Quantum jump dynamics in cavity QED

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We study the stochastic dynamics of the electromagnetic field in a lossless cavity interacting with a beam of two-level atoms, given that the atomic states are measured after they have crossed the cavity. The atoms first interact at the exit of the cavity with a classical laser field $\mathcal{E}$ and then enter into a detector which measures their states. Each measurement disentangles the field and the atoms and changes in a random way the state $|\psi(t)\rangle$ of the cavity field. For weak atom-field coupling, the evolution of $|\psi(t)\rangle$ when many atoms cross the cavity and the detector is characterized by a succession of quantum jumps occurring at random times, separated by quasi-Hamiltonian evolutions, both of which depend on the laser field $\mathcal{E}$. For $\mathcal{E}=0$, the dynamics is the same as in the Monte Carlo wave function model of Dalibard *et al.* [Phys. Rev. Lett. **68**, 580 (1992)] and Carmichael, *An Open System Approach to Quantum Optics*, Lecture Notes in Physics Vol. 18 (Springer, Berlin, 1991)]. The density matrix of the quantum field, obtained by averaging the projector $|\psi(t)\rangle\langle\psi(t)|$ over all results of the measurements, is independent of $\mathcal{E}$ and follows the master equation of the damped harmonic oscillator at finite temperature. We provide numerical evidence showing that for large $\mathcal{E}$, an arbitrary initial field state $|\psi(0)\rangle$ evolves under the monitoring of the atoms and the measurements toward squeezed states $|\alpha, re^{2i\phi}\rangle$, moving in the α-complex plane but with almost constant squeezing parameters r and ϕ. The values of r and ϕ are determined analytically. On the other hand, for $\mathcal{E}=0$, the dynamics transforms the initial state into Fock states $|n\rangle$ with fluctuating numbers of photons n, as shown in Kist *et al.* [J. Opt. B: Quantum Semiclassical Opt. **1**, 251 (1999)]. In the last part, we derive the quantum jump dynamics from the linear quantum jump model proposed in Spehner and Bellissard [J. Stat. Phys. **104**, 525 (2001)], for arbitrary open quantum systems having a Lindblad-type evolution. A careful derivation of the infinite jump rates limit, where the dynamics can be approximated by a diffusion process of the quantum state, is also presented. © *2002 American Institute of Physics.*
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I. INTRODUCTION

The dissipative dynamics of an open quantum system S can be described in two different ways. The first one consists in coupling S with a reservoir R and assuming that the total system $S+R$ is isolated. Since one is concerned by the dynamics of S only, one traces out the degrees of freedom of R in the equation of motion of $S+R$. Within the Markov approximation, the reduced density matrix of S follows a first-order differential equation with time-independent coefficients. In many cases, a separation of time scales between the Hamiltonian (R-independent) and dissipative (R-dependent) evolutions allows one to perform a local averaging in time, which kills nonresonant terms.[1,2] The coarse-grained master equation obtained in this way has the Lindblad form.[3] An alternative approach to this density matrix description has been developed in the last two

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decades in quantum optics[4–8] (see, e.g., Ref. 9), quantum measurement theory,[10–13] quantum and classical stochastic calculus,[14–17] and electronic transport in solids.[18–20] This approach is based on stochastic evolutions of pure states. The system is described by a random wave function (RW) evolving according to a linear or nonlinear *stochastic Schrödinger equation*. Consistency with the density matrix approach requires that the pure state evolution gives the master equation back after averaging over the dynamical noise. Apart from being intuitively appealing, the RW models provide quite efficient tools for solving master equations numerically, since Schrödinger equations have N components, whereas master equations have $N\times N$ components, N being the dimension of the Hilbert space of S. However, the RW models are more than simple mathematical or numerical tools: they describe the *real* evolution of single quantum systems under continuous monitoring by measurements (photon counting, homodyne or heterodyne detections).[5,8,9] In recent years, the attainment of low temperature and low dissipation regimes, as well as the improvements of detection techniques, has allowed the investigation of the dynamics of such continuously monitored systems. Remarkable examples of these are single ions[21] and Bose–Einstein condensates[22] in electromagnetic traps, probed by laser beams, and electromagnetic fields in high Q-cavities,[23] probed by beams of highly excited atoms (Rydberg atoms). This new generation of experiments, combined with the difficulties usually encountered in solving the master equation, has strongly stimulated the developments of the RW approach in quantum optics.

The aim of this paper is to investigate a specific physical realization, which could be in principle realizable experimentally (although this question is not addressed here), of a class of RW models based on quantum jumps. The system we consider is the electromagnetic field of a high Q-cavity interacting with a beam of two-level atoms, which forms the reservoir of temperature T. The states of the atoms leaving the cavity are measured by a detector. A laser field $\mathcal{E}$ is placed between the cavity and the detector. The corresponding master equation, obtained by averaging over the results of the measurement on the atoms, is, for weak atom-field coupling, the equation of the damped harmonic oscillator with finite temperature. The same problem has been considered in Ref. 24 in the reverse situation where one knows exactly the state of each atom before it crosses the cavity and no measurement is performed on it at the exit (its final state thus being unknown). It has been shown in this reference that the cavity field evolves at large times to a state which is completely controlled by the atomic initial states.

