Lecture 2:
Bipartite systems coupled to two baths:
Time-evolution of Entanglement

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Ex. of 2-qubit separable and classical states

\( \Diamond \) 2-qubit states with max. disordered marginals \( \rho_{A/B} = \frac{1}{2} \) can be written (up to conjugation by a local unitary \( U_A \otimes U_B \)) as

\[
\rho = \frac{1}{4} \left( 1 \otimes 1 + \sum_{m=1}^{3} c_m \sigma_m \otimes \sigma_m \right)
\]

\( \bar{c} \in T \) tetrahedron with vertices

\( F_{\pm} = (\pm 1, \mp 1, 1), \ G_{\pm} = (\pm 1, \pm 1, -1) \)

\( \Diamond \) \( \rho \) separable \( \iff \bar{c} \in \) octahedron \( IJKLMN \)

\( \Diamond \) Quantum discords

\[
\delta_A(\rho) = \sum_{\nu=0}^{3} p_\nu \ln p_\nu + \ln 4 - \frac{1 - |c|}{2} \ln(1 - |c|) - \frac{1 + |c|}{2} \ln(1 + |c|)
\]

\[
D_A(\rho) = 2 \left( 1 - \sqrt{\frac{1 + b_+ + b_-}{2}} \right), \ b_\pm = \frac{1}{2} \sqrt{(1 \pm c_1)^2 - (c_2 + c_3)^2}
\]

if \( |c| = \max_m |c_m| = |c_1| \) (else circular permut. of \( (c_1, c_2, c_3) \))

\( p_\nu = \) euclidean distance of the origin \( O \) to the faces of \( T \)
Outlines

- Evolution of the concurrence and quantum discord for the 2 spin-boson model
- Average concurrence for quantum trajectories
- Protecting entanglement with quantum trajectories
- Qubits coupled to a common bath
- Conclusions & Perspectives

Joint work with: Sylvain Vogelsberger
The 2 spin-boson model

[Yu and Eberly (’04); Merkli et al. (’10); Merkli (’11), ...]

• Consider two spins $A$ and $B$ coupled to two independent free-boson reservoirs $R_A$ and $R_B$. There are no interactions between $A$ & $B$. The total Hamiltonian is:

$$H_{\text{tot}} = H_A + H_B + H_{R_A} + H_{R_B} + \lambda H_{\text{int}}$$

$$H_A = \omega_A \sigma^A_z, \quad H_B = \omega_B \sigma^B_z$$

$$H_{R_A} = \sum_k \mu_k a_k^\dagger a_k$$

$$H_{R_B} = \sum_k \nu_k b_k^\dagger b_k$$

• Initial state: $\rho_{\text{tot}}(0) = \rho_{AB}(0) \otimes \rho_{R_A} \otimes \rho_{R_B}$

• MODEL 1: spin-boson coupling given by Jaynes-Cummings:

$$H_{\text{int}} = \sum_k \left( g_k \sigma^A_+ \otimes a_k + f_k \sigma^B_+ \otimes b_k + \text{h.c.} \right)$$
The 2 spin-boson model (2)

• $\rho_{R_k}$ Gibbs states with inverse temperatures $\beta_k < \infty$
  $\Rightarrow$ the 2-spin state converges at large times to
  $$\rho_{AB}(\infty) = Z^{-1} e^{-\beta_A \omega_A \sigma_z^A} \otimes e^{-\beta_B \omega_B \sigma_z^B} \text{ product state}$$

• $\rho_{AB}(\infty)$ is in the interior of the set of separable states $S$.
  **Reason:** $\rho_{AB} \in \partial S \iff \rho_{AB}^{T_B}$ has at least one zero eigenvalue
  (by the Peres-Horodecki criterium).
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  **Reason:** $\rho_{AB} \in \partial S \Leftrightarrow \rho_{AB}^{T_B}$ has at least one zero eigenvalue (by the Peres-Horodecki criterium).

