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LETTER TO THE EDITOR

Weyl expansion of a circle billiard in a magnetic field

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Abstract. We compute the high orders of the Weyl expansion for the heat kernel of a circle billiard in the presence of a uniform and perpendicular magnetic field. It is shown, in accordance with a conjecture made in Narevich *et al* (1998 *J. Phys. A: Math. Gen.* **31** 4277), that some terms of this expansion can be identified with those of the Weyl expansion of a semi-infinite cylinder. The boundary correction to the Landau diamagnetic susceptibility of a non-degenerate electron gas in the billiard is determined.

Consider a spinless particle of charge $-e$ ($e > 0$) and mass m constrained by a hard wall to move inside a disc of radius a. A uniform magnetic field $\vec{B}$ is applied perpendicularly to the disc. Let $\omega = e\|\vec{B}\|/mc$, $l_B = \sqrt{\hbar/m\omega}$ and $R = \sqrt{2}a/l_B = \sqrt{2ma^2\omega/\hbar}$ be respectively the cyclotron frequency, the magnetic length and the dimensionless radius. In the symmetric gauge $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$, the Hamiltonian of the particle is:

$$H = \hbar\omega\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\left(-\mathrm{i}\partial_\theta + \frac{r^2}{4}\right)^2\right) \tag{1}$$

where (r, θ) are the dimensionless polar coordinates defined by $r = \sqrt{2}\|\vec{x}\|/l_B$; θ is the angle between $\vec{x}$ and a fixed vector $\vec{e}_x$ parallel to the plane of the disc, counted positively if the triad $(\vec{e}_x, \vec{x}, \vec{B})$ is right-handed. The eigenfunctions $\psi_n(r, \theta)$ of H are required to be finite as $r \to 0$ and to satisfy the Dirichlet boundary condition $\psi_n(R, \theta) = 0$, $0 \leqslant \theta < 2\pi$. In this letter, we describe an algorithm to compute the Weyl asymptotic expansion of the heat kernel $P(t) = \mathrm{tr}\, \mathrm{e}^{-(t/\hbar)H}$ for this system. This also determines the Weyl expansions of other simply related spectral quantities, like e.g. the density of states [3]. The first few terms of the asymptotic expansion of $P(t)$ as $\tau = \omega t \to 0$ are:

$$\begin{aligned}
P\left(t = \frac{\tau}{\omega}\right) \sim \frac{R^2}{4}&\left(\frac{1}{\tau} - \frac{\tau}{24} + \frac{7\tau^3}{5760} - \frac{31\tau^5}{945 \times 2^{10}} + \cdots\right)\\
&-\frac{\sqrt{\pi}R}{4}\left(\tau^{-\frac{1}{2}} - \frac{3\tau^{\frac{3}{2}}}{64} + \frac{25\tau^{\frac{7}{2}}}{2^{14}} - \frac{7309\tau^{\frac{11}{2}}}{315 \times 2^{19}} + \cdots\right)\\
&+\frac{1}{6}\left(1 - \frac{3\tau^2}{56} + \frac{757\tau^4}{3003 \times 2^7} - \frac{104\,971\tau^6}{1616\,615 \times 2^{10}} + \cdots\right)\\
&+\frac{\sqrt{\pi}}{2^7 R}\left(\tau^{\frac{1}{2}} - \frac{7\tau^{\frac{5}{2}}}{2^7} + \frac{83\tau^{\frac{9}{2}}}{5 \times 2^{13}} - \cdots\right)
\end{aligned}$$

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$$+\frac{2}{315R^2}\left(\tau-\frac{69\tau^3}{1144}+\frac{14\,431\tau^5}{46\,189\times 2^7}-\cdots\right)$$
$$+\frac{37\sqrt{\pi}}{2^{14}R^3}\left(\tau^{\frac{3}{2}}-\frac{393\tau^{\frac{7}{2}}}{5920}+\cdots\right)+\frac{136}{45\,045R^4}\left(\tau^2-\frac{3203\tau^4}{43\,928}+\cdots\right)+\cdots. \quad (2)$$