We first introduce in Sec. II the class of quantum jump models studied in this work, for arbitrary open systems having a Lindblad-type evolution. The experimental scheme is presented in Sec. III, where we also compute the random evolution of the cavity field and its corresponding average evolution. We focus in Sec. IV on single quantum trajectories, i.e., single realizations of the measurements. The numerical simulations and analytical results presented in this section show that for $T>0$ and large laser fields $\mathcal{E}$, the state of the cavity field localizes at large times to squeezed states with an almost constant squeezing amplitude r which depends only on T. Section V is devoted to the derivation, for arbitrary open systems, of the nonlinear quantum jump schemes from the corresponding linear ones introduced in Ref. 18. Their relation with the so-called quantum state diffusion stochastic Schrödinger equations[10–12,15–17] is also established. Our conclusions are presented in Sec. VI.

II. THE QUANTUM JUMP SCHEMES

Let us first recall briefly a few basic facts about the master equation approach to open quantum systems. Consider an open system S interacting with a reservoir R. The density matrix σ of the total system $S+R$ is assumed to follow the Liouville–von Neumann equation of closed systems. A state of S is specified by the reduced density matrix ρ, defined as the partial trace of σ over the reservoir's Hilbert space. By tracing out the degrees of freedom of R in the Liouville–von Neumann equation, one obtains an integro-differential equation for ρ (Nakajima–Zwanzig equation).[25] Using a Born–Markov approximation and a local time averaging on a time scale much larger than the inverse Bohr frequencies of S, this equation is transformed into a simpler first-order linear differential equation, called the *master equation*.[1,25] This coarse-grained equation has in most cases the Lindblad form:[3]

$$\frac{d\rho}{dt} = -i[H,\rho] + \frac{1}{2}\sum_m ([L_m\rho, L_m^\dagger] + [L_m, \rho L_m^\dagger]). \tag{1}$$

H is the Hamiltonian of S (including the energy shifts due to the coupling with the reservoir), and L_m are some operators acting on the Hilbert space of S. The sum over the discrete indices m can be finite or infinite, depending on the nature of the problem.

We now describe the random wave function approach of Dalibard, Castin, and Mølmer[4] and Carmichael.[5] This approach, called in the former reference the Monte Carlo wave function method, has been introduced independently by several other authors.[6,7,13] It is based on quantum jumps, i.e., on discontinuous random evolutions of the wave function of S.

At some random times, quantum jumps (QJ) occur as a result of some continuous measurement on the system S (e.g., a detection of a photon emitted by a system constituted by an atom). If a jump occurs, the wave function $|\psi\rangle$ of S is modified discontinuously as follows:

$$\text{jump } m: \quad |\psi\rangle \to \frac{L_m|\psi\rangle}{\|L_m|\psi\rangle\|}, \tag{2}$$

where L_m are the Lindblad operators appearing in (1). The probability of occurrence of a jump of type m in the time interval $[t, t+\delta t]$ is

$$\delta p_m(t) = \|L_m|\psi(t)\rangle\|^2 \delta t. \tag{3}$$

One must choose δt small enough so that $\delta p_m(t) \ll 1$ for any $|\psi\rangle$ and all m's (this is fulfilled if δt^{-1} is much bigger than the damping constants γ_m appearing in the master equation, contained in the operators L_m). If no jump occurs between t and $t+\delta t$, the wave function evolves between these two times according to Schrödinger's equation with an effective Hamiltonian $H+K$, and is then normalized at $t+\delta t$:

$$\begin{aligned} |\psi(t+\delta t)\rangle &= \frac{|\varphi(t+\delta t)\rangle}{\|\varphi(t+\delta t)\|}, \\ |\varphi(t+\delta t)\rangle &= e^{-i\delta t(H+K)}|\psi(t)\rangle. \end{aligned} \tag{4}$$

K can be computed in special cases by first determining perturbatively the wave function of the total system $S+R$ and then projecting it onto the subspace corresponding to the no-jump measurement.[4,9] An easier (though less fundamental) way to compute K is to ask directly that the average $\mathsf{M}|\psi(t)\rangle\langle\psi(t)|$ satisfies the master equation (1) (see the following). This gives[4]

$$K = K_0 \equiv \frac{1}{2i}\sum_m L_m^\dagger L_m. \tag{5}$$

Note that K_0 is *not* self-adjoint. Hence the norm of the wave function is not conserved by the evolution operator $e^{-i\delta t(H+K_0)}$. This can be interpreted by invoking the gain of information on the system provided by the measurement, namely, by the knowledge that no jump occurred between t and $t+\delta t$. For instance, in the case of an atom coupled to the quantized electromagnetic field, we may infer from a no-photon detection that the atom has not emitted spontaneously a photon.

Since the wave function is normalized at each step δt in (2) and (4), the random dynamics is norm-preserving on the time resolution δt. These normalizations make the stochastic quantum evolution nonlinear. We will see in Sec. V that it is possible to define an equivalent linear model, in which the random wave function is not normalized.[18,14] The map $t \in [0,\infty[\mapsto |\psi(t)\rangle$ for a given outcome of the jumps is called a *quantum trajectory*.[5]

Let us consider the average density matrix $\rho(t) = \mathsf{M}|\psi(t)\rangle\langle\psi(t)|$, where M is the average over all realizations of the jumps. It can be easily shown[4] that $\rho(t)$ obeys the master equation (1) to lowest order in $\|L_m\|^2\delta t$. Actually, taking $H=0$ for simplicity, one has

$$\rho(t+\delta t)=\mathsf{M}_{t+\delta t}|\psi(t+\delta t)\rangle\langle\psi(t+\delta t)|$$
$$=\mathsf{M}_t\left(\left(1-\sum_m \delta p_m(t)\right)\frac{e^{-i\delta tK_0}|\psi(t)\rangle\langle\psi(t)|e^{i\delta tK_0^\dagger}}{\|e^{-i\delta tK_0}|\psi(t)\rangle\|^2}+\sum_m \delta p_m(t)\frac{L_m|\psi(t)\langle\psi(t)|L_m^\dagger}{\|L_m|\psi(t)\rangle\|^2}\right).$$

