Cavity QED: a Quantum Trajectory Point of View

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Outlines

- Open quantum systems
- Dynamics and asymptotic states
- Quantum trajectories in optical cavities
- Localization towards squeezed states
- Conclusions

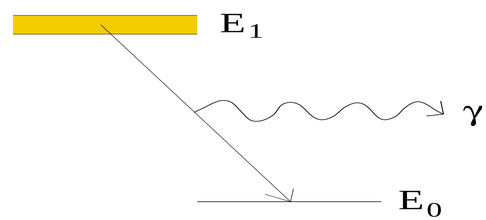
Joint work with M. Orszag

Outlines

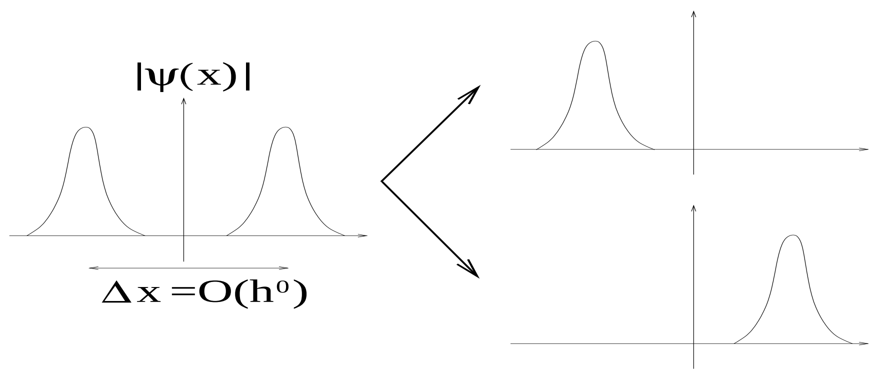
- Open quantum systems

Examples of Open Quantum Systems

1. Atom coupled to the electromagnetic field.



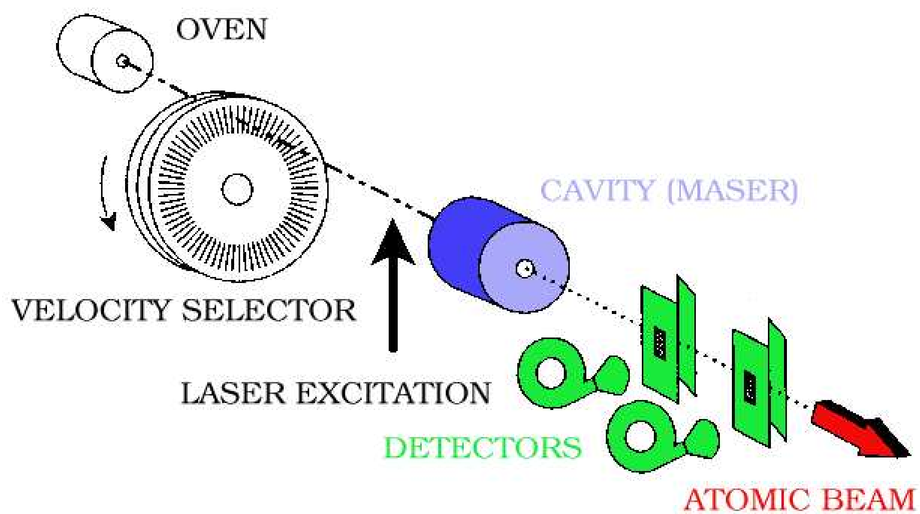
2. Macroscopic body coupled to its environment
   $\hookrightarrow$ decoherence effects: *quantum* $\longrightarrow$ *classical*.



3. Electrons in solids coupled to phonons
   $\hookrightarrow$ (energy) dissipation and decoherence in electronic transport.

Examples of Open Quantum Systems (2)

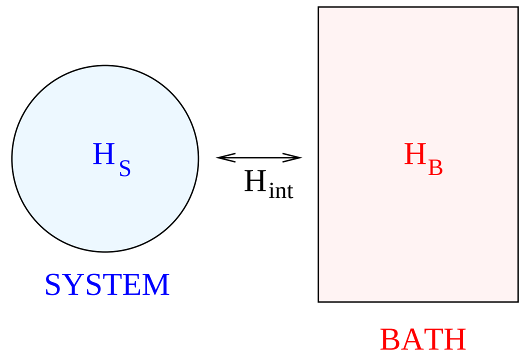
4. Electromagnetic field in an optical cavity crossed by an atomic beam



Micromaser experiments in Walther's and Haroche's groups
(Munich, Paris)

- Rydberg atoms enter one by one the cavity in excited state $|e\rangle$
- an atomic transition $e \to g$ is in resonance with one mode of the field in the cavity $\Rightarrow$ two-level atoms with states $|g\rangle, |e\rangle$
- a detector measures the state of each atom leaving the cavity.

