sup>1</sup> Instituto de Física, Pontificia Universidad Católica de Chile - Casilla 306, Santiago 22, Chile

<sup>2</sup> Université Grenoble Alpes, Institut Fourier UMR5582 - F-38402 Saint-Martin d'Hères, France

<sup>3</sup> Université Grenoble Alpes, LPMMC - F-38000 Grenoble, France

received 30 May 2014; accepted in final form 9 October 2014 published online 30 October 2014

Abstract – We study the generation of entanglement between two species of bosons living on a ring lattice, where each group of particles can be described by a *d*-dimensional Hilbert space (qudit). Gauge fields are exploited to create an entangled state between the pair of qudits. Maximally entangled eigenstates are found for well-defined values of the Aharonov-Bohm phase, which are zero-energy eigenstates of *both* the kinetic and interacting parts of the Bose-Hubbard Hamiltonian, making them quite exceptional and robust. We propose a protocol to reach the maximally entangled state (MES) by starting from an initially prepared ground state. Also, an indirect method to detect the MES by measuring the current of the particles is proposed.



Introduction. - Entanglement is one of the most important and unique features of quantum mechanics leading to many applications such as teleportation and quantum cryptography [1]. It may also be exploited for quantum metrology applications such as enhancing the precision of atomic interferometers used for time measurements and ultra-small signal detection [2]. Entanglement is a key ingredient in quantum information processing where the fundamental units are the so-called quantum bits (qubits), corresponding to simple two-level systems. Generalizations of this concept to systems with d-dimensional Hilbert spaces (qudits) have been the subject of great interest in recent years. For instance, it has been demonstrated that maximally entangled states among a pair of qudits violate local realism theories stronger than qubits [3]. Entanglement is a delicate quantum feature so its generation, protection, propagation, and distribution have been the subject of intense research over the last decades [4]. In particular, it is well known (see, e.g., [5]) that entangled states are much more fragile than uncorrelated states, making their use in applications difficult. Consequently, it remains a major challenge to generate robust entangled states. Engineering experimental setups to observe quantum correlations requires high coherence and a precise control on the Hamiltonian.

Recent experimental advances in the manipulation of ultracold atoms in optical lattices make them one of the most promising candidates due to the controllability of the

system dimension [6] and precise tunability of the interaction among particles [7]. In particular, ultracold atom systems have proven to be ideal devices for high precision interferometry [8]. For instance, entangling atoms in an optical lattice can reduce the noise in a clock measurement [9]. In addition, current developments have demonstrated that Raman-assisted tunneling can be used to implement synthetic magnetic fields for neutral atoms in optical lattices [10–15]. The atoms in the lattice can therefore acummulate a non-adiabatic Berry phase as they hop from site to site, usually referred to as the Peierls phase [16]. Similarly, such a phase can be engineered by applying a suitable periodic force [17]. These experimental tools open new routes for the observation of striking physics such as the quantum Hall effect [18,19] and novel topological phenomena [15].

In the present work we demonstrate the existence of maximally entangled eigenstates (MES) in a onedimensional Bose-Hubbard system with a Peierls phase. We study the entanglement between two distinct interacting species of bosons moving on a ring-shaped lattice, where each species has a fixed particle number and constitutes a qudit. Furthermore, we develop a protocol by which the MES can be prepared through an adequate time variation of the Peierls phase. Such a scenario could be realized experimentally by using ultracold atoms in an optical lattice with a tunable synthetic gauge field. We also propose a way to estimate the amount of entanglement between the two groups of bosons by looking at the particle current in the ring.

**The model.** – Consider two species of interacting bosons moving along a ring. In its simplest form, the dynamics of the system is governed by a Bose-Hubbard Hamiltonian,

$$\hat{H} = \hat{K}_A + \hat{K}_B + \hat{H}_{int}.$$
(1)