This equation is simplified by taking

$$e^{-i\delta tK_0}=\left(1-\delta t\sum_m L_m^\dagger L_m\right)^{1/2}. \tag{6}$$

Then, by (3):

$$\rho(t+\delta t)=\left(1-\delta t\sum_m L_m^\dagger L_m\right)^{1/2}\rho(t)\left(1-\delta t\sum_m L_m^\dagger L_m\right)^{1/2}+\delta t\sum_m L_m\rho(t)L_m^\dagger. \tag{7}$$

Expanding the square roots and keeping only terms of order one in δt, one obtains (1) and K_0 is given by (5). Note that, for an arbitrary time interval δt between consecutive measurements, $\rho(t+\delta t)$ is *not* given by integrating (1) from t to $t+\delta t$ (a different result is already obtained at the next order δt^2). This should be kept in mind when dealing with real or numerical experiments, where δt is always finite.

Consider a transformation $L_m\to L'_m$ on the operators L_m which does not change (1). The quantum jumps and the effective Hamiltonian K may be modified by this replacement. This leads to a different stochastic dynamics, which unravels the same master equation. A particular transformation leaving (1) invariant is[26]

$$L_m\to L'_m=L_m+\lambda_m,\quad H\to H'=H+\frac{1}{2i}\sum_m(\lambda_m^* L_m-\lambda_m L_m^\dagger)=H'^\dagger, \tag{8}$$

where λ_m's are complex numbers. This invariance of the Lindblad equation is not related to a particular symmetry of the system or its coupling with the reservoir. It simply expresses that the separation between the Hamiltonian part $-i[H,\rho]$ and the remaining dissipative part in (1) is not unique. The transformation (8) generates a whole family of distinct QJ models depending on the set of numbers λ_m. The modification of the wave function at a jump m is now given by

$$\text{jump } m:\quad |\psi\rangle\to\frac{W_m|\psi\rangle}{\|W_m|\psi\rangle\|} \tag{9}$$

with the jump operators W_m proportional to $(L_m+\lambda_m)$:

$$W_m=\gamma_m^{-1/2}(L_m+\lambda_m). \tag{10}$$

The new generalized Hamiltonian $H+K$ is obtained by replacing L_m by L'_m in (5) and adding to it $H'-H$:

$$K=\frac{1}{2i}\sum_m(L_m^\dagger L_m+2\lambda_m^* L_m+|\lambda_m|^2). \tag{11}$$

The last term in the sum, proportional to the identity operator, is written only for convenience: it is irrelevant because of the normalization in Eq. (4) giving the evolution between jumps. The probability of occurrence of a jump of type m becomes

$$\delta p_m(t)=\|(L_m+\lambda_m)|\psi(t)\rangle\|^2\delta t=\gamma_m\delta t\,\|W_m|\psi(t)\rangle\|^2. \tag{12}$$

Note that it increases like $|\lambda_m|^2$ for large λ_m.

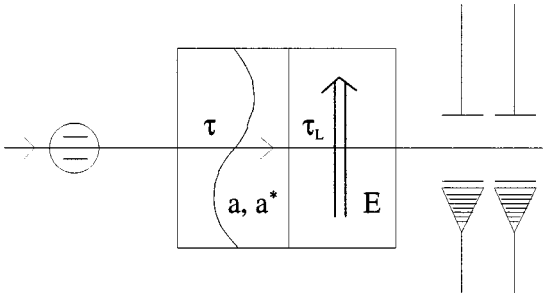

FIG. 1. The two-level atoms of the beam cross one by one a cavity containing the studied quantum field a, a second cavity containing a laser field $\mathcal{E}$, and a detector measuring their states.

III. FIELD MODE IN A CAVITY

It is shown in this section that the QJ schemes described in Sec. II can be physically realized by an atomic beam crossing a cavity with perfectly reflecting walls and interacting with the quantum field inside, which forms the system S. The measurements are performed on the outgoing atoms, after they have interacted with a classical laser field $\mathcal{E}$ placed between the cavity and the measuring apparatus.

A. Experimental scheme

Let us consider one mode of the quantized electromagnetic field of a lossless cavity coupled to its environment. The environment is a beam of atoms prepared in one of two Rydberg states $|g\rangle$ (''ground state'') and $|e\rangle$ (''excited state'') in resonance with the frequency ω of the mode. The fluxes r_g and r_e of atoms crossing the cavity, prepared, respectively, in states $|g\rangle$ and $|e\rangle$, are assumed to be such that at most one atom is in the cavity at any time. The time interval between the crossing of two consecutive atoms in the cavity is $\delta t=(r_g+r_e)^{-1}$. To simplify, all the atoms of the beam are supposed to have the same speed. They thus spend the same time $\tau<\delta t$ in the cavity, interacting with the field mode. The atom-mode interaction Hamiltonian is in the interaction picture (rotating-wave approximation):[1,2]

$$H_{\rm int}=-i\,(\lambda^*|g\rangle\langle e|\,a^\dagger-\lambda|e\rangle\langle g|\,a), \tag{13}$$

where $a^\dagger$ and a are the creation and annihilation operators of a photon. The coupling constant λ is equal to $\sqrt{q^2\omega/2\varepsilon_0 V}\,\vec{d}_{ge}\cdot\vec{\sigma}$, where $\vec{d}_{ge}$ is the matrix element of the atomic dipole, $\vec{\sigma}$ the polarization vector of the field mode, q the charge of the electron, and V the volume of the cavity.