- By continuity of $t \mapsto \rho_{AB}(t)$, the 2-spin state
  \[ \rho_{AB}(t) = \text{tr}_{R_A,R_B} (e^{-itH_{\text{tot}}} \rho_{\text{tot}}(0) e^{itH_{\text{tot}}}) \]
  become **separable** (cross $\partial S$) **after a finite time** $t_{ESD}$. 
The 2 spin-boson model (2)

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  $\Rightarrow$ the 2-spin state converges at large times to
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- $\rho_{AB}(\infty)$ is in the interior of the set of separable states $S$.
  **Reason:** $\rho_{AB} \in \partial S \iff \rho_{AB}^{TB}$ has at least one zero eigenvalue (by the Peres-Horodecki criterium).

- By continuity of $t \mapsto \rho_{AB}(t)$, the 2-spin state
  $$\rho_{AB}(t) = \text{tr}_{R_A, R_B} (e^{-itH_{\text{tot}}} \rho_{\text{tot}}(0) e^{itH_{\text{tot}}})$$
  become separable (cross $\partial S$) after a finite time $t_{ESD}$.

$\leftrightarrow$ can be checked by computing the concurrence of $\rho_{AB}(t)$ in the weak coupling limit [Yu and Eberly ('04),...], or by using the resonance perturbation theory [Merkli et al. ('10)].

$C(t)$ vs. $t$
2 spin-boson model: pure phase dephasing

- **MODEL 2:**

\[ H_{\text{int}} = \sum_k (g_k \sigma_z^A \otimes (a_k + a_k^\dagger) + f_k \sigma_z^B \otimes (b_k + b_k^\dagger)) \]

Weak coupling limit (but not necessary)

\[ \frac{d}{dt} \rho_{AB} = \gamma_z^A (\sigma_z^A \rho_{AB}(t)\sigma_z^A - \rho_{AB}(t)) + (A \leftrightarrow B) \]

\[ \rightarrow \text{no convergence to a NESS.} \]
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- **MODEL 2:**

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Weak coupling limit (but not necessary)

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\frac{d}{dt} \rho_{AB} = \gamma_z^A (\sigma_z^A \rho_{AB}(t) \sigma_z^A - \rho_{AB}(t)) + (A \leftrightarrow B)
\]

→ no convergence to a NESS.

Initial 2-spin state with maximally mixed marginals

\[
\rho_{AB}(0) = \frac{1}{4} \left( 1 \otimes 1 + \sum_{m=1}^{3} c_m \sigma_m \otimes \sigma_m \right), \quad \vec{c} \in \mathcal{T}
\]

↔ remains of this form at all times \( t \geq 0 \), with

\[
c_1(t) = e^{-2\gamma t} c_1, \quad c_2(t) = e^{-2\gamma t} c_2 \quad \text{and} \quad c_3(t) = c_3
\]

\[
\gamma = \gamma_z^A + \gamma_z^B.
\]
2 spin-boson model: pure phase dephasing

- **Concurrence:**

\[ C[\rho_{AB}(t)] = \frac{1}{2} \max\{0, |c_1 \pm c_2|e^{-2\gamma t} - 1 \mp c_3\} \]

Initial state entangled \(\iff |c_1 \pm c_2| > 1 \pm c_3 \) (± or −).

- For \(c_3 = \mp 1\) (e.g. initial Bell state \(|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\)),

\[ C[\rho_{AB}(t)] = |c_1 \pm c_2|e^{-2\gamma t} \text{ vanishes asymptotically.} \]

Otherwise entanglement is lost after a finite time

\[ t_{ESD} = \frac{1}{2\gamma} \ln\left(\max\{|c_1 \pm c_2|, |1 \pm c_3|\}\right). \]

- For \(c_1 = \pm 1, c_2 = \mp c_3\) with \(|c_3| < 1\),

\[ \delta_A[\rho_{AB}(t)] = \text{const. for} \quad 0 \leq t \leq -\frac{\ln|c_3|}{2\gamma}. \]

[Mazzola, Piilo & Maniscalco ('10)]

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Entanglement sudden death and birth

Entanglement typically disappears before coherences are lost!

It can disappear after a finite time

- always the case if the qubits relax to a Gibbs state $\rho_\infty$ at positive temperature
- otherwise depends on the initial state.

[Diosi ’03], [Dodd & Halliwell PRA 69 (’04)], [Yu et Eberly PRL 93 (’04)]
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\[ \rho_\infty \]

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- always the case if the qubits relax to a Gibbs state \( \rho_\infty \) at positive temperature
- otherwise depends on the initial state.