As mentioned before, the inclusion of a gauge field results in an acquired quantum-mechanical phase by the particles as they hop from site to site. Then, the kinetic part for each species D = A, B is written as

$$\hat{K}_D = -J \sum_{j=1}^{L} (e^{i\phi_D} \hat{d}_j^{\dagger} \hat{d}_{j+1} + e^{-i\phi_D} \hat{d}_{j+1}^{\dagger} \hat{d}_j), \qquad (2)$$

where J is the tunneling strength,  $\phi_D$  is the Peierls phase for species D, L is the number of sites and  $\hat{d}_j^{\dagger}$  ( $\hat{d} = \hat{a}, \hat{b}$ ) is the particle creation operator at site j. Since the particles live on a ring they obey the periodic boundary condition j + L = j, so that  $\hat{d}_{L+1}^{\dagger} = \hat{d}_1^{\dagger}$ . The interaction part of the Hamiltonian is given by

$$H_{int} = \frac{U_A}{2} \sum_{j=1}^{L} \hat{n}_{Aj}^2 + \frac{U_B}{2} \sum_{j=1}^{L} \hat{n}_{Bj}^2 - V \sum_{j=1}^{L} \hat{n}_{Aj} \hat{n}_{Bj}, \quad (3)$$

where  $\hat{n}_{Dj} = \hat{d}_j^{\dagger} \hat{d}_j$  is the particle number operator for species D (see fig. 1). Since we consider fixed particle numbers for both species, the linear terms in the interaction Hamiltonian have been omitted as they only contribute to a constant energy shift.

This setting may be accomplished with ultracold atoms in an optical lattice, either by using the same isotope with different internal states [20] or by using two kinds of atoms, such as <sup>87</sup>Rb and <sup>41</sup>K [21]. Interactions can be accurately tuned experimentally via Feschbach resonances [7,21] and synthetic magnetic fields can be engineered to include the Peierls phase [10–15].

We will now demonstrate that, for a suitable choice of the interactions, maximally entangled eigenstates are found at zero energy for specific values of the Peierls phase  $\phi$ . Consider a fixed number N of bosons for each species, such that both Hilbert spaces for the particles A and B have dimension

$$d = \begin{pmatrix} N+L-1\\ N \end{pmatrix}.$$

The two groups of particles (A and B) living on the ring are the parts of a bi-partite system. A maximally entangled state (MES) of such a system has the form [22]

$$|MES\rangle = \sum_{q=1}^{d} e^{i\theta_q} |q\rangle_A |q\rangle_B, \qquad (4)$$



Fig. 1: (Color online) Schematic picture of particles moving on a ring-shaped lattice. Two different species are represented by red (A) and white (B) circles. As they hop to neighboring sites a Peierls phase  $\phi$  is accumulated. Curly lines indicate different types of interactions among particles (see eq. (3)).

with arbitrary phases  $\theta_q$ , where  $\{|q\rangle_D\}$  is an orthonormal basis for species D. In fact, it is easy to check that the reduced density matrix of  $\hat{\rho}_{AB} = |MES\rangle\langle MES|$  becomes proportional to the identity:

$$\hat{\rho}_A = Tr_B(\hat{\rho}_{AB}) = \frac{1}{d} \sum_{q=1}^d |q\rangle_{AA} \langle q| \equiv \mathbb{1}/d, \qquad (5)$$

implying that A and B are maximally entangled.

Now, let us consider the Fock states  $|q\rangle_A |q\rangle_B = |n_1, \ldots, n_L\rangle_A |n_1, \ldots, n_L\rangle_B$  where  $n_j$  is the number of particles on site j. This state is an eigenstate of the interaction Hamiltonian with eigenvalue

$$E_{int}(n_1, \dots, n_L) = \left(\frac{U_A + U_B}{2} - V\right) \sum_{j=1}^L n_j^2.$$

We choose  $2V = U_A + U_B$  in order to achieve a degeneracy in the interaction energy for these states, *i.e.*  $E_{int}(n_1, \ldots, n_L) = 0$  for any  $\{n_j\}$ .

Let us analyze now what happens when the kinetic part of the Hamiltonian acts on  $|MES\rangle$ . Notice that in general the MES is not an eigenstate of  $\hat{K} = \hat{K}_A + \hat{K}_B$ , since  $\hat{d}_j^{\dagger} \hat{d}_{j+1} |q\rangle_A |q\rangle_B$  does not belong to the subspace spanned by the states  $|q\rangle_A |q\rangle_B$ . Nevertheless, it is still possible to have  $\hat{K}|MES\rangle = 0$ , that is, the MES can be a zero-energy eigenstate. Let us start by considering a more general state of the form

$$|\psi\rangle = \sum_{\vec{n}} c_{\{n_1,\dots,n_L\}} |n_1,\dots,n_L\rangle_A |n_1,\dots,n_L\rangle_B,$$

where the sum is over all  $\vec{n} = \{n_1, \ldots, n_L\}$  such that  $\sum_j n_j = N$ . Assuming  $\phi_A = \phi_B = \phi$ , it is straightforward