At the exit of the cavity, the atoms enter into a second cavity, identical to the first but containing a classical laser field $\vec{\mathcal{E}}$ (Fig. 1). They spend a time $\tau_L<\delta t$ there, interacting with the laser field. Under the dipolar and rotating-wave approximations, the atom-laser interaction Hamiltonian is in the interaction picture:[1,2]

$$H_L=-\frac{i}{2}(\Omega^*|g\rangle\langle e|-\Omega|e\rangle\langle g|), \tag{14}$$

where $\Omega=i\,\vec{d}_{ge}\cdot\vec{\mathcal{E}}$ is the Rabi frequency. The Hamiltonian (14) describes the atom-laser interaction for a laser field in resonance with the atomic transition. The more general situation of nonzero detuning of the laser frequency will be discussed in the following.

Finally, the state of each atom at the exit of the second cavity is measured by a detector, telling us if it is in its ''ground'' or in its ''excited'' state. The corresponding experimental scheme is presented in Fig. 1. It has been considered in Ref. 27 without the laser field $\mathcal{E}$. The flight times of the atoms between the two cavities and between the second cavity and the detector are taken sufficiently small so that spontaneous emission of photons by the atoms can be neglected.

B. Stochastic dynamics of the field mode

Let us compute the evolution of the state of the field mode in the first cavity, for a given result of the measurements. If one looks at it with a time resolution equal or bigger than the time δt separating the entrance of consecutive atoms, the field mode is continuously monitored by the measurements on the atoms (continuous measurement). The coupling constant λ, the Rabi frequency Ω, and the interaction times τ, τ_L are assumed to satisfy

$$|\lambda|\tau \ll 1, \quad |\Omega|\tau_L \ll 1. \tag{15}$$

Let us first determine the change of the wave function $|\psi(t)\rangle$ of the field mode when one atom, initially in state $|i\rangle$, $i=g$ or e, crosses the two cavities and the detector. At the time t just prior to the entrance of the atom in the first cavity, the wave function $|\Psi(t)\rangle$ of the total system "atom+field mode" is a tensor product state $|\Psi(t)\rangle = |i\rangle|\psi(t)\rangle$ (the atom and the field did not yet interact). Since the two fields are in separated cavities, the atom interacts with the quantum field *before* interacting with the classical field $\mathcal{E}$. The total wave function at the exit of the second cavity (before the measurement) is thus, in the interaction picture,

$$|\Psi_{\text{ent}}\rangle = e^{-i\,\tau_L H_L}\, e^{-i\,\tau H_{\text{int}}} |i\rangle|\psi(t)\rangle. \tag{16}$$

Note that the evolution would be different if the laser field was placed in the first cavity (in this case, the product of exponentials must be replaced by the exponential of $H_{\text{int}}+H_L$). Because of their interaction, the quantum field and the atom are now entangled, i.e., $|\Psi_{\text{ent}}\rangle$ is not a product state. After the measurement on the atom has been performed, the field and the atom again become disentangled and the wave function of the total system is

$$|\Psi(t+\delta t)\rangle = |j\rangle|\psi(t+\delta t)\rangle, \quad |\psi(t+\delta t)\rangle = \frac{\langle j|\Psi_{\text{ent}}\rangle}{\|\langle j|\Psi_{\text{ent}}\rangle\|}, \tag{17}$$

with $j=g$ if the atom is detected in its ground state and $j=e$ if it is detected in its excited state. We have assumed for simplicity that the measurement is performed at a time $t+\tau_{\text{mes}} < t+\delta t$ earlier than the entrance of the next atom in the first cavity. Since we work in the interaction picture, $|\Psi(t+\tau_{\text{mes}})\rangle$ is then equal to $|\Psi(t+\delta t)\rangle$. The wave function $|\psi(t+\delta t)\rangle$ of the mode at the time $t+\delta t$ is thus well-defined and given by

$$\begin{aligned} |\psi(t+\delta t)\rangle &= \frac{|\varphi(t+\delta t)\rangle}{\|\varphi(t+\delta t)\|}, \\ |\varphi(t+\delta t)\rangle &= \langle j|e^{-i\tau_L H_L} e^{-i\tau H_{\text{int}}}|i\rangle|\psi(t)\rangle. \end{aligned} \tag{18}$$

The probability that the atom is detected in state $|j\rangle$, given that it enters in the cavity in state $|i\rangle$, is

$$p_{i\to j} = \|\langle j|\Psi_{\text{ent}}\rangle\|^2 = \|\varphi(t+\delta t)\|^2. \tag{19}$$

A straightforward computation leads to

$$\begin{aligned} e^{-i\tau H_{\text{int}}} = &\sum_{p=0}^{\infty} \frac{(-|\lambda|^2\tau^2)^p}{(2p)!}\left(|g\rangle\langle g|\, n^p + |e\rangle\langle e|\,(n+1)^p\right) \\ &+ \sum_{p=0}^{\infty} \frac{(-|\lambda|^2\tau^2)^p}{(2p+1)!}\left(\lambda\tau|e\rangle\langle g|\, a\, n^p - \lambda^*\tau|g\rangle\langle e|\, n^p\, a^\dagger\right), \end{aligned}$$

where $n=a^\dagger a$ is the number operator. Denoting by $n^{1/2}$ its square root, this formula can be rewritten in a more compact form:

$$e^{-i\tau H_{\rm int}} = |g\rangle\langle g| \cos(|\lambda|\tau n^{1/2}) + |e\rangle\langle e| \cos(|\lambda|\tau(n+1)^{1/2}) + \lambda\tau|e\rangle\langle g|\, a\, \mathrm{sinc}(|\lambda|\tau n^{1/2}) - \lambda^*\tau|g\rangle\langle e|\, \mathrm{sinc}(|\lambda|\tau n^{1/2})\, a^\dagger \tag{20}$$

with $\mathrm{sinc}(u) = (\sin u)/u$. Similarly,

$$e^{-i\tau H_L} = \cos\left(\frac{|\Omega|\tau_L}{2}\right) + \frac{\Omega\tau_L}{2}|e\rangle\langle g|\, \mathrm{sinc}\left(\frac{|\Omega|\tau_L}{2}\right) - \frac{\Omega^*\tau_L}{2}|g\rangle\langle e|\, \mathrm{sinc}\left(\frac{|\Omega|\tau_L}{2}\right).$$

Let us introduce the operator

$$\tilde{a} = a\, \mathrm{sinc}(|\lambda|\tau n^{1/2}) \tag{21}$$

and the complex numbers

$$\eta = \lambda\tau, \quad \epsilon = \frac{\Omega \tan(|\Omega|\tau_L/2)}{|\Omega|\lambda\tau}. \tag{22}$$

Four cases must be considered.

(1) The atom enters the first cavity in its ground state and is detected in the same state: $i=j=g$. Then,

$$|\varphi(t+\delta t)\rangle = (1+|\eta\epsilon|^2)^{-1/2}(\cos(|\eta|n^{1/2}) - |\eta|^2\epsilon^*\tilde{a})|\psi(t)\rangle \equiv (1 - i\,\delta t\, K_g)|\psi(t)\rangle. \tag{23}$$

(2) The atom enters the first cavity in its ground state and is detected in its excited state: $i=g$, $j=e$. Then,

$$|\varphi(t+\delta t)\rangle = \eta\,(1+|\eta\epsilon|^2)^{-1/2}\, W_-|\psi(t)\rangle, \quad W_- \equiv \tilde{a} + \epsilon\cos(|\eta|n^{1/2}). \tag{24}$$

(3) The atom enters the first cavity in its excited state and is detected in its ground state: $i=e$, $j=g$. Then,

$$|\varphi(t+\delta t)\rangle = -\,\eta^*(1+|\eta\epsilon|^2)^{-1/2}\, W_+|\psi(t)\rangle, \quad W_+ \equiv \tilde{a}^\dagger + \epsilon^*\cos(|\eta|(n+1)^{1/2}). \tag{25}$$

(4) The atom enters the first cavity in its excited state and is detected in the same state: $i=j=e$. Then,

$$|\varphi(t+\delta t)\rangle = (1+|\eta\epsilon|^2)^{-1/2}(\cos(|\eta|(n+1)^{1/2}) - |\eta|^2\epsilon\,\tilde{a}^\dagger)|\psi(t)\rangle \equiv (1 - i\,\delta t\, K_e)|\psi(t)\rangle. \tag{26}$$

Cases (2) and (3) correspond, respectively, to the absorption and the emission of a photon of the cavity mode or of the laser field by the atom. In order to have quantum jumps separated by ''continuous'' Hamiltonian evolutions on time scales bigger than δt, the probabilities of these events must be very small. The probability $\delta p_-(t)$ of case (2) is equal to $p_{g\to e}$ multiplied by the probability $r_g\,\delta t$ that the atom enters in the first cavity in state $|g\rangle$. With the help of (19) and (24), one gets

$$\delta p_-(t) = \frac{r_g\,\delta t\,|\eta|^2}{1+|\eta\epsilon|^2}\|W_-|\psi(t)\rangle\|^2 = r_g\,\delta t(\|\sin(|\eta|n^{1/2})|\psi(t)\rangle\|^2 + \mathcal{O}(\eta\epsilon)). \tag{27}$$

The probability $\delta p_+(t)$ of case (3) is given by a similar formula, replacing r_g by r_e, W_- by W_+, and n by $n+1$. The probabilities of cases (1) and (4) are, respectively, $r_g\,\delta t - \delta p_-(t)$ and $r_e\,\delta t - \delta p_+(t)$. We see that $\delta p_\pm(t)$ is small if $|\eta| = |\lambda|\tau \ll 1$ and $|\eta\epsilon| \simeq |\Omega|\tau_L/2 \ll 1$, provided that also

$$\langle\psi(t)|(|\eta|n^{1/2} - k\pi)^{2q}|\psi(t)\rangle = \mathcal{O}(\eta^q), \quad q = 1,2,\ldots, \tag{28}$$

with k a non-negative integer. This condition is met for $k=0$ if the maximal number of photons in the cavity is much smaller than $|\eta|^{-2}$ (of order $|\eta|^{-1}$ or smaller). This corresponds to the perturbative regime. If the condition is met for $k\geqslant 1$, the atom makes almost $k/2$ Rabi oscillations in the cavity and leaves it, with a high probability, nearly in the same state as it entered it. Indeed, neglecting terms of order η^2 and $\eta^4\epsilon^4$, it follows from (22), (23), and (26):

$$\begin{aligned}\delta t K_g &\simeq \frac{|\eta|^2}{2i}(\tilde{a}^\dagger\tilde{a}+2(-1)^k\epsilon^*\tilde{a}+|\epsilon|^2),\\ \delta t K_e &\simeq \frac{|\eta|^2}{2i}(\tilde{a}\,\tilde{a}^\dagger+2(-1)^k\epsilon\,\tilde{a}^\dagger+|\epsilon|^2),\end{aligned} \tag{29}$$

