J. Phys. A: Math. Gen. **32** (1999) L227–L230. Printed in the UK PII: S0305-4470(99)00514-4

LETTER TO THE EDITOR

Weyl expansion of a circle billiard in a magnetic field

R Narevich† and D Spehner‡

† Department of Physics, University of Maryland, College Park, MD 20742, USA
‡ Laboratoire de Physique Quantique, Université Paul Sabatier, F-31062 Toulouse Cedex 4, France

Received 23 December 1998, in final form 12 March 1999

Abstract. We compute the high orders of the Weyl expansion for the heat kernel of a circle billiard in the presence of a uniform and perpendicular magnetic field. It is shown, in accordance with a conjecture made in Narevich *et al* (1998 *J. Phys. A: Math. Gen.* **31** 4277), that some terms of this expansion can be identified with those of the Weyl expansion of a semi-infinite cylinder. The boundary correction to the Landau diamagnetic susceptibility of a non-degenerate electron gas in the billiard is determined.

Consider a spinless particle of charge $-e$ ($e > 0$) and mass m constrained by a hard wall to move inside a disc of radius a. A uniform magnetic field $\vec{B}$ is applied perpendicularly to the disc. Let $\omega = e\|\vec{B}\|/mc$, $l_B = \sqrt{\hbar/m\omega}$ and $R = \sqrt{2}a/l_B = \sqrt{2ma^2\omega/\hbar}$ be respectively the cyclotron frequency, the magnetic length and the dimensionless radius. In the symmetric gauge $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$, the Hamiltonian of the particle is:

$$H = \hbar\omega \left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\left(-\mathrm{i}\partial_\theta + \frac{r^2}{4} \right)^2 \right) \tag{1}$$

where (r, θ) are the dimensionless polar coordinates defined by $r = \sqrt{2}\|\vec{x}\|/l_B$; θ is the angle between $\vec{x}$ and a fixed vector $\vec{e}_x$ parallel to the plane of the disc, counted positively if the triad $(\vec{e}_x, \vec{x}, \vec{B})$ is right-handed. The eigenfunctions $\psi_n(r, \theta)$ of H are required to be finite as $r \to 0$ and to satisfy the Dirichlet boundary condition $\psi_n(R, \theta) = 0, 0 \leqslant \theta < 2\pi$. In this letter, we describe an algorithm to compute the Weyl asymptotic expansion of the heat kernel $P(t) = \operatorname{tr} \mathrm{e}^{-(t/\hbar)H}$ for this system. This also determines the Weyl expansions of other simply related spectral quantities, like e.g. the density of states [3]. The first few terms of the asymptotic expansion of $P(t)$ as $\tau = \omega t \to 0$ are:

$$\begin{aligned} P\left(t = \frac{\tau}{\omega}\right) \sim{}& \frac{R^2}{4}\left(\frac{1}{\tau} - \frac{\tau}{24} + \frac{7\tau^3}{5760} - \frac{31\tau^5}{945 \times 2^{10}} + \cdots \right) \\ & - \frac{\sqrt{\pi}R}{4}\left(\tau^{-\frac{1}{2}} - \frac{3\tau^{\frac{3}{2}}}{64} + \frac{25\tau^{\frac{7}{2}}}{2^{14}} - \frac{7309\tau^{\frac{11}{2}}}{315 \times 2^{19}} + \cdots \right) \\ & + \frac{1}{6}\left(1 - \frac{3\tau^2}{56} + \frac{757\tau^4}{3003 \times 2^7} - \frac{104\,971\tau^6}{1616\,615 \times 2^{10}} + \cdots \right) \\ & + \frac{\sqrt{\pi}}{2^7 R}\left(\tau^{\frac{1}{2}} - \frac{7\tau^{\frac{5}{2}}}{2^7} + \frac{83\tau^{\frac{9}{2}}}{5 \times 2^{13}} - \cdots \right) \end{aligned}$$

0305-4470/99/190227+04$19.50 © 1999 IOP Publishing Ltd

$$+\frac{2}{315R^2}\left(\tau-\frac{69\tau^3}{1144}+\frac{14\,431\tau^5}{46\,189\times 2^7}-\cdots\right)$$
$$+\frac{37\sqrt{\pi}}{2^{14}R^3}\left(\tau^{\frac{3}{2}}-\frac{393\tau^{\frac{7}{2}}}{5920}+\cdots\right)+\frac{136}{45\,045R^4}\left(\tau^2-\frac{3203\tau^4}{43\,928}+\cdots\right)+\cdots. \quad (2)$$