20010-p2



Fig. 2: (Color online) (a) Spectrum of the system for L = 3, N = 2 and J = U/10 = V/10 for a range of values of  $\phi$ . Energy E is shown in units of J. The inset illustrates the energies in the subspace spanned by  $|n_1, n_2, n_3\rangle_A |n_1, n_2, n_3\rangle_B$ , with  $E_{int}(n_1, n_2, n_3) = 0$ . (b) Continuous lines show families of maximally entangled states for L = 3, with zero energy at  $(\tilde{\phi}_A + \tilde{\phi}_B)/2 = (2l - 1)\pi/6$  (*l* integer). Dashed line corresponds to  $\phi_A = \phi_B$ .

to show that

$$\hat{K}|\psi\rangle = -J \sum_{j=1}^{L} \sum_{\vec{n}} (e^{i\phi} c_{\{\dots,n_j,n_{j+1},\dots\}} + e^{-i\phi} c_{\{\dots,n_j+1,n_{j+1}-1,\dots\}}) \sqrt{n_{j+1}} \sqrt{n_j+1} \\
\times (|\dots,n_j+1,n_{j+1}-1,\dots\rangle_A|\dots,n_j,n_{j+1},\dots\rangle_B + |\dots,n_j,n_{j+1},\dots\rangle_B),$$

which implies that in order to have  $\hat{K}|\psi\rangle = 0$  the coefficients must fulfill

$$e^{i\bar{\phi}}c_{\{\dots,n_j,n_{j+1},\dots\}} + e^{-i\bar{\phi}}c_{\{\dots,n_j+1,n_{j+1}-1,\dots\}} = 0$$

for all  $\vec{n}$ , all j = 1, ..., L, and a certain  $\tilde{\phi}$ . Furthermore, taking into account the periodic boundary conditions, we find

$$\tilde{\phi} = m\frac{\pi}{L} - \frac{\pi}{2},\tag{6}$$

where *m* is an integer. The fact that  $|\psi_m\rangle$  is a zero-energy eigenstate of the kinetic Hamiltonian  $\hat{K}$  is a consequence of a destructive interference effect between the hopping of one particle of the species *A* from site *j* to site *j*+1 and the hopping of one particle of the species *B* in the opposite direction from *j*+1 to *j*. In conclusion, when  $2V = U_A + U_B$ and the phases  $\phi_A = \phi_B = \tilde{\phi}$  in the Hamiltonian fulfill (6), there exists a zero-energy eigenstate of the form

$$|\psi_m\rangle = \frac{1}{\sqrt{d}} \sum_{\vec{n}} e^{i\frac{2\pi m}{L}p_{\{n_1,\dots,n_L\}}} |n_1,\dots,n_L\rangle_A |n_1,\dots,n_L\rangle_B,$$

where

$$p_{\{\dots,n_j+1,n_{j+1}-1,\dots\}} = p_{\{\dots,n_j,n_{j+1},\dots\}} + 1.$$

Notice that not only  $|\psi_m\rangle$  is a zero-energy eigenstate of both the kinetic and interacting parts of the Hamiltonian [23], but it is also a maximally entangled state of the form (4). Furthermore, the energy of this state remains constant for any value of the hopping parameter. Thus, an arbitrary time dependence of J(t) cannot influence the dynamics of this *MES*, making it quite exceptional since even strong fluctuations of the tunneling rate have no effect over it.

In what follows we consider the special case of a threesite ring (L = 3) and  $U_A = U_B = U = V$ . As shown in fig. 2(a), zero-energy eigenstates are observed for  $\tilde{\phi} = (2l-1)\pi/6$  (*l* integer), in agreement with eq. (6). In fig. 2(b) it is observed that for different phases  $\phi_A$  and  $\phi_B$  of the two species, there are families of zero-energy MES along the lines  $(\tilde{\phi}_A + \tilde{\phi}_B)/2 = (2l-1)\pi/6$ . This can be justified by generalizing the above calculation to the case  $\tilde{\phi}_A \neq \tilde{\phi}_B$ . Further exploration of the entanglement in the (U, V) parameter space for fixed  $\tilde{\phi} = \pi/2$  reveals that maximally entangled eigenstates are found only when V = U as displayed in fig. 4(a). Interestingly, other fringes of high entanglement are observed for  $V \simeq 0.55U$ and  $V \simeq 0.25U$ .