up to an irrelevant additional phase $k\pi$. When the atom is measured in the same state as its initial state [cases (1) and (4)], the normalized wave function of the mode is thus modified, up to a sign, by an amount of order $|\eta|+|\eta^{3/2}\epsilon|$. In the opposite cases (2) and (3), the mode wave function is modified by an amount of order 1. It suffers a quantum jump:

$$|\psi(t+\delta t)\rangle = \frac{W_\pm|\psi(t)\rangle}{\|W_\pm|\psi(t)\rangle\|}, \tag{30}$$

with the jump operators

$$\begin{aligned}W_- &\simeq \tilde{a}+(-1)^k\epsilon,\\ W_+ &\simeq \tilde{a}^\dagger+(-1)^k\epsilon^*.\end{aligned} \tag{31}$$

The signs $-$ and $+$ correspond to case (2) (absorption of a photon) and case (3) (emission of a photon), respectively. The probability that a jump $\pm$ occurs is of order $|\eta|+|\eta^{3/2}\epsilon|+|\eta\epsilon|^2$, see (27) and (28). It is given approximately by

$$\delta p_\pm(t) \simeq \gamma_\pm\,\delta t\|W_\pm|\psi(t)\rangle\|^2, \tag{32}$$

where we have introduced the damping rates:

$$\begin{aligned}\gamma_- &= r_g|\eta|^2 = r_g|\lambda|^2\tau^2,\\ \gamma_+ &= r_e|\eta|^2 = r_e|\lambda|^2\tau^2.\end{aligned} \tag{33}$$

The operator $\tilde{a}$ is given, up to terms of order $\eta^{1/2}$, by

$$\tilde{a} \simeq \begin{cases} a & \text{if } k=0\\ (-1)^k a\left(\dfrac{|\eta|n^{1/2}-k\pi}{k\pi}-\dfrac{(|\eta|n^{1/2}-k\pi)^2}{(k\pi)^2}\right) & \text{if } k\geqslant 1.\end{cases} \tag{34}$$

If $|\psi(t)\rangle=|n\rangle$ is a Fock state with $n=(k\pi/|\eta|)^2$ photons, the crossing of the atom of initial state $|i\rangle$ has no effect on the field if $i=g$ (since $\tilde{a}|\psi(t)\rangle=0$), and a small effect if $i=e$ and $|\eta|\ll 1$.

If condition (28) is not met, the atom strongly modifies the state of the field mode in all cases (1)–(4). Hence there is no ''continuous-like'' Hamiltonian evolution changing weakly the state of the mode, separated by unlikely jumps. For $\epsilon=0$, the evolutions for $i=g$ [Eq. (23)] and $i=e$ [Eq. (26)] have the form (6) (the jump operators $W_\pm$ are proportional to $\tilde{a}^\dagger$ and $\tilde{a}$, respectively, and $\cos^2(|\eta|n^{1/2})=1-|\eta|^2\tilde{a}^\dagger\tilde{a}$, $\cos^2(|\eta|(n+1)^{1/2})=1-|\eta|^2\tilde{a}\tilde{a}^\dagger$). Although there is *a priori* no conceptual difficulty in treating this case, the corresponding dynamics becomes cumbersome when many atoms cross the cavities. We thus restrict ourself in what follows to the simpler situation in which conditions (15) and (28) are fulfilled.

Let us determine the change of the wave function of the field mode when many atoms cross the cavities. Let Δt be such that the number $\Delta t/\delta t$ of atoms crossing the cavities in a time interval $[t,t+\Delta t]$ is large but much smaller than the inverse of the coupling strength $|\eta|$:

$$1 \ll \frac{\Delta t}{\delta t} \ll |\eta|^{-1}, |\eta^{3/2}\epsilon|^{-1}, |\eta\epsilon|^{-2}. \tag{35}$$

Assuming that r_g and r_e are of the same order of magnitude, the numbers of atoms entering in the first cavity in state $|g\rangle$ and $|e\rangle$ between t and $t+\Delta t$ are large and approximately equal to $r_g\Delta t$ and $r_e\Delta t$, respectively. Since the probabilities $\delta p_\pm(t)$ of occurrence of jumps when one atom crosses the cavities are assumed to be as small as $|\eta|+|\eta^{3/2}\epsilon|+|\eta\epsilon|^2$, the probability of occurrence of two or more jumps between t and $t+\Delta t$ is negligible. Suppose first that no jump occur between these two times, i.e., that only cases (1) and (4) occur for all atoms. Then,

$$|\varphi(t+\Delta t)\rangle = \cdots(1-i\,\delta t K_g)\cdots(1-i\,\delta t K_e)\cdots(1-i\,\delta t K_g)\cdots|\psi(t)\rangle. \tag{36}$$