[Diosi ’03], [Dodd & Halliwell PRA 69 (’04)], [Yu et Eberly PRL 93 (’04)]

If the two qubits are coupled to a common bath, entanglement can also suddenly reappear

\[ C(t) \]

\( t \rightarrow \text{sudden birth} \)

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Outlines

- Evolution of the concurrence and quantum discord for the 2 spin-boson model
- Average concurrence for quantum trajectories
Quantum trajectories

As a result of continuous measurements on the environment, the bipartite system remains in a pure state $|\psi(t)\rangle$ at all times $t > 0$

\[ t \in \mathbb{R}_+ \mapsto |\psi(t)\rangle \quad \text{quantum trajectory} \]

**Reason:** each measurement disentangle the system and the environment (by wavepacket reduction).
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Averaging over the measurements, one gets the density matrix:

\[ \rho(t) = \frac{|\psi(t)\rangle\langle\psi(t)|}{\int \text{d}p[\psi] |\psi\rangle\langle\psi|} = \int \text{d}p[\psi] |\psi(t)\rangle\langle\psi(t)| . \]
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$$t \in \mathbb{R}_+ \mapsto |\psi(t)\rangle \text{ quantum trajectory}$$

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Averaging over the measurements, one gets the density matrix:

$$\rho(t) = |\psi(t)\rangle \langle \psi(t)| = \int dp[\psi] |\psi(t)\rangle \langle \psi(t)| .$$

In general this decomposition is **NOT THE OPTIMAL one,**

$$\overline{E_{\psi(t)}} \geq \overline{E_{\rho(t)}} \quad [\text{Nha & Carmichael PRL 98 ('04)].}$$

But for specific models, one can find measurement schemes with

$$\overline{C_{\psi(t)}} = \overline{C_{\rho(t)}} \quad \forall t \geq 0 \quad \text{with } C = \text{Wootters concurrence for 2 qubits}$$

[Carvalho et al. PRL 98 ('07), Viviescas et al. ('10)].
Two 2-level atoms (qubits) initially in state \( |\psi\rangle = \sum_{s,s'=0,1} c_{ss'} |s\rangle |s'\rangle \) are coupled to independent modes of the electromagnetic field initially in the vacuum.

Two perfect photon counters make a click when a photon is emitted by the atom \( i (i = A, B) \)
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Two perfect photon counters make a click when a photon is emitted by the atom $i$ ($i = A, B$)

- If $D_i$ detects a photon between $t$ and $t + dt$, the qubits suffer a quantum jump [occurs with proba. $\gamma_i ||\sigma_i^- |\psi(t)\rangle||^2 dt$]
  
  $|\psi(t)\rangle \longrightarrow \sigma_i^- |\psi(t)\rangle = |0\rangle_i \otimes |\phi(t)\rangle \rightsquigarrow \text{separable.}$
Photon counting

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- If no click occurs between $t_0$ and $t$ [proba. $\|e^{-itH_{\text{eff}}} |\psi(t_0)\rangle\|^2$]

  $$|\psi(t)\rangle = \frac{e^{-i(t-t_0)H_{\text{eff}}} |\psi(t_0)\rangle}{\|e^{-itH_{\text{eff}}} |\psi(t_0)\rangle\|}, \quad H_{\text{eff}} = H_0 - \frac{i}{2} \sum_{i=A,B} \gamma_i \sigma_+^i \sigma_-^i.$$
Photon counting (2)

Concurrence:

$$C_{\psi(t)} = |\langle \psi(t) | \sigma_y \otimes \sigma_y T | \psi(t) \rangle|$$

$$T = \text{complex conjugation op.}$$

$$\sigma_y = \text{Pauli matrix}$$

$$E_{\psi(t)} = h(C_{\psi(t)}), h \text{ convex} \uparrow$$

- Trajectories with 1 or more jumps between 0 and t have a concurrence $C_{\psi(t)} = 0$ (since $|\psi(t)\rangle$ separable after 1 jump).