State preparation. – In general, bipartite entanglement of qudits is not easy to generate, and even once it is generated, it can be easily destroyed by noise, spontaneous emission, atomic collisions, etc. Nevertheless, as discussed above, in our proposed setup robust MESs exist for certain values of the Peierls phase. As evidenced on fig. 2(a) the MES is not the ground state of the system. Thus, the natural question that arises is how to prepare the system in such state. To this aim we propose a protocol that takes advantage of the navigation through the spectrum as the phase  $\phi$ , which can be varied continuously in experiments [17], is changed in time. First, let us observe in fig. 3(a) what happens to the spectrum when choosing J = U. Note that if initially we set  $\phi_i = 0$ , we could easily prepare the system in its ground state, since it is well separated from the first excited state. Now, let us consider a linear variation in time of the Peierls phase,  $\phi(t) = \alpha t$ . According to the adiabatic theorem, for a slow variation of  $\phi$  the system stays in the corresponding eigenstate (black curve in fig. 3(a)). In contrast, for large enough  $\alpha$  there is a transition to an excited state (red curve



Fig. 3: (Color online) (a) Energy spectrum E in units of J vs.  $\phi$  for L = 3, N = 2 and J = U = V. The two bands involved in the state preparation process are highlighted in black and in red. The blue arrows indicate the path followed to reach the zero-energy MES located at  $\phi = \pi/2$ . (b) Fidelity  $\mathcal{F}$  as a function of the velocity parameter  $\alpha$ . Dotted and dashed lines for J = 8U and J = 4U, respectively. Diamond and square mark the maximum fidelity in each case. The time dependence of the entanglement (Schmidt number (7)) during the preparation process for the values of  $\alpha$  and U corresponding to the diamond and square marks is shown in the inset. The right panel displays the optimal fidelity as a function of the interspecies interaction V in units of J. The squares ( $\Box$ ) and circles ( $\circ$ ) correspond to N = 2 and N = 4 particles per species, respectively.



Fig. 4: (Color online) (a) Entanglement ( $\mathcal{K}$ ) of the eigenstate with the highest entanglement as a function of the interaction parameters U and V. (b) Renormalized current ( $\tilde{\mathcal{J}}$ ) of the same eigenstate as in (a). In both plots  $\phi = \pi/2$  and the dashed line corresponds to U = V. On this line, the diamond ( $\diamond$ ) marks the point J = 8U, whereas the square ( $\Box$ ) indicates J = 4U.

in fig. 3(a)) as one passes through the avoided crossing located at  $\phi_c \simeq \pi/3$ . The probability of finding the system in the excited state after going through the avoided crossing is given approximately by the Landau-Zener probability  $P_{LZ} = e^{-\frac{2\pi\Delta^2}{\alpha\hbar}}$  [24] where  $\Delta$  is the gap at the avoided crossing. Thus, for  $\alpha \gg 2\pi\Delta^2/\hbar$  the transition probability is  $P_{LZ} \simeq 1$  and after passing through the avoided crossing the wave function of the system will overlap strongly with the excited state. As a result, if the velocity  $\alpha$  is set to zero when  $\phi(t) = \pi/2$ , the final state  $|\psi_f\rangle$  will be very similar to the desired MES (see fig. 3(a)). A good measure of the quality of the preparation of the target state (MES) is provided by the fidelity  $\mathcal{F} = |\langle MES | \psi_f \rangle|^2$ . The velocity parameter  $\alpha$  can be tuned to maximize  $\mathcal{F}$ . For instance, choosing J = 8U an optimal value of  $\mathcal{F} \simeq 0.9$  is reached for  $\alpha \simeq 0.02 J/\hbar$ , as shown in fig. 3(b).