There are $r_g\Delta t$ factors $(1-i\,\delta t K_g)$ and $r_e\Delta t$ factors $(1-i\,\delta t K_e)$. Let us expand the products and neglect terms of order $(\Delta t/\delta t)^2|\eta^{(4+m)/2}\epsilon^m|$, $m=0,\ldots,4$, which are small by (35). We find:

$$|\psi(t+\Delta t)\rangle = \frac{|\varphi(t+\Delta t)\rangle}{\|\varphi(t+\Delta t)\|},$$
$$|\varphi(t+\Delta t)\rangle \simeq (1-i\Delta t K)|\psi(t)\rangle, \tag{37}$$

with $K = r_g\delta t K_g + r_e\delta t K_e$. Note that the change $|\varphi(t+\Delta t)\rangle - |\varphi(t)\rangle$ is proportional to Δt. The effective Hamiltonian K is the average of the operators K_i, which describe the evolution of the quantum field as one *given* atom crosses the cavities without absorbing or emitting a photon, over its (unknown) initial state $|i\rangle$ ($i=g$ with probability $r_g\delta t$, and $i=e$ with probability $r_e\delta t$). This averaging is related to our assumption that many atoms cross the cavities between t and $t+\Delta t$; it is an average over the initial states of the atoms, and must be distinguished from the average over the results of the measurements. Equation (29) gives

$$K = \frac{1}{2i}(\gamma_-(\tilde{a}^\dagger\tilde{a} + 2(-1)^k\epsilon^*\tilde{a} + |\epsilon|^2) + \gamma_+(\tilde{a}\tilde{a}^\dagger + 2(-1)^k\epsilon\tilde{a}^\dagger + |\epsilon|^2)). \tag{38}$$

If a jump $\pm$ occurs between t and $t+\Delta t$, the change of the wave function is approximately

$$|\psi(t+\Delta t)\rangle = \frac{W_\pm|\psi(t)\rangle}{\|W_\pm|\psi(t)\rangle\|}. \tag{39}$$

Actually, the change due to atoms crossing the cavities without modifying their states [cases (1) and (4)] between t and $t+\Delta t$ is of order $(\Delta t/\delta t)(|\eta|+|\eta^{3/2}\epsilon|)$. It can be neglected with respect to the change due to an atom having emitted or absorbed a photon, of order 1. Similarly, the probability $\Delta p_\pm(t)$ of occurrence of a jump $\pm$ in the time interval $[t,t+\Delta t]$ is obtained by replacing δt by Δt in (32).

The above coarse-grained stochastic dynamics coincides for $\gamma_+ = r_e = 0$ with that considered by Wiseman and Milburn[8] for a damped mode monitored by homodyne detection. In the Schrödinger picture, the mode wave function $|\psi_S(t+\Delta t)\rangle$ is given by (39), in which $W_\pm$ is replaced by $W_\pm^S(t) = e^{-itH}W_\pm e^{itH}$, if a jump $\pm$ occurs between t and $t+\Delta t$. In the opposite (and much more probable) case, it is given by (37), with K replaced by $H+K_S(t)$, $K_S(t) = e^{-itH}Ke^{itH}$. Here $H = \omega(a^\dagger a + 1/2)$ is the free Hamiltonian of the field mode, and we have assumed $\omega\Delta t \ll 1$. The jump operators in the Schrödinger picture are, up to irrelevant phase factors, $W_-^S(t) = \tilde{a} + (-1)^k\epsilon e^{-i\omega t}$ and $W_+^S(t) = \tilde{a}^\dagger + (-1)^k\epsilon^* e^{i\omega t}$. Like the Hamiltonian $H_{\rm int}$, $W_\pm$ depends on time in the Schrödinger picture and is time-independent in the interaction picture. The dynamics in the

Schrödinger picture is the same as in the interaction picture provided that ϵ is replaced by an oscillating field $\epsilon(t)=\epsilon e^{-i\omega t}$. This is also true for the dynamics between jumps, $K_S(t)$ being obtained from (38) by the same rule.

For $\epsilon=0$, the random evolution of $|\psi_S(t)\rangle$ coincides with the random evolution in the model of Dalibard *et al.*,[4] in agreement with the results of Ref. 27. For $\epsilon\neq 0$, it coincides with the QJ evolution of Sec. II with the two Lindblad operators $L_-=\sqrt{\gamma_-}\tilde{a}$ and $L_+=\sqrt{\gamma_+}\tilde{a}^\dagger$, and with $\lambda_- = \sqrt{\gamma_-}(-1)^k\epsilon e^{-i\omega t}$, $\lambda_+=\sqrt{\gamma_+}(-1)^k\epsilon^* e^{i\omega t}$.

All the results of this section are actually valid for an arbitrary Hamiltonian H_L—describing the interaction of the two-level atoms with an arbitrary external field placed between the first cavity and the detector—such that $\|H_L\|\tau_L \ll 1$. In fact, if we define $\epsilon = \langle e|e^{-i\tau_L H_L}|g\rangle/\lambda\tau\langle e|e^{-i\tau_L H_L}|e\rangle$, the only change in the calculation is the replacement of the prefactor $(1-|\eta\epsilon|^2)^{-1/2}$ by $\langle g|e^{-i\tau_L H_L}|g\rangle$ in (23) and (25) and by $\langle e|e^{-i\tau_L H_L}|e\rangle$ in (24) and (26). In particular, the stochastic dynamics is not modified if the laser frequency is detuned from the atomic frequency by $\delta=\omega_L-\omega$, with $|\delta|\tau_L\ll 1$. This is because ϵ is independent of δ in leading order in $\|H_L\|\tau_L$, i.e., the interaction time τ_L is too short for the atoms to feel the frequency shift from the atomic transition.

C. Average dynamics of the field mode

In order to relate the random wave function dynamics with the familiar density matrix approach, let us compute the master equation satisfied by the field mode density matrix:

$$\rho(t)=\mathsf{M}|\psi(t)\rangle\langle\psi(t)|, \tag{40}$$

where M denotes the mean value over the results of the measurements on the atoms (i.e., over all quantum trajectories).