System coupled to a bath



Evolution operator of the total system ‘S+B’:

$$U(t) = e^{-it(H_S + H_B + H_{\text{int}})}$$

Initial density matrix: $\rho_{S+B} = \rho_S \otimes \rho_B$ (no coupling at $t < 0$)

Reduced density matrix of ‘S’: $\rho_S(t) = \underset{B}{\mathrm{tr}}\big(U^\dagger(t)\rho_{S+B}\, U(t)\big)$

$$|\Psi_{S+B}\rangle = |\psi\rangle \otimes |\phi\rangle \quad \longrightarrow \quad U(t)|\Psi_{S+B}\rangle = \underbrace{\sum_j c_j(t)\, |\psi_j\rangle \otimes |\phi_j\rangle}_{\text{entangled state}}$$

$$\rho_S = |\psi\rangle\langle\psi| \quad \longrightarrow \quad \rho_S(t) = \underbrace{\sum_j |c_j(t)|^2\, |\psi_j\rangle\langle\psi_j|}_{\text{impure state}}$$

Properties of the bath

- contains infinitely many particles (*thermodynamic limit*)
- initially at thermal equilibrium with temperature $T = 1/\beta$:

$$\langle\,\cdot\,\rangle = \lim_{N\to\infty} \mathrm{tr}\big(\rho_B\,\cdot\,\big) \quad , \quad \rho_B = Z_B^{-1} e^{-\beta H_B}$$

- $H_{\text{int}} = \lambda\, S\, B$ $=$ system-bath interaction Hamiltonian.

  The (auto)correlation function

$$g(\tau) = \langle e^{i\tau H_B}\, B\, e^{-i\tau H_B}\, B\rangle - \langle B\rangle^2$$

  satisfies $|g(\tau)| \le C\, e^{-\tau/\tau_c}$ with a **correlation time** $\tau_c \ge \hbar\beta$

- the central limit theorem applies for the sum $B = \sum_n B_n$:

$$\langle e^{iBx}\rangle = \exp\Big(-\frac{1}{2}\langle B^2\rangle\, x^2\Big) \quad \text{(here } \langle B\rangle = 0 \text{ for simplicity)}$$

All these properties are satisfied for the free boson bath.

Times scales

H_S Hamiltonian of the system, eigenenergies $E_0 < ... < E_i < ...$

$H_{\text{int}} = \lambda\, S\, B$ interaction Hamiltonian

$$S|s\rangle = s|s\rangle$$

TIME SCALES

1. τ_c correlation time of the bath
2. $\tau_S \sim \hbar(E_{i+1} - E_i)^{-1}$ Heisenberg time of the system
   = time scale of the dynamics of the **uncoupled system**:
   $$\rho_S^{(0)}(t) = e^{-itH_S}\rho_S\, e^{itH_S}$$
3. $\tau_{\text{rel}} \propto \lambda^2$ inverse damping rate
   = time scale of evolution of the diagonal elements $\langle s|\tilde{\rho}_S(t)|s\rangle$
   of the density matrix $\tilde{\rho}_S(t)$ in the interaction picture w.r.t. H_S.
4. τ_{dec} decoherence time = time scale of the evolution of the
   off-diagonal elements $\langle s|\rho_S(t)|s'\rangle$

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- Dynamics and asymptotic states

Dynamics at small times: $t \ll \tau_c, \tau_S$

System-bath interaction Hamiltonian: $H_{\text{int}} = \lambda\, S\, B$

Evolution operator of 'S+B':

$$
\begin{aligned}
U(t) &= e^{-it(H_S + H_B + \lambda S B)} \\
&= e^{-it\lambda S B} e^{-itH_S} e^{-itH_B} \underbrace{e^{-t^2\lambda [H_S + H_B, SB]/2 + \dots}}_{\simeq 1}
\end{aligned}
$$

$$
\begin{aligned}
\langle s|\rho_S(t)|s'\rangle &= \langle s| \operatorname*{tr}_B \Big(U(t) \rho_S \otimes \rho_B\, U^\dagger(t) \Big) |s'\rangle \\
&\simeq \langle s| \operatorname*{tr}_B \Big(e^{-it\lambda s B} e^{-itH_S} \rho_S\, e^{itH_S} \otimes \rho_B\, e^{it\lambda s' B} \Big) |s'\rangle \\
&= \Big\langle e^{-it\lambda(s-s')B} \Big\rangle_\beta \langle s|\rho_S^{(0)}(t)|s'\rangle
\end{aligned}
$$

$$
\langle s|\rho_S(t)|s'\rangle \simeq \exp\Big(-\frac{t^2}{t_{\text{dec}}^2}\Big) \langle s|\rho_S^{(0)}(t)|s'\rangle \quad , \quad t_{\text{dec}} = \frac{\sqrt{2}\,\hbar}{\lambda \sqrt{\langle B^2 \rangle} |s - s'|}
$$

(Strunz, Haake & Braun '01)

Decoherence

$$\langle s|\rho_S(t)|s'\rangle \simeq \exp\left(-\frac{t^2}{t_{\text{dec}}^2}\right)\langle s|\rho_S^{(0)}(t)|s'\rangle \quad , \quad t_{\text{dec}} = \frac{\sqrt{2}\,\hbar}{\lambda\sqrt{\langle B^2\rangle}|s-s'|}$$

Assume Schrödinger cat pure state at $t=0$:

$$\rho_S = \big(\alpha\langle s| + \alpha'\langle s'|\big)\big(\alpha|s\rangle + \alpha'|s'\rangle\big)$$

with **macroscopically distinct states** $|s\rangle$ and $|s'\rangle$, i.e., $|s-s'| \gg 1$.

$$t_{\text{dec}} \ll \tau_c \ , \ \tau_S \ , \ \tau_{\text{rel}}$$

decoherence is the fastest process for large enough $|s-s'|$.