The Weyl expansion in the zero field is obtained by keeping only the first term in each parenthesis and coincides with known results [2, 3]. The terms proportional to R^2 (first parenthesis) give the Weyl expansion of the heat kernel $P_\infty(t)$ associated with the Landau spectrum (infinite plane geometry), whose full asymptotic expansion can be easily calculated [3, 1]. The terms proportional to R (second parenthesis) coincide with the first terms of the Weyl expansion of $P_{per}(t)-P_\infty(t)$, here $P_{per}(t)$ is the heat kernel of a semi-infinite cylinder of radius a in a uniform magnetic field [1]. This result is in agreement with a conjecture made in [1], according to which each term of the Weyl expansion of a billiard with a smooth boundary in zero field becomes multiplied by a universal billiard-independent function of $\tau=\omega t$ if a uniform magnetic field is applied perpendicularly. The first two functions, multiplying respectively the area term $(R^2/4)\tau^{-1}$ and the perimeter term $-(\sqrt{\pi}R/4)\tau^{-1/2}$, have been found in [1] to be $(\tau/2)/\sinh(\tau/2)$ and

$$2\sqrt{\frac{\tau}{\pi}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{d}\epsilon}{2\mathrm{i}\pi}\mathrm{e}^{\epsilon\tau}\int_{-\infty}^{\infty}\mathrm{d}x\,(\partial_\epsilon\ln D_{-\epsilon-\frac{1}{2}}(x)+\tfrac{1}{2}\psi(\epsilon+\tfrac{1}{2}))$$

where $c>0$ and $D_{-\epsilon-1/2}$, ψ are respectively the parabolic cylinder and the digamma functions. We use this opportunity to correct an error made in the Weyl expansion of $P_{per}(t)$ in [1], formula (29): the correct power of 2 in the denominator of the term of order $\tau^{11/2}$ should be 19 as in (2), instead of 20.

Since $P(t)$ for $t=\hbar\beta$ is the canonical partition function, one can determine from (2) the magnetic susceptibility χ of an ideal non-degenerate gas in the disc at inverse temperature β. If N is the number of particles per unit area and $\lambda_T=\sqrt{\pi\hbar^2\beta/2m}\ll N^{-1/2}$ is the de Broglie thermal length, we obtain an expansion of χ in powers of λ_T/a which begins as follows:

$$\frac{\chi}{\chi_\infty}=1-\frac{\lambda_T}{8a}-\left(\frac{1}{8}-\frac{4}{21\pi}\right)\frac{\lambda_T^2}{a^2}-\left(\frac{1}{8}-\frac{3049}{10\,752\pi}\right)\frac{\lambda_T^3}{a^3}$$
$$-\left(\frac{1}{8}-\frac{1329}{3584\pi}+\frac{248}{2145\pi^2}\right)\frac{\lambda_T^4}{a^4}+\cdots \quad (3)$$

where $\chi_\infty=-N\beta e^2\hbar^2/12m^2c^2$ is the Landau susceptibility. We have found that at each order in λ_T/a up to the fifth order, the corrections to the Landau diamagnetic susceptibility are paramagnetic.

To show (2), we use a Green function approach [1–3]. The Green function $G(E;r,\theta;r',\theta')$ is given by:

$$(H+E)G(E;r,\theta;r',\theta')=\frac{2}{rl_B^2}\delta(r-r')\delta(\theta-\theta') \quad (4)$$

(note the + sign in front of the energy E). It satisfies the boundary condition: $G(E;R,\theta;r',\theta')=G(E;r,\theta;R,\theta')=0$, and we require moreover that it be finite as $r\to 0$ and $r'\to 0$. One defines similarly the 'infinite plane' Green function $G_\infty(E;r,\theta;r',\theta')$ which satisfies the same equation, is finite at the origin, and is such that $r\mapsto rG_\infty(E;r,\theta;r',\theta')$ and $r'\mapsto r'G_\infty(E;r,\theta;r',\theta')$ be integrable on $\mathbb{R}_+$. Set $\epsilon=E/\hbar\omega$ and let $f_l^{\pm}(\epsilon,r)$ be two independent solutions of

$$\left(\partial_r^2+\frac{1}{r}\partial_r+Q_l^2(\epsilon,r)\right)f_l(\epsilon,r)=0 \qquad Q_l^2(\epsilon,r)=-\epsilon-\frac{1}{r^2}\left(l+\frac{r^2}{4}\right)^2. \quad (5)$$