The Weyl expansion in the zero field is obtained by keeping only the first term in each parenthesis and coincides with known results [2, 3]. The terms proportional to R^2 (first parenthesis) give the Weyl expansion of the heat kernel $P_\infty(t)$ associated with the Landau spectrum (infinite plane geometry), whose full asymptotic expansion can be easily calculated [3, 1]. The terms proportional to R (second parenthesis) coincide with the first terms of the Weyl expansion of $P_{per}(t)-P_\infty(t)$, here $P_{per}(t)$ is the heat kernel of a semi-infinite cylinder of radius a in a uniform magnetic field [1]. This result is in agreement with a conjecture made in [1], according to which each term of the Weyl expansion of a billiard with a smooth boundary in zero field becomes multiplied by a universal billiard-independent function of $\tau = \omega t$ if a uniform magnetic field is applied perpendicularly. The first two functions, multiplying respectively the area term $(R^2/4)\tau^{-1}$ and the perimeter term $-(\sqrt{\pi}R/4)\tau^{-1/2}$, have been found in [1] to be $(\tau/2)/\sinh(\tau/2)$ and

$$2\sqrt{\frac{\tau}{\pi}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{d}\epsilon}{2\mathrm{i}\pi}\mathrm{e}^{\epsilon\tau}\int_{-\infty}^{\infty}\mathrm{d}x\,\left(\partial_\epsilon \ln D_{-\epsilon-\frac{1}{2}}(x)+\tfrac{1}{2}\psi(\epsilon+\tfrac{1}{2})\right)$$

where $c > 0$ and $D_{-\epsilon-1/2}$, ψ are respectively the parabolic cylinder and the digamma functions. We use this opportunity to correct an error made in the Weyl expansion of $P_{per}(t)$ in [1], formula (29): the correct power of 2 in the denominator of the term of order $\tau^{11/2}$ should be 19 as in (2), instead of 20.

Since $P(t)$ for $t = \hbar\beta$ is the canonical partition function, one can determine from (2) the magnetic susceptibility χ of an ideal non-degenerate gas in the disc at inverse temperature β. If N is the number of particles per unit area and $\lambda_T = \sqrt{\pi\hbar^2\beta/2m} \ll N^{-1/2}$ is the de Broglie thermal length, we obtain an expansion of χ in powers of λ_T/a which begins as follows:

$$\frac{\chi}{\chi_\infty} = 1-\frac{\lambda_T}{8a}-\left(\frac{1}{8}-\frac{4}{21\pi}\right)\frac{\lambda_T^2}{a^2}-\left(\frac{1}{8}-\frac{3049}{10\,752\pi}\right)\frac{\lambda_T^3}{a^3}$$
$$-\left(\frac{1}{8}-\frac{1329}{3584\pi}+\frac{248}{2145\pi^2}\right)\frac{\lambda_T^4}{a^4}+\cdots \quad (3)$$

where $\chi_\infty = -N\beta e^2\hbar^2/12m^2c^2$ is the Landau susceptibility. We have found that at each order in λ_T/a up to the fifth order, the corrections to the Landau diamagnetic susceptibility are paramagnetic.

To show (2), we use a Green function approach [1–3]. The Green function $G(E; r, \theta; r', \theta')$ is given by:

$$(H+E)G(E; r, \theta; r', \theta') = \frac{2}{rl_B^2}\delta(r-r')\delta(\theta-\theta') \quad (4)$$

(note the + sign in front of the energy E). It satisfies the boundary condition: $G(E; R, \theta; r', \theta') = G(E; r, \theta; R, \theta') = 0$, and we require moreover that it be finite as $r \to 0$ and $r' \to 0$. One defines similarly the 'infinite plane' Green function $G_\infty(E; r, \theta; r', \theta')$ which satisfies the same equation, is finite at the origin, and is such that $r \mapsto rG_\infty(E; r, \theta; r', \theta')$ and $r' \mapsto r'G_\infty(E; r, \theta; r', \theta')$ be integrable on $\mathbb{R}_+$. Set $\epsilon = E/\hbar\omega$ and let $f_l^\pm(\epsilon, r)$ be two independent solutions of

$$\left(\partial_r^2+\frac{1}{r}\partial_r+Q_l^2(\epsilon, r)\right)f_l(\epsilon, r) = 0 \qquad Q_l^2(\epsilon, r) = -\epsilon-\frac{1}{r^2}\left(l+\frac{r^2}{4}\right)^2. \quad (5)$$