Dynamics at long times: markovian master equations

- **Born-Markov approximation:** $\lambda\tau_c \ll 1\ ,\ \tau_c \ll t$

  Redfield equation:

$$\frac{d\rho_S}{dt} = -i\left[H_S, \rho_S\right] + \lambda^2 \Big(\left[T\rho_S\,,\,S^\dagger\right] + \left[S\,,\,\rho_S\,T^\dagger\right] \Big)$$

$$T = \int_0^\infty d\tau\, g(-\tau)\, e^{-i\tau H_S} S e^{i\tau H_S}$$

  2^{nd} *perturbation theory + neglect memory effects.*

- **Adiabatic (or Rotating Wave) approximation:** $\tau_S \ll \tau_{\text{rel}}$

  The coarse-grained dynamics on **intermediate scale δt,**

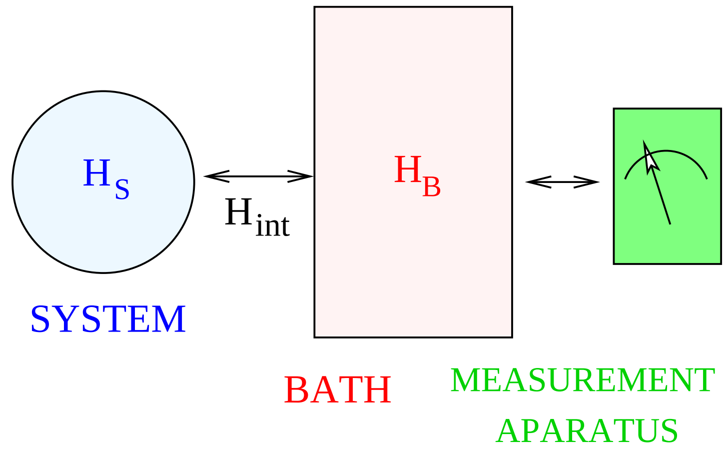
  $\tau_S \ll \delta t \ll \tau_{\text{rel}}$ is given by a **Lindbad equation**

$$\frac{\delta\rho}{\delta t} = -i\left[H_S, \rho\right] + \frac{1}{2}\sum_m \Big(\left[L_m\,\rho_S\,,\,L_m^\dagger\right] + \left[L_m\,,\,\rho_S\,L_m^\dagger\right] \Big)$$

  e.g., for the damped harmonic oscillator at finite temperature:

$$L_- = \sqrt{\gamma}\,a \quad , \quad L_+ = \sqrt{e^{-\beta\,\omega}\gamma}\,a^\dagger\ .$$

Quantum trajectories



(1) Between $t = 0$ and δt, 'S' and 'B' interact and **get entangled:**

$$|\Psi_{S+B}\rangle = |\psi\rangle \otimes |\phi\rangle \longrightarrow |\Psi_{\text{ent}}\rangle = \sum_j c_j\, |\psi_j\rangle \otimes |\phi_j\rangle$$

(2) At $t = \delta t$, **measurement on 'B'** $\Rightarrow \begin{cases} \textbf{state of 'S' modified} \\ \textbf{'S' and 'B' disentangled} \end{cases}$

$$|\Psi_{\text{ent}}\rangle \longrightarrow |\psi_j\rangle \otimes |\phi_j\rangle\ ,\quad j = \text{result of the measurement}$$

Evolution of the wavefunction of 'S', $|\psi_S(t)\rangle$, by repeating (1) & (2)

GIVEN a sequence of measurement results = QUANTUM TRAJECTORY.

Quantum jump model

(Dalibard-Castin-Mølmer, Carmichael '93)

Idea: describe the system by a **random wavefunction** $|\psi(t)\rangle$ instead of a density matrix ρ_S.