The conditions on these functions are that $\lim_{r\to 0} f_l^-(\epsilon, r) < \infty$ and that $r \mapsto r f_l^+(\epsilon, r)$ be integrable on $[c, \infty[$ for any $c > 0$. We are interested in solutions of (5) of the following form:

$$f_l^{\pm}(\epsilon, r) = (r q_l(\epsilon, r))^{-\frac{1}{2}} \mathrm{e}^{\pm \mathrm{i} \int_{r_0}^{r} \mathrm{d}r' \, q_l(\epsilon, r')} \tag{6}$$

where $r_0 > 0$ and the imaginary parts of the complex functions $q_l(\epsilon, r)$ tend to $+\infty$ as $r \to \infty$ or $r \to 0$. Using (1) and expanding the Green functions as Fourier series in $\theta - \theta'$, one gets:

$$\begin{aligned} G_\infty(E; r, \theta; r', \theta') &= -\frac{m}{\pi\hbar^2} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}l(\theta-\theta')}}{W_l(\epsilon)} f_l^-(\epsilon, \min\{r, r'\}) f_l^+(\epsilon, \max\{r, r'\}) \\ G(E; r, \theta; r', \theta') &= G_\infty(E; r, \theta; r', \theta') + \frac{m}{\pi\hbar^2} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}l(\theta-\theta')} f_l^+(\epsilon, R)}{W_l(\epsilon) f_l^-(\epsilon, R)} f_l^-(\epsilon, r) f_l^-(\epsilon, r'). \end{aligned} \tag{7}$$

The Wronskian $W_l = r f_l^- \partial_r f_l^+ - r f_l^+ \partial_r f_l^-$ in the denominator is independent of r. The Laplace transform $\Delta g(E)$ of $\Delta P(t) = P(t) - P_\infty(t)$ is related to $G(E; r, \theta; r', \theta)$ and $G_\infty(E; r, \theta; r', \theta)$ by [2]:

$$\begin{aligned} \Delta g(E) &= \int_0^\infty \frac{\mathrm{d}t}{\hbar} \mathrm{e}^{-Et/\hbar} \Delta P(t) \\ &= \frac{l_B^2}{2} \int_0^R r \, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \, (G(E; r, \theta; r, \theta) - G_\infty(E; r, \theta; r, \theta)). \end{aligned} \tag{8}$$

Manipulating equation (5) in a standard way one shows that

$$\int_0^R r \, \mathrm{d}r \; f_l^-(\epsilon, r)^2 = R(f_l^- \partial_R \partial_\epsilon f_l^- - (\partial_R f_l^-)(\partial_\epsilon f_l^-))(\epsilon, R). \tag{9}$$

Using (6)–(9) and the Poisson summation formula, one obtains after some algebra:

$$\hbar\omega \Delta g(E) = \sum_{\nu=-\infty}^{\infty} \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}l \, \mathrm{e}^{2\mathrm{i}\pi\nu l} \left(-\partial_\epsilon + \frac{\mathrm{i}}{2q_l} \partial_R \partial_\epsilon \right) \ln q_l(\epsilon, R). \tag{10}$$

The small t expansion of $\Delta P(t)$ can easily be found from the large E expansion of $\Delta g(E)$ and the reciprocal of Watson's lemma [4]. In order to determine this large E expansion, we set $y_l(\epsilon, r) = r^{1/2} f_l(\epsilon, r)$ in (5), and solve asymptotically the resulting equation by means of an improved version of the Wentzel–Kramers–Brillouin (WKB) method due to Fröman and Fröman [5] (the calculation has also been done using the WKB method, with the same results). The functions $q_l(\epsilon, r)$ in (6) are expanded as follows:

$$q_l(\epsilon, R) = Q_l(\epsilon, R) \sum_{n=0}^{\infty} Y_l^{(2n)}(\epsilon, R) \tag{11}$$

with $Y_l^{(0)}(\epsilon, R) = 1$. The $Y_l^{(2n)}$'s can be calculated recursively by replacing (11) and (6) in (5), giving (see [5]):

$$\begin{aligned} Y_l^{(2)} &= \tfrac{1}{2} Q_l^{-\frac{3}{2}} \partial_R^2 Q_l^{-\frac{1}{2}} + \tfrac{1}{8} Q_l^{-2} R^{-2} \\ Y_l^{(2n)} &= \sum_{p,q=0}^{n-1} \left(Y_l^{(2)} Y_l^{(2p)} Y_l^{(2q)} + \frac{3Q^{-2}}{8} (\partial_R Y_l^{(2p)})(\partial_R Y_l^{(2q)}) - \frac{Q^{-1}}{4} Y_l^{(2p)} \partial_R Q^{-1} \partial_R Y_l^{(2q)} \right) \\ &\quad \times \delta_{p+q,n-1} - \tfrac{1}{2} \sum_{p,q,i,j=0}^{n-1} Y_l^{(2p)} Y_l^{(2q)} Y_l^{(2i)} Y_l^{(2j)} \delta_{p+q+i+j,n} (1 - \delta_{i+j,0}) \end{aligned} \tag{12}$$

if $n \geqslant 1$. The asymptotic expansion of $\Delta g(E)$ is obtained by keeping only the term $\nu = 0$ in (10). Let $u = l/R + R/4$ and set $Z^{(2n)}(\epsilon, u, R) = Y_l^{(2n)}(\epsilon, R)$ and $z(\epsilon, u) = -\mathrm{i} Q_l(\epsilon, R) =$