- If no jump occurs between 0 and t, one finds for $H_0 = 0$:

  $$C_{\text{no jump}}(t) = N_t^{-2} C_0 e^{-(\gamma_A + \gamma_B)t} \text{ with } N_t = ||e^{-itH_{\text{eff}}} |\psi\rangle||.$$
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Average concurrence over all trajectories:

\[ \overline{C_{\psi(t)}} = \text{proba (no jump in } [0, t]) \times C_{\text{no jump}}(t) \]
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Average concurrence over all trajectories:

\[ \overline{C_{\psi(t)}} = \text{proba (no jump in } [0, t]) \times C_{\text{no jump}}(t) = C_0 e^{-(\gamma_A + \gamma_B) t} . \]

\( \leftrightarrow \overline{C_{\psi(t)}} \) vanishes asymptotically \( \Rightarrow \) sudden death of entanglement never occurs for quantum trajectories!
General quantum jump dynamics

Consider 2 noninteracting qubits coupled to *independent baths* monitored by means of *local measurements*

⇒ the jump operators $J = J^A \otimes 1$ or $1_A \otimes J_B$ are *local*.

- *The no-jump trajectories have a non-vanishing concurrence $C_{nj}(t) > 0$ at all finite times* (if $C_0 > 0$).

**Proof:** assume the contrary, i.e. $|\psi_{nj}(t)\rangle$ separable, then $|\psi(0)\rangle \propto e^{itH_{eff}} |\psi_{nj}(t)\rangle$ would be separable since $e^{itH_{eff}}$ is a tensor product of two local operators acting on each qubits.
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  **Proof:** assume the contrary, i.e. $|\psi_{nj}(t)\rangle$ separable, then $|\psi(0)\rangle \propto e^{itH_{eff}} |\psi_{nj}(t)\rangle$ would be separable since $e^{itH_{eff}}$ is a tensor product of two local operators acting on each qubits.

- **The average concurrence over all trajectories is**

  $$\overline{C_{\psi}(t)} = C_0 e^{-\kappa t}$$

  *where $\kappa \geq 0$ depends on the measurement scheme only (but not on initial state).*  
  **[Vogelsberger & D.S, PRA ('10)]**

**Note:** $\overline{E_{\psi(t)}} \geq h(\overline{C_{\psi(t)}})$ by convexity of $h$. 

Quantum state diffusion

• The result $\overline{C_\psi(t)} = C_0 e^{-\kappa t}$ is not only true for quantum jump dynamics but also for quantum state diffusion, e.g. for trajectories given by the stochastic Schrödinger equation

$$|d\psi\rangle = \left[ (-iH_0 - K)dt + \sum_{J \text{ local}} \gamma_J \left( \Re \langle J \rangle \psi \, J - \frac{1}{2} (\Re \langle J \rangle \psi)^2 \right) dt + \sum \sqrt{\gamma_J} \left( J - \Re \langle J \rangle \psi \right) dw \right] |\psi\rangle$$

which describes homodyne detection.

• The disentanglement rates $\kappa$ are different for photon-counting, homodyne, and heterodyne detections:

$$\kappa_{QJ} = \frac{1}{2} \sum_{J} \gamma_J \left( \text{tr}(J^\dagger J) - 2|\text{det}(J)|\right)$$

$$\kappa_{ho} = \frac{1}{2} \sum_{J} \gamma_J \left( \text{tr}(J^\dagger J) - 2\Re \text{det}(J) - (\Im \text{tr}(J))^2 \right)$$

$$\kappa_{het} = \frac{1}{2} \sum_{J} \gamma_J \left( \text{tr}(J^\dagger J) - \frac{1}{2} |\text{tr}(J)|^2 \right).$$

Adjusting the laser phases $J \rightarrow e^{-i\theta} J$ yields $\kappa_{ho} \leq \kappa_{QJ}, \kappa_{het}$. 
Discussion

It is **not possible** to have $C_{\psi}(t) = C_{\rho}(t)$ if one measures locally the independent environments of the qubits (since $C_{\rho}(t)$ may vanish at a finite time $t_{ESD}$, whereas $\bar{C}_{\psi}(t) > 0 \ \forall \ t$).