The conditions on these functions are that $\lim_{r\to 0} f_l^-(\epsilon, r) < \infty$ and that $r \mapsto r f_l^+(\epsilon, r)$ be integrable on $[c, \infty[$ for any $c > 0$. We are interested in solutions of (5) of the following form:

$$f_l^{\pm}(\epsilon, r) = (r q_l(\epsilon, r))^{-\frac{1}{2}} \mathrm{e}^{\pm \mathrm{i} \int_{r_0}^{r} \mathrm{d}r' \, q_l(\epsilon, r')} \tag{6}$$

where $r_0 > 0$ and the imaginary parts of the complex functions $q_l(\epsilon, r)$ tend to $+\infty$ as $r \to \infty$ or $r \to 0$. Using (1) and expanding the Green functions as Fourier series in $\theta - \theta'$, one gets:

$$\begin{aligned} G_\infty(E; r, \theta; r', \theta') &= -\frac{m}{\pi \hbar^2} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}l(\theta-\theta')}}{W_l(\epsilon)} f_l^-(\epsilon, \min\{r, r'\}) f_l^+(\epsilon, \max\{r, r'\}) \\ G(E; r, \theta; r', \theta') &= G_\infty(E; r, \theta; r', \theta') + \frac{m}{\pi \hbar^2} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}l(\theta-\theta')} f_l^+(\epsilon, R)}{W_l(\epsilon) f_l^-(\epsilon, R)} f_l^-(\epsilon, r) f_l^-(\epsilon, r'). \end{aligned} \tag{7}$$

The Wronskian $W_l = r f_l^- \partial_r f_l^+ - r f_l^+ \partial_r f_l^-$ in the denominator is independent of r. The Laplace transform $\Delta g(E)$ of $\Delta P(t) = P(t) - P_\infty(t)$ is related to $G(E; r, \theta; r', \theta)$ and $G_\infty(E; r, \theta; r', \theta)$ by [2]:

$$\begin{aligned} \Delta g(E) &= \int_0^\infty \frac{\mathrm{d}t}{\hbar} \mathrm{e}^{-Et/\hbar} \Delta P(t) \\ &= \frac{l_B^2}{2} \int_0^R r \, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \, (G(E; r, \theta; r, \theta) - G_\infty(E; r, \theta; r, \theta)). \end{aligned} \tag{8}$$

Manipulating equation (5) in a standard way one shows that

$$\int_0^R r \, \mathrm{d}r \, f_l^-(\epsilon, r)^2 = R(f_l^- \partial_R \partial_\epsilon f_l^- - (\partial_R f_l^-)(\partial_\epsilon f_l^-))(\epsilon, R). \tag{9}$$

Using (6)–(9) and the Poisson summation formula, one obtains after some algebra:

$$\hbar \omega \Delta g(E) = \sum_{\nu=-\infty}^{\infty} \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}l \, \mathrm{e}^{2\mathrm{i}\pi \nu l} \left(-\partial_\epsilon + \frac{\mathrm{i}}{2 q_l} \partial_R \partial_\epsilon \right) \ln q_l(\epsilon, R). \tag{10}$$

The small t expansion of $\Delta P(t)$ can easily be found from the large E expansion of $\Delta g(E)$ and the reciprocal of Watson's lemma [4]. In order to determine this large E expansion, we set $y_l(\epsilon, r) = r^{1/2} f_l(\epsilon, r)$ in (5), and solve asymptotically the resulting equation by means of an improved version of the Wentzel–Kramers–Brillouin (WKB) method due to Fröman and Fröman [5] (the calculation has also been done using the WKB method, with the same results). The functions $q_l(\epsilon, r)$ in (6) are expanded as follows:

$$q_l(\epsilon, R) = Q_l(\epsilon, R) \sum_{n=0}^{\infty} Y_l^{(2n)}(\epsilon, R) \tag{11}$$

with $Y_l^{(0)}(\epsilon, R) = 1$. The $Y_l^{(2n)}$'s can be calculated recursively by replacing (11) and (6) in (5), giving (see [5]):

$$\begin{aligned} Y_l^{(2)} &= \tfrac{1}{2} Q_l^{-\frac{3}{2}} \partial_R^2 Q_l^{-\frac{1}{2}} + \tfrac{1}{8} Q_l^{-2} R^{-2} \\ Y_l^{(2n)} &= \sum_{p,q=0}^{n-1} \left(Y_l^{(2)} Y_l^{(2p)} Y_l^{(2q)} + \frac{3Q^{-2}}{8} (\partial_R Y_l^{(2p)})(\partial_R Y_l^{(2q)}) - \frac{Q^{-1}}{4} Y_l^{(2p)} \partial_R Q^{-1} \partial_R Y_l^{(2q)} \right) \\ &\quad \times \delta_{p+q,n-1} - \tfrac{1}{2} \sum_{p,q,i,j=0}^{n-1} Y_l^{(2p)} Y_l^{(2q)} Y_l^{(2i)} Y_l^{(2j)} \delta_{p+q+i+j,n} (1 - \delta_{i+j,0}) \end{aligned} \tag{12}$$