To track the evolution of the degree of entanglement during the preparation process we use the so-called Schmidt number  $\mathcal{K}_0 = 1/Tr_B(\rho_B^2)$ , where  $\rho_B$  is the reduced density matrix for subsystem *B* [25,26]. For convenience, we define the normalized Schmidt number

$$\mathcal{K} = (\mathcal{K}_0 - 1)/(d - 1), \quad 0 \le \mathcal{K} \le 1,$$
 (7)

so that the maximum degree of entanglement is found at  $\mathcal{K} = 1$ . As shown in fig. 3(b) an appropriate choice of parameters leads to high levels of entanglement of the order of  $\mathcal{K} \simeq 0.95$ . Such optimal value of the fidelity increases as the interaction V/J is decreased (right panel in fig. 3(b)). Nevertheless, as the interspecies interaction is lowered, longer preparation times are required to reach high entanglement. This correlation between low interactions and long preparation times is expected since

the interspecies interaction is responsible for the entanglement formation. Furthermore, it can be inferred from our numerical simulations that the preparation time becomes inversely proportional to the interaction  $(t_{prep} \sim 1/V)$ . The right panel in fig. 3(b)) also shows that the fidelity decreases with the particle number. This is a result of the increasing number of avoided crossings between energy bands, which also scales rapidly with the size L of the ring. In this regard, the trimer with few particles appears to be the best choice for experimental observation of a MES using the above protocol.

Entanglement detection. - The amount of entanglement is not a directly observable quantity. In this regard it has been recently shown in a similar setup that high entanglement can be linked to a low current of particles [27]. The current operator for particles in the ring is  $\hat{\mathcal{J}} = \hat{\mathcal{J}}_A \otimes \hat{\mathbf{1}}_B + \hat{\mathbf{1}}_A \otimes \hat{\mathcal{J}}_B$ , where  $\hat{\mathcal{J}}_{A,B}$  are the respective current operators for each species, defined as  $\hat{\mathcal{J}}_D = -\frac{iJ}{\hbar L} \sum_{j=1}^L (e^{i\phi} \hat{d}_j^{\dagger} \hat{d}_{j+1} - e^{-i\phi} \hat{d}_{j+1}^{\dagger} \hat{d}_j)$  [28]. In fig. 4(b) we plot the renormalized current  $\tilde{\mathcal{J}} = 1 - \mathcal{J}\hbar/J$  corresponding to the eigenstate with the highest entanglement as a function of the interaction parameters U and V. Namely, a state with low current has  $\tilde{\mathcal{J}} \simeq 1$ . Comparison with fig. 4(a) shows a remarkable correlation between high entanglement and low current regions. In fact, the intensity plot for  $\mathcal{J}$  displays a fringe pattern very similar to the one observed for the entanglement in fig. 4(a). Notably, at  $2V = U_A + U_B$  the current vanishes for the corresponding MES. Furthermore, from fig. 4 it can be concluded that to detect highly entangled states the precision of the current measurement should be  $\Delta \mathcal{J} \lesssim 10^{-2} J/\hbar$ . It is worth mentioning that to enhance the signal in an experimental setup it could be necessary to replicate the system many times. In this regard, time of flight techniques [29] could be used to estimate the entanglement between two species of ultracold atoms in an array of rings, as suggested in [27].

**Conclusions.** – We have demonstrated analytically the existence of non-trivial maximally entangled eigenstates between two species of bosons living on a lattice ring with a Peierls phase. It is remarkable to find an exact eigenstate of such a strongly interacting system, which could potentially belong to a more general class of models with similar quantum states. Moreover, our results show that gauge fields can be exploited to create a pair of maximally entangled qudits as inter-species and intra-species interactions and the Peierls phase are appropriately tuned. These states exhibit vanishing current and are zero-energy eigenstates of both the kinetic and interacting parts of the Hamiltonian, making them quite exceptional and robust. Moreover, we have proposed a protocol to reach the MES based on an appropriate manipulation of the Peierls phase. Finally, an indirect detection method of the MES is proposed by measuring the current of the particles.

It is worth mentioning that recent studies in Jaynes-Cummings ring lattices have implemented synthetic gauge fields within a Bose-Hubbard framework [30]. Furthermore, a theoretical proposal has shown that similar physics could be observed by considering two coupled rings [31].

\* \* \*

SR would like to thank SEBASTIAN EGGERT for fruitful discussions. SR is supported by the FONDECYT project No. 11110537. LMM and SR acknowledge financial support from the FONDECYT project No. 1110671 and MO is supported by the FONDECYT project No. 1140994. DS is supported by the French project No. ANR-09-BLAN-0098-01.