Random time evolution:

- **Quantum jumps** $|\psi\rangle \longrightarrow \frac{L_m|\psi\rangle}{\|L_m|\psi\rangle\|}$ occur at **random times** as result of measurements. Proba of a jump m between t and $t+\delta t$:

$$\delta p_m(t) = \|L_m|\psi(t)\rangle\|^2\,\delta t$$

- If no jump between t and $t+\delta t$, $|\psi(t+\delta t)\rangle = \frac{e^{-i\delta t(H_S+K)}|\psi(t)\rangle}{\|\cdots\|}$

  $K = (2i)^{-1}\sum_m L_m^\dagger L_m$ is **not self-adjoint!**

$\rho(t) = \mathsf{M}\,|\psi(t)\rangle\langle\psi(t)|$, M $=$ mean over the measurement results

$\hookrightarrow$ if $\delta p_m(t) \ll 1$, then $\rho(t)$ satisfies the Lindblad master equation:

$$\frac{\delta\rho}{\delta t} = -i\big[H_S,\rho\big] + \frac{1}{2}\sum_m\Big(\big[L_m\,\rho_S\,,\,L_m^\dagger\big] + \big[L_m\,,\,\rho_S\,L_m^\dagger\big]\Big)$$

Asymptotic states for large t

- **The total system 'S+B' is isolated:**

  $\hookrightarrow$ the state of 'S+B' converges to equilibrium

$$\Rightarrow \quad \rho_S(t) \to \frac{1}{Z} \operatorname*{tr}_B \left(e^{-\beta(H_S + H_B + H_{\mathsf{int}})} \right)$$

- **The system is driven by some external force:**

  the state of 'S' converges to a *non equilibrium steady state*

  with a nonzero (energy, charge,...) current.

- **The system is driven by continuous measurements on the bath:**

?

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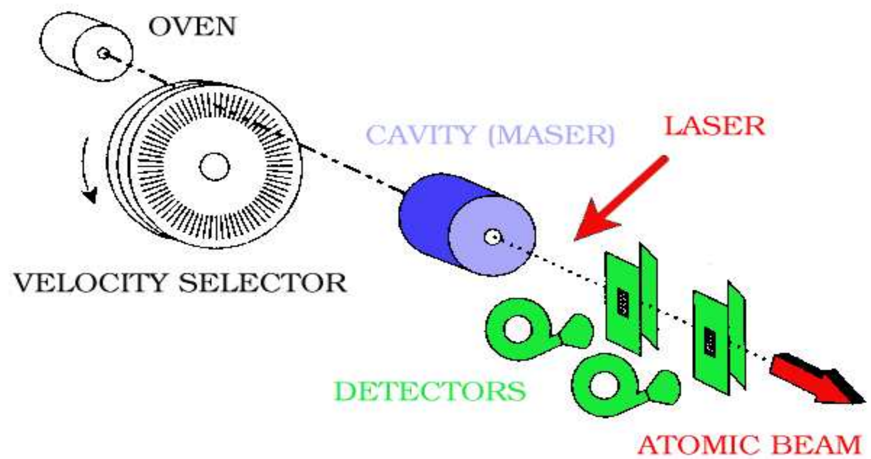
Modifi ed micromaser experiment



- The atoms enter **one by one** the cavity in either state $|g\rangle$ or $|e\rangle$, with fluxes r_g and $r_e = e^{-\hbar\omega/kT} r_g$ ($T =$ temperature of the atoms)
- Each atom interacts during time τ_{int} with the field via the coupling:

$$H_{\text{int}} = -i\lambda\big(|g\rangle\langle e|\, a^\dagger - |e\rangle\langle g|\, a\big)$$

- Each atom leaving the cavity first **interacts with a laser field**, e.g. via

$$H_L = -i\big(\Omega|g\rangle\langle e| - \Omega^*|e\rangle\langle g|\big)/2$$

then its **state is measured by a detector.**

$\Big\}$ *measurement in a rotated basis*

One atom crosses the cavity

State of the total system (**'atom + field'**) at the exit of the cavity:

$$|\Psi_{\text{ent}}\rangle = e^{-i\tau_L H_L}\, e^{-i\tau_{\text{int}} H_{\text{int}}} |\psi_{\text{field}}\rangle \otimes |i\rangle$$

$|\psi_{\text{field}}\rangle$ initial state of the field

$|i\rangle$ initial state of the atom

H_L, τ_L atom-laser Hamiltonian and interaction time.

State of the total system **after the measurement:**

$$|\Psi'\rangle = \underbrace{|\psi'_{\text{field}}\rangle \otimes |j\rangle}_{\textbf{atom and field disantangled}} \quad , \quad |\psi'_{\text{field}}\rangle = \frac{\langle j|\Psi_{\text{ent}}\rangle}{\|\langle j|\Psi_{\text{ent}}\rangle\|}$$

$|j\rangle$ measured state of the atom.

The atom enters the cavity in $|i\rangle$ and is measured $|j\rangle$ with probability

$$p_{i\to j} = r_i(r_g + r_e)^{-1} \|\langle j|\Psi_{\text{ent}}\rangle\|^2.$$

One atom crosses the cavity (2)

Dimensionless parameters of the problem:

$$\eta = \lambda\,\tau_{\text{int}} \qquad , \qquad \epsilon = \eta^{-1}\,\frac{\Omega}{|\Omega|}\,\tan\left(\frac{|\Omega|\,\tau_L}{2}\right)$$

$\Omega =$ Rabi frequency of the laser.

One must distinguish the **4 cases** $i, j = g, e$.