if $n \geqslant 1$. The asymptotic expansion of $\Delta g(E)$ is obtained by keeping only the term $\nu = 0$ in (10). Let $u = l/R + R/4$ and set $Z^{(2n)}(\epsilon, u, R) = Y_l^{(2n)}(\epsilon, R)$ and $z(\epsilon, u) = -\mathrm{i} Q_l(\epsilon, R) =$

$\sqrt{\epsilon + u^2}$. Making this change of variables in (10), we obtain:

$$\Delta g(E) \simeq \frac{R}{\hbar\omega} \int_{-\infty}^{\infty} \mathrm{d}u \left(-\frac{1}{4z^2} - \frac{\sum_{n=1}^{\infty} \partial_\epsilon Z^{(2n)}}{2\sum_{n=0}^{\infty} Z^{(2n)}} + \frac{2u^2 - uR}{8Rz^5 \sum_{n=0}^{\infty} Z^{(2n)}} \right.$$
$$\left. + \frac{\sum_{n=1}^{\infty} \partial\partial_\epsilon Z^{(2n)}}{4z(\sum_{n=0}^{\infty} Z^{(2n)})^2} - \frac{\sum_{n,m=1}^{\infty} \partial_\epsilon Z^{(2n)} \partial Z^{(2m)}}{4z(\sum_{n=0}^{\infty} Z^{(2n)})^3} \right) \tag{13}$$

with $\partial = (-u/R + \frac{1}{2})\partial_u + \partial_R$. The integrand in the right-hand side is expanded in the form $\sum_{0\leqslant i\leqslant j-2} d_{i,j}(R) u^i z^{-j}$, giving by a simple integration and change of indices:

$$\Delta P(\tau) \sim R \sum_{r=1}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(\frac{2p+1}{2})}{\Gamma(\frac{2p+1+r}{2})} d_{2p,2p+r+1}(R) \tau^{r/2-1} \qquad \tau = \omega t \to 0. \tag{14}$$

The two first terms in (13) give contributions to $d_{i,j}(R)$ for j even and the three last terms contribute to $d_{i,j}(R)$ for odd j. The first $Z^{(2n)}$, found by performing the change of variables $(\epsilon, l, R) \to (\epsilon, u, R)$ in (12), are, for example, given by:

$$\begin{aligned} Z^{(2)} &= -\frac{5}{8z^6}\left(\frac{u^4}{R^2} - \frac{u^3}{R} + \frac{u^2}{4}\right) + \frac{1}{8z^4}\left(\frac{6u^2}{R^2} - \frac{3u}{R} + \frac{1}{2}\right) - \frac{1}{8R^2z^2} \\ Z^{(4)} &= -\frac{1}{2}(Z^{(2)})^2 + \frac{\partial z^{-1}\partial Z^{(2)}}{4z} \\ Z^{(6)} &= -(Z^{(2)})^3 - 3Z^{(2)}Z^{(4)} - \frac{3(\partial Z^{(2)})^2}{8z^2} + \frac{Z^{(2)}\partial z^{-1}\partial Z^{(2)}}{4z} + \frac{\partial z^{-1}\partial Z^{(4)}}{4z}. \end{aligned} \tag{15}$$

We calculated the coefficients $d_{2p,j}(R)$ with the `Mathematica` computing system (version 3.0).

The Weyl expansion that we have derived here could be valuable in calculating various spectral quantities for cavities in a magnetic field. Recently, non-congruent planar regions were constructed that have identical spectra [6, 7]. Turning on the magnetic field, one can expect from a perturbation theory argument that these cavities do not remain isospectral. According to the conjecture in [1], they could however possess identical Weyl series. The circular billiard also provides an interesting example to study the generalization in the presence of a magnetic field of a conjecture made by Berry and Howls [3] about the high orders of the Weyl expansion of billiards in a zero field. This question will be addressed in a future project.

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