$\rightarrow$ To prepare the separable pure states in the decomp. of $\rho(t)$ at time $t_{ESD}$, one must necessarily perform nonlocal (joint) measurements on the 2 environments!
Discussion

It is **not possible** to have \( C_\psi(t) = C_\rho(t) \) if one measures locally the independent environments of the qubits (since \( C_\rho(t) \) may vanish at a finite time \( t_{\text{ESD}} \), whereas \( C_\psi(t) > 0 \ \forall \ t \)).

\[ \rightarrow \text{To prepare the separable pure states in the decomp. of } \rho(t) \text{ at time } t_{\text{ESD}}, \text{ one must necessarily perform nonlocal (joint) measurements on the 2 environments!} \]

* This raises the question: **is ESD observable?**

  [Almeida et al., Science 316 ('07)]. \[ \rightarrow \text{simulation of master eq.} \]
  [Viviescas et al., arXiv:1006.1452]. \[ \rightarrow \text{YES with some nonlocal measurements } \Rightarrow \text{ require additional quantum channels...} \]

* For **A-B entanglement**, “ignoring” the environment state is not the same as measuring it without reading the results.

  [Mascararenhas et al., arXiv:1006.1233].
Outlines

- Evolution of the concurrence and quantum discord for the 2 spin-boson model
- Average concurrence for quantum trajectories
- Protecting entanglement with quantum trajectories
Entanglement protection

One may use the continuous monitoring by the measurements to protect the qubits against disentanglement.

- For ex., for pure phase dephasing ($J^i = u_i \cdot \sigma^i$, $i = A, B$),
  $\kappa_{QJ} = \kappa_{ho} = \kappa_{het} = 0$ so that $C_\psi(t) = C_0 = \text{const.}$

Bell initial state

$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + e^{-i\varphi} |\downarrow\downarrow\rangle)$

$C_0 = 1 \Rightarrow C_\psi(t) = 1$ for all quantum trajectories and all times

$\leftrightarrow$ perfect entanglement protection!
Entanglement protection (2)

One may use the continuous monitoring by the measurements to protect the qubits against disentanglement.

- For ex., for pure phase dephasing ($J^i = u_i \cdot \sigma^i$, $i = A, B$), $\kappa_{QJ} = \kappa_{ho} = \kappa_{het} = 0$ so that $C_\psi(t) = C_0 = \text{const.}$

- For two baths at inv. temper. $\beta_i < \infty$, the smallest rate is $\kappa_{QJ} = \sum_{i=A,B} \gamma^i (e^{\beta_i \omega_0/2} - 1)^2$ (jump op. $J \propto \sqrt{\gamma_-^i \sigma_-^i + \sqrt{\gamma_+^i \sigma_+^i}}$)

Bell initial state

$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - i|\downarrow\downarrow\rangle)$

$\overline{C_\psi(t)} = e^{-\kappa t}$

$\leftrightarrow$ perfect entanglement protection only possible at infinite temperature!
Outlines

- Evolution of the concurrence and quantum discord for the 2 spin-boson model
- Average concurrence for quantum trajectories
- Protecting entanglement with quantum trajectories
- Qubits coupled to a common bath
Two 2-level atoms (qubits) initially in state
\[ |\psi\rangle = \sum_{s,s'} c_{ss'} |s\rangle |s'\rangle \]
are coupled to the same modes of the electromagnetic field initially in the vacuum.

\[
\overline{C}_\psi(t) = \frac{1}{2} |c_-^2 - c_+^2 e^{-2\gamma t} + 4c_{11}c_{00} e^{-\gamma t}| + 2|c_{11}|^2 \gamma t e^{-2\gamma t}
\]

with \( c_\pm = c_{11} \pm c_{00} \).

- If \( c_{11} = 0 \) then \( \overline{C}_\psi(t) = C_\rho(t) \).
- If \( c_{11} > 0 \) then \( \overline{C}_\psi(t) \) increases at small times.
Outlines

• Evolution of the concurrence and quantum discord for the 2 spin-boson model
• Average concurrence for quantum trajectories
• Protecting entanglement with quantum trajectories
• Qubits coupled to a common bath
• Conclusions & Perspectives
Conclusions & Perspectives

• The mean concurrence $\overline{C(t)}$ of two qubits coupled to independent baths monitored by continuous local measurements decays exponentially with a rate depending on the measurement scheme only.

• Measuring the baths helps to protect entanglement, sometimes perfectly!

• For two qubits coupled to a common bath, the time behavior of the mean concurrence depends strongly on the initial state. One may have $\overline{C(t)} = C_\rho(t)$.

Open problems: non-Markov unravelings, multipartite systems,...