Non-perturbative solution:

e.g., for $i = g, j = e$ (absorption of a photon),

$$|\psi'_{\text{field}}\rangle \;\propto\; \epsilon^{-1}\Big(\widetilde{a} + \epsilon\,\cos(|\eta|\sqrt{a^\dagger a})\Big)|\psi_{\text{field}}\rangle \;,\; \widetilde{a} = a\,\operatorname{sinc}(|\eta|\sqrt{a^\dagger a})$$

Weak coupling limit: $|\eta|\sqrt{\langle a^\dagger a\rangle} \ll 1 \quad , \quad |\epsilon| \gg 1 \quad , \quad |\eta\,\epsilon| \approx 1$

$\hookrightarrow$ the crossing of one atom perturbs weakly the field

$$|\psi'_{\text{field}}\rangle \;\propto\; \left(1 + W_{g\to e}\right)|\psi_{\text{field}}\rangle \quad , \quad W_{g\to e} = \epsilon^{-1} a - \frac{|\eta|^2}{2} a^\dagger a\,.$$

Many atoms cross the cavity

Weak coupling limit $\quad |\eta|\sqrt{\langle a^\dagger a\rangle} \ll 1 \quad , \quad |\epsilon| \gg 1 \quad , \quad |\eta\,\epsilon| \approx 1$

Coarse-grained field dynamics with time resolution Δt,

$$\underbrace{(r_e + r_g)^{-1}}_{\text{time between consecutive atoms}} \quad \ll \;\Delta t\; \ll \quad |\eta|^{-1/2}(r_e + r_g)^{-1}$$

$$|\psi_{\text{fi eld}}(t+\Delta t)\rangle \;\propto\; \Big(1 + \Delta N_{g\to e}\, W_{g\to e} + 3 \text{ other terms}\Big)|\psi_{\text{fi eld}}(t)\rangle$$

$\Delta N_{g\to e}$: $\sharp$ atoms entering the cavity in $|g\rangle$ detected in $|e\rangle$ during Δt.

Quantum state diffusion (Itô)

$$\begin{aligned}|d\psi_{\text{fi eld}}\rangle \quad = \quad & \Big[\sqrt{\gamma_+}\big(e^{i\theta}\,a^\dagger - \mathrm{Re}\langle e^{i\theta}a^\dagger\rangle_t\big)\,dw_+ + \sqrt{\gamma_-}\big(e^{-i\theta}\,a - \mathrm{Re}\langle e^{-i\theta}a\rangle_t\big)\,dw_- \\ & + \mathrm{Re}\langle e^{i\theta}a^\dagger\rangle_t\Big(\gamma_+\,e^{i\theta}a^\dagger + \gamma_-\,e^{-i\theta}a - \frac{\gamma_- + \gamma_+}{2}\,\mathrm{Re}\langle e^{i\theta}a^\dagger\rangle_t\Big)dt \\ & - \frac{1}{2}\Big(\gamma_+\,aa^\dagger + \gamma_-\,a^\dagger a\Big)dt\Big]|\psi_{\text{fi eld}}(t)\rangle\end{aligned}$$

$\theta = \arg\epsilon \quad , \quad \gamma_\pm = r_{e,g}|\eta|^2 \quad , \quad dw_\pm =$ real Wiener processes.

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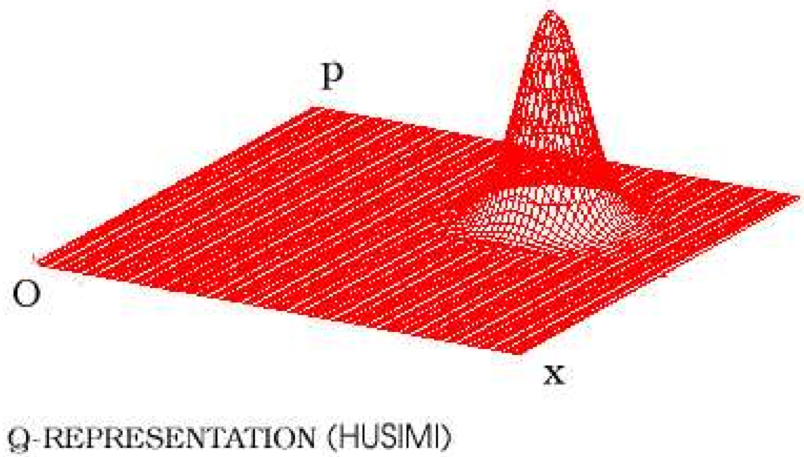
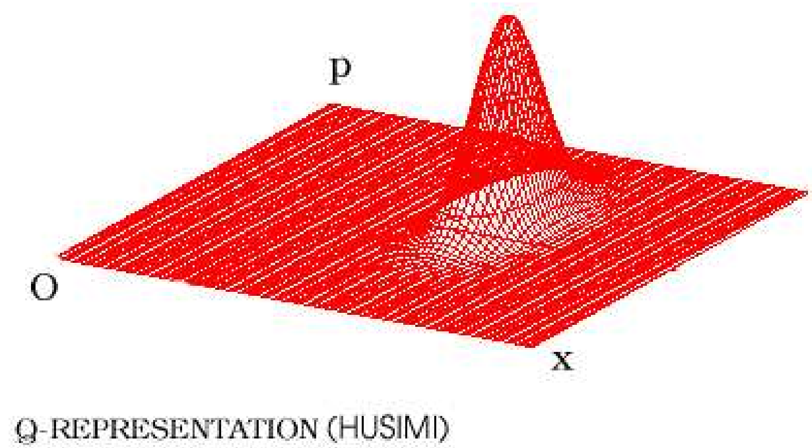
Squeezing