## REFERENCES

- BENNETT C. H., BRASSARD G., CREPEAU C., JOZSA R., PERES A. and WOOTTERS W. K., *Phys. Rev. Lett.*, **70** (1993) 1895; EKERT A. K., *Phys. Rev. Lett.*, **67** (1991) 661; DEUTSCH D., EKERT A. K., JOZSA R., MACCHIAVELLO C., POPESCU S. and SANPERA A., *Phys. Rev. Lett.*, **77** (1996) 2818.
- [2] GIOVANNETTI V., LLOYD S. and MACONE L., *Phys. Rev. Lett.*, **96** (2006) 010401; CRONIN A. D., SCHMIEDMAYER J. and PRITCHARD D. E., *Rev. Mod. Phys.*, **81** (2009) 1051; SIZE S. *et al.*, *J. Phys. B*, **38** (2005) S449.
- [3] KASZLIKOWSKI D., GNACIŃSKI P., ZUKOWSKI M., MIKLASZEWSKI W. and ZEILINGER A., *Phys. Rev. Lett.*, 85 (2000) 4418.
- [4] HORODECKI R., HORODECKI P., HORODECKI M. and HORODECKI K., *Rev. Mod. Phys.*, 81 (2009) 865.
- [5] HUELGA S. F., MACCHIAVELO C., PELLIZZARI T., EKERT A. K., PLENIO M. B. and CIRAC J. I., *Phys. Rev. Lett.*, 79 (1997) 3865.
- [6] MORSCH O. and OBERTHALER M., Rev. Mod. Phys., 78 (2006) 179.
- [7] INOUYE S. et al., Nature, **392** (1998) 151.
- [8] GROSS C. et al., Nature, 464 (2010) 1165; RIEDEL M. F. et al., Nature, 464 (2010) 1170.
- [9] WEINSTEIN J. D., BELOY K. and DEREVIANKO A., Phys. Rev. A, 81 (2010) 030302.
- [10] JAKSCH D. and ZOLLER P., New J. Phys., 5 (2003) 56.
- [11] DALIBARD J., GERBIER F., JUZELIUNAS G. and OHBERG P., *Rev. Mod. Phys.*, 83 (2011) 1523.
- [12] AIDELSBURGER M. et al., Phys. Rev. Lett., 107 (2011) 255301.
- [13] LIN Y.-J. et al., Nature, 462 (2009) 628.
- [14] GOLDMAN N. et al., Phys. Rev. Lett., 103 (2009) 035301.
- [15] GOLDMAN N. et al., Phys. Rev. Lett., 105 (2010) 255302.
- [16] JIMÉNEZ-GARCÍA K. et al., Phys. Rev. Lett., 108 (2012) 225303.
- [17] STRUCK J. et al., Phys. Rev. Lett., 108 (2012) 225304.
- [18] MIYAKE H. et al., Phys. Rev. Lett., 111 (2013) 185302.
- [19] AIDELSBURGER M. et al., Phys. Rev. Lett., **111** (2013) 185301.
- [20] HE Q. Y. et al., Phys. Rev. Lett., **106** (2011) 120405; BAR-GILL N., GROSS C., MAZETS I., OBERTHALER M. and KURIZKI G., Phys. Rev. Lett., **106** (2011) 120404.

- [21] THALHAMMER G. et al., Phys. Rev. Lett., **100** (2008) 210402.
- [22] CERF N. J., J. Mod. Opt., 47 (2000) 187; DURT T., KASZLIKOWSKI D. and ZUKOWSKI M., Phys. Rev. A, 64 (2001) 024101.
- [23] SHEMESH D., Linear Algebra Appl., **62** (1984) 11.
- [24] LANDAU L. D., Physik. Z. Sowjet., 2 (1932) 46; ZENER
   C., Proc. R. Soc. A, 137 (1932) 696.
   [25] CROPP B. BEATTWAY K. and EDEDIM J. H. J. Phys.
- [25] GROBE R., RZAZEWSKI K. and EBERLY J. H., J. Phys. B, 27 (1994) L503.
- [26] ORSZAG M., Quantum optics (Springer) 2008.

- [27] MORALES-MOLINA L., REYES S. A. and ORSZAG M., Phys. Rev. A, 86 (2012) 033629.
- [28] AMICO L., OSTERLOH A. and CATALIOTTI F., Phys. Rev. Lett., 95 (2005) 063201.
- [29] SALGER T. et al., Science, **326** (2009) 1241.
- [30] NUNNENKAMP A., KOCH JENS and GIRVIN S. M., New J. Phys., 13 (2003) 095008.
- [31] AMICO L., AGHAMALYAN D., CREPAZ H., AUKSZTOL F., DUMKE R. and KWEK L.-C., *Sci. Rep.*, 4 (2014) 4298, arXiv:1304.4615.