$\sqrt{\epsilon+u^2}$. Making this change of variables in (10), we obtain:

$$\Delta g(E) \simeq \frac{R}{\hbar\omega}\int_{-\infty}^{\infty} \mathrm{d}u \left(-\frac{1}{4z^2} - \frac{\sum_{n=1}^{\infty} \partial_\epsilon Z^{(2n)}}{2\sum_{n=0}^{\infty} Z^{(2n)}} + \frac{2u^2 - uR}{8Rz^5 \sum_{n=0}^{\infty} Z^{(2n)}} + \frac{\sum_{n=1}^{\infty} \partial\partial_\epsilon Z^{(2n)}}{4z(\sum_{n=0}^{\infty} Z^{(2n)})^2} - \frac{\sum_{n,m=1}^{\infty} \partial_\epsilon Z^{(2n)} \partial Z^{(2m)}}{4z(\sum_{n=0}^{\infty} Z^{(2n)})^3} \right) \tag{13}$$

with $\partial = (-u/R + \frac{1}{2})\partial_u + \partial_R$. The integrand in the right-hand side is expanded in the form $\sum_{0\leqslant i\leqslant j-2} d_{i,j}(R) u^i z^{-j}$, giving by a simple integration and change of indices:

$$\Delta P(\tau) \sim R \sum_{r=1}^{\infty}\sum_{p=0}^{\infty} \frac{\Gamma(\frac{2p+1}{2})}{\Gamma(\frac{2p+1+r}{2})} d_{2p,2p+r+1}(R)\tau^{r/2-1} \qquad \tau = \omega t \to 0. \tag{14}$$

The two first terms in (13) give contributions to $d_{i,j}(R)$ for j even and the three last terms contribute to $d_{i,j}(R)$ for odd j. The first $Z^{(2n)}$, found by performing the change of variables $(\epsilon, l, R) \to (\epsilon, u, R)$ in (12), are, for example, given by:

$$\begin{aligned} Z^{(2)} &= -\frac{5}{8z^6}\left(\frac{u^4}{R^2} - \frac{u^3}{R} + \frac{u^2}{4}\right) + \frac{1}{8z^4}\left(\frac{6u^2}{R^2} - \frac{3u}{R} + \frac{1}{2}\right) - \frac{1}{8R^2z^2} \\ Z^{(4)} &= -\frac{1}{2}(Z^{(2)})^2 + \frac{\partial z^{-1}\partial Z^{(2)}}{4z} \\ Z^{(6)} &= -(Z^{(2)})^3 - 3Z^{(2)}Z^{(4)} - \frac{3(\partial Z^{(2)})^2}{8z^2} + \frac{Z^{(2)}\partial z^{-1}\partial Z^{(2)}}{4z} + \frac{\partial z^{-1}\partial Z^{(4)}}{4z}. \end{aligned} \tag{15}$$

We calculated the coefficients $d_{2p,j}(R)$ with the `Mathematica` computing system (version 3.0).

The Weyl expansion that we have derived here could be valuable in calculating various spectral quantities for cavities in a magnetic field. Recently, non-congruent planar regions were constructed that have identical spectra [6, 7]. Turning on the magnetic field, one can expect from a perturbation theory argument that these cavities do not remain isospectral. According to the conjecture in [1], they could however possess identical Weyl series. The circular billiard also provides an interesting example to study the generalization in the presence of a magnetic field of a conjecture made by Berry and Howls [3] about the high orders of the Weyl expansion of billiards in a zero field. This question will be addressed in a future project.

We thank S Fishman, B Georgeot, R Prange and U Smilansky for helpful discussions and useful comments. This work was supported in its early stages by the fund for the promotion of research at the Technion. RN acknowledges the support by NSF DMR 9624559 and the US–Israel BSF 95-00067-2.

References

[1] Narevich R, Spehner D and Akkermans E 1998 *J. Phys. A: Math. Gen.* **31** 4277
[2] Stewartson K and Waechter R T 1971 *Proc. Camb. Phil. Soc.* **69** 353
[3] Berry M V and Howls C J 1994 *Proc. R. Soc.* A **447** 527
[4] Wong R 1989 *Asymptotic Approximations of Integrals* (Boston, MA: Academic)
[5] Fröman N and Fröman P O 1995 *Phase-Integral Method: Allowing Nearlying Transition Points (Springer Tracts in Natural Philosophy vol 40)* (Berlin: Springer)
[6] Gordon C, Webb D and Wolpert S 1992 *Inv. Math.* **110** 1
[7] Chapman S J 1995 *Am. Math. Mon.* **102** 124