initial field state = coherent state: $\Delta x = \Delta p$

5000 atoms crossed the cavity:

field state = squeezed state

$\Delta x < \Delta p$

Squeezed state: $|\alpha, \xi\rangle = \underbrace{\exp\left(\alpha\, a^\dagger - \alpha^* a\right)}_{\text{displacement}} \underbrace{\exp\left(-\frac{\xi}{2} a^{2\dagger} + \frac{\xi^*}{2} a^2\right)}_{\text{squeezing}} |0\rangle$

The squeezing is non-random!

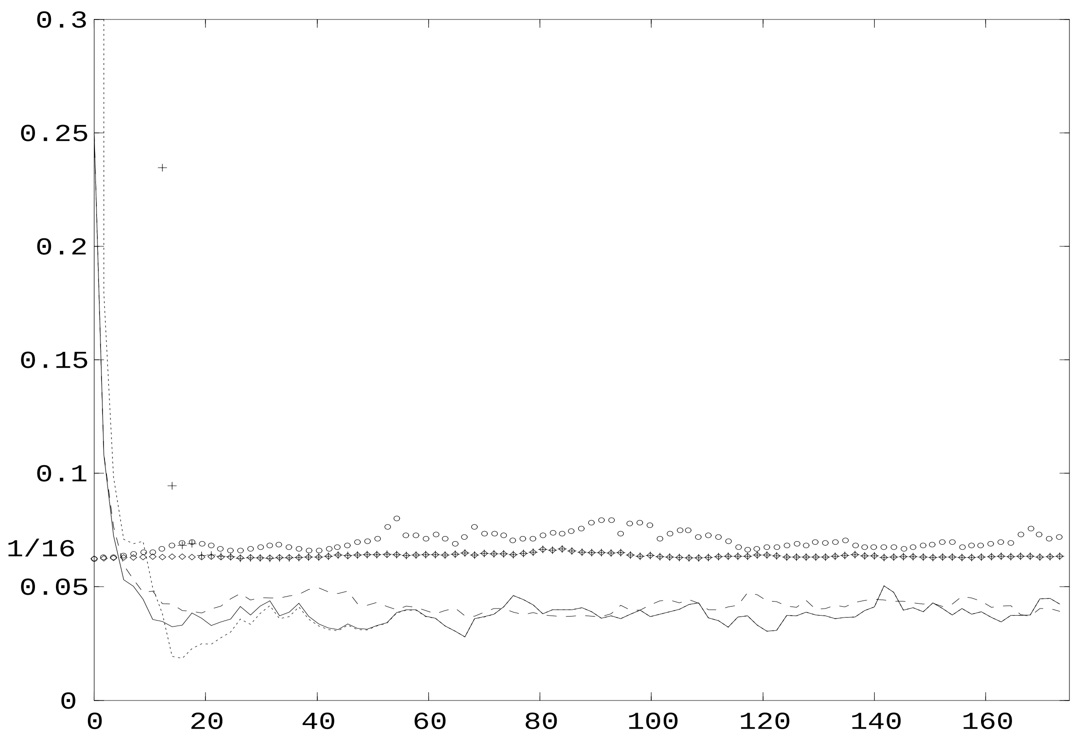
Squeezed states $|\psi(t)\rangle = |\alpha(t), \xi(t)\rangle$ are **special solutions** of the stochastic Schrödinger eq. for the coarse-grained dynamics, with

$$\begin{cases} \alpha(t) & \textbf{random} \\ \xi(t) & \textbf{deterministic} \end{cases}$$

(Rigo & Gisin, '96)

At large times, $\arg \xi(t) \to 2\,\theta \quad , \quad \tanh |\xi(t)| \to \dfrac{\gamma_+}{\gamma_-} = e^{-\hbar\omega/kT}$

Localization effect: consider trajectories with initial states $|\psi\rangle \neq |\alpha, \xi\rangle$



Δx^2 and $\Delta x^2 \Delta p^2$ vs t for trajectories with different initial states

$\epsilon = 20, \eta \simeq 0.06$,

$\gamma_+/\gamma_- = 3/4$

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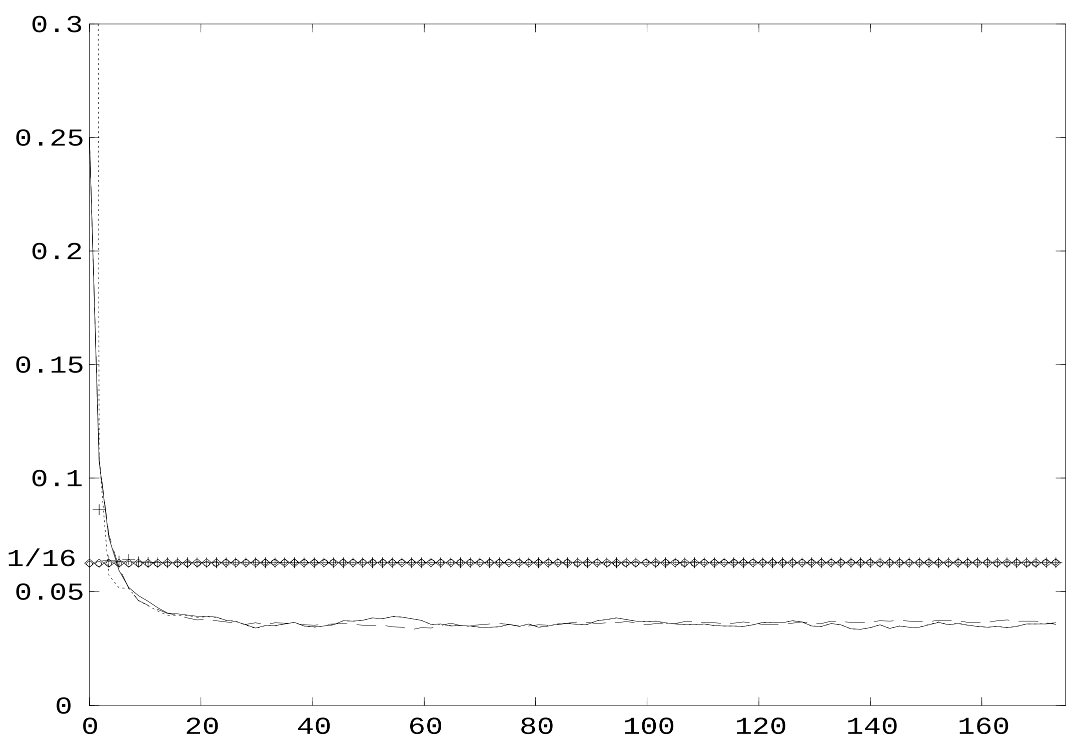
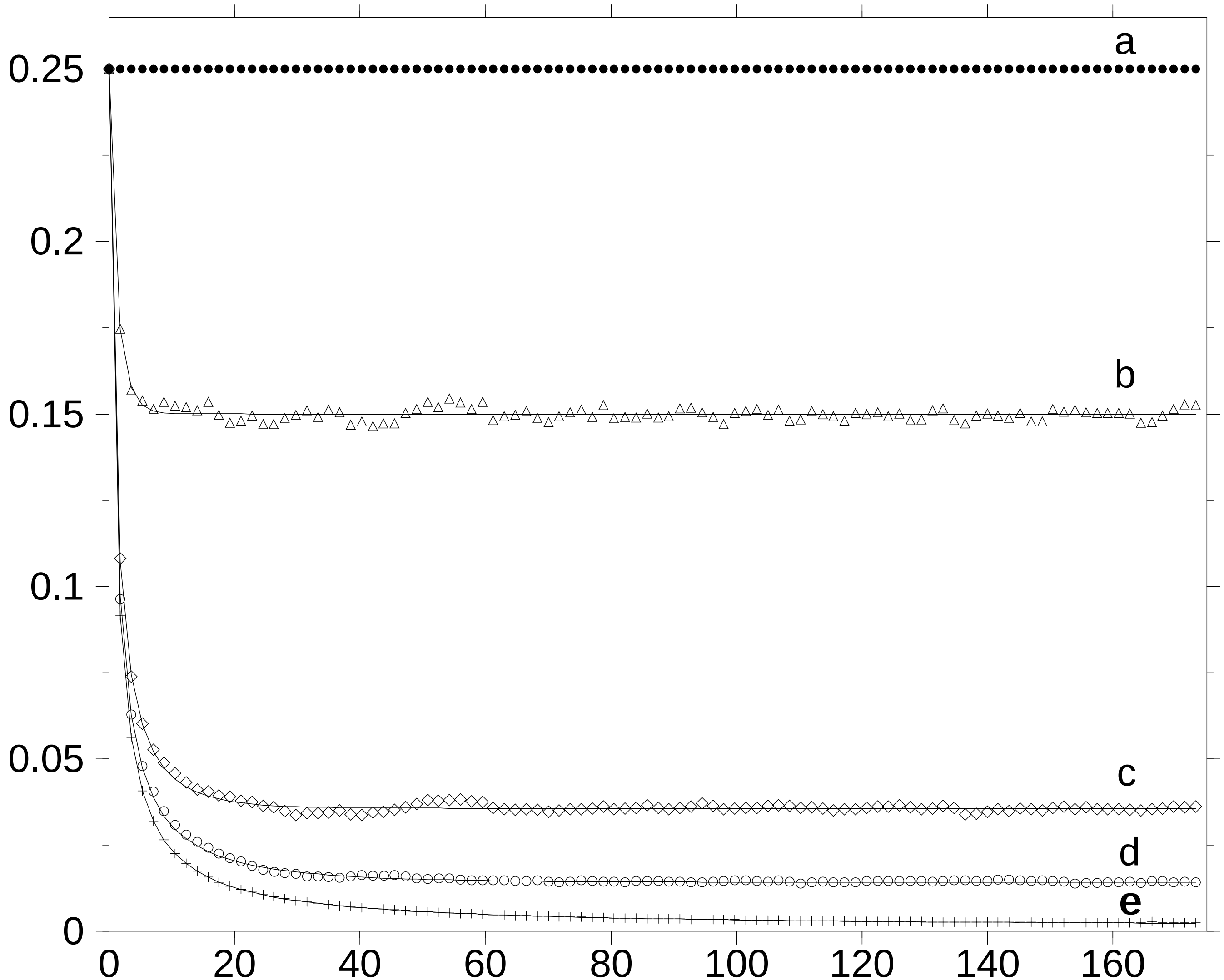


Δx^2 and $\Delta x^2 \Delta p^2$ vs t for trajectories with different initial states

$\epsilon = 100, \eta \simeq 0.01$

$\gamma_+/\gamma_- = 3/4$

Trajectories with different temperatures



Values of $\frac{r_e}{r_g}$ $a : 0$ $b : 0.25$ $c : 0.75$ $d : 0.89$ $d : 0.99$.

The fi nal squeezing depends only on T

Averaging over the results of the measurements

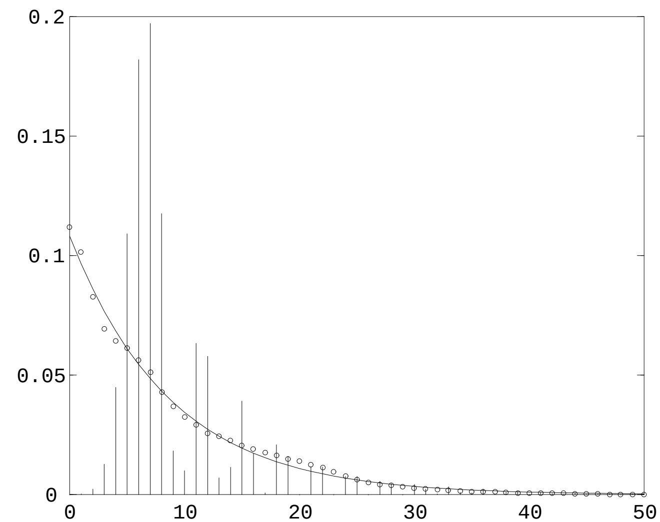
$\Leftrightarrow$ remove the detector (**no measurement**)

$\hookrightarrow$ **thermalization** of the field

$$\rho_{\text{fi eld}}(t) \equiv \mathrm{M}\ |\psi(t)\rangle\langle\psi(t)| \to \rho^{\text{eq}} \equiv \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\omega(n+0.5)/k_B T}|n\rangle\langle n|\,.$$

Photon number statistics:

$$\mathcal{P}_n(t) = \left|\langle n|\psi(t)\rangle\right|^2$$



ERGODICITY

$$\lim_{t\to\infty} \mathrm{M}\,\mathcal{P}_n(t) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}t'\,\mathcal{P}_n(t') \propto \left(e^{-\omega/k_B T}\right)^n$$

For $n \gg 1$, $\left|\langle n|\alpha,\xi\rangle\right|^2 \sim \underbrace{f_n(|\xi|,\alpha)}_{\text{rational function of } n}\ \underbrace{\cos^2\Phi_n(|\xi|,\alpha)}_{\text{oscillations}}\ \underbrace{\left(\tanh|\xi|\right)^n}_{\text{exponential decay}}$

$|\psi(t)\rangle \simeq |\alpha(t),\xi(t)\rangle$, $\xi(t)$ non-random $\Rightarrow \tanh|\xi(t)| \to e^{-\hbar\omega/k_B T}$.

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Conclusions

$N = (r_g + r_e)\,t \gg 1$ atoms cross the cavity and the detector

$$\begin{cases} \textbf{weak atom-field coupling:} & |\eta|\sqrt{\langle a^\dagger a\rangle} \ll 1 \\ \textbf{large atom-laser coupling:} & \epsilon \gg 1\ ,\ |\eta\,\epsilon| \approx 1\ (|\Omega|\tau_L \text{ finite}). \end{cases}$$

MAIN RESULTS:

- The **field** evolves to a **squeezed state indep. of the initial state**

$$N \gtrsim \eta^{-2} \quad \Rightarrow \quad |\psi(t)\rangle \simeq |\alpha(t), \xi(t)\rangle$$

- The **squeezing amplitude** $|\xi(t)|$ and **phase** $\arg \xi(t)$ are **independent of the results of the measurements** $\longleftrightarrow$ $\alpha(t)$ wanders randomly in the complex plane around 0.

- The **squeezing increases** with the **temperature** T **of the atoms**. Our theory predicts $\Delta x \to 0$ as $T \to \infty$ but for very small Δx one leaves the weak coupling regime $\hookrightarrow$ smallest reachable Δx.