As is well known, averaging the projector $|\psi(t)\rangle\langle\psi(t)|$ over all results of the measurements is the same as not performing any measurement. One may therefore equivalently determine the differential equation for the reduced density matrix $\tilde{\rho}(t)$ of the photon mode, in the same experimental scheme as in Fig. 1 but *without the detector*. The reduced density matrix $\tilde{\rho}(t)$ is defined as the partial trace over the atomic Hilbert spaces of the density matrix $\sigma(t)$ of the total system "atoms+field mode" (see Sec. II):

$$\tilde{\rho}(t)=\operatorname*{tr}_A(\sigma(t)). \tag{41}$$

In order to justify the above-mentioned statement, let us show that the changes of $\rho(t)$ and $\tilde{\rho}(t)$ when one atom crosses the cavities are the same. The atom arrives in the first cavity in state $|i\rangle$, $i=g$ or $i=e$. Before it enters in the cavity, the total density matrix $\sigma(t)$ is a tensor product $|i\rangle\langle i|\otimes\tilde{\rho}(t)$. We assume that $\tilde{\rho}(t)=\rho(t)=\mathsf{M}|\psi(t)\rangle\langle\psi(t)|$. It follows from the second equality in (17), from (19), and from the fact that the measurements on the atoms are independent:

$$\rho(t+\delta t)=\mathsf{M}_{t+\delta t}|\psi(t+\delta t)\rangle\langle\psi(t+\delta t)|=\operatorname*{tr}_A(\mathsf{M}_t|\Psi_{\mathrm{ent}}\rangle\langle\Psi_{\mathrm{ent}}|), \tag{42}$$

where $|\Psi_{\mathrm{ent}}\rangle$ is given by (16). Thus,

$$\rho(t+\delta t)=\operatorname*{tr}_A(e^{-i\tau_L H_L}e^{-i\tau H_{\mathrm{int}}}|i\rangle\langle i|\rho(t)e^{i\tau H_{\mathrm{int}}}e^{i\tau_L H_L})=\tilde{\rho}(t+\delta t)=\operatorname*{tr}_A(e^{-i\tau H_{\mathrm{int}}}\sigma(t)e^{i\tau H_{\mathrm{int}}}). \tag{43}$$

Hence the changes of $\rho(t)$ and $\tilde{\rho}(t)$ are identical as one atom crosses the cavities. The two exponentials of H_L in (43) disappear by cyclicity of the trace, showing that $\tilde{\rho}(t+\delta t)=\rho(t+\delta t)$ does not depend on the laser field $\mathcal{E}$. This last point is actually clear, since if no measurement on the atoms is performed, their interaction with the laser field in the second cavity has no effect on the field in the first cavity.

Since they have the same time evolution, we identify in what follows $\rho(t)$ and $\tilde{\rho}(t)$ and compute the master equation for the usual density matrix (41). Two cases must be distinguished.

(1) The atom enters in the first cavity in its ground state: $i=g$. By replacing (20) into (43),

$$\rho(t+\delta t)=(1-|\eta|^2\tilde{a}^\dagger\tilde{a})^{1/2}\rho(t)(1-|\eta|^2\tilde{a}^\dagger\tilde{a})^{1/2}+|\eta|^2\tilde{a}\rho(t)\tilde{a}^\dagger\equiv(1+\delta t\mathcal{L}_g)\rho(t), \tag{44}$$

where $\tilde{a}$ is given by (21).

(2) The atom enters in the first cavity in its excited state: $i=e$. Then,

$$\rho(t+\delta t)=(1-|\eta|^2\tilde{a}\tilde{a}^\dagger)^{1/2}\rho(t)(1-|\eta|^2\tilde{a}\tilde{a}^\dagger)^{1/2}+|\eta|^2\tilde{a}^\dagger\rho(t)\tilde{a}\equiv(1+\delta t\mathcal{L}_e)\rho(t). \tag{45}$$

For a given initial state of the atom, the evolution in (44) and (45) has the general form (7). Under the conditions (15) and (28), one has

$$\begin{aligned}\delta t\mathcal{L}_g(\rho)&\simeq|\eta|^2(\tilde{a}\rho\tilde{a}^\dagger-\tfrac{1}{2}\{\tilde{a}^\dagger\tilde{a},\rho\}),\\ \delta t\mathcal{L}_e(\rho)&\simeq|\eta|^2(\tilde{a}^\dagger\rho\tilde{a}-\tfrac{1}{2}\{\tilde{a}\tilde{a}^\dagger,\rho\}),\end{aligned} \tag{46}$$

where the curly brackets denote the anticommutator $\{A,B\}=AB+BA$.

Let us determine the coarse-grained evolution of $\rho(t)$ on the time scale Δt satisfying (35). By the same arguments as in Sec. III B,

$$\rho(t+\Delta t)=\cdots(1+\delta t\mathcal{L}_g)\cdots(1+\delta t\mathcal{L}_e)\cdots(1+\delta t\mathcal{L}_g)\cdots\rho(t),$$

with $r_g\Delta t$ factors $(1+\delta t\mathcal{L}_g)$ and $r_e\Delta t$ factors $(1+\delta t\mathcal{L}_e)$. One can retain only the terms of order one in Δt in the expansion of the product:

$$\rho(t+\Delta t)\simeq\{1+\Delta t(r_g\delta t\mathcal{L}_g+r_e\delta t\mathcal{L}_e)\}\rho(t). \tag{47}$$

The superoperator inside the parentheses is the average of the superoperator $\mathcal{L}_i$ over the initial atomic states $|i\rangle$, $i=e$ or g. Writing $\Delta\rho/\Delta t=\mathrm{d}\rho/\mathrm{d}t$, the coarse-grained master equation in the interaction picture is therefore

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\gamma_-(\tilde{a}\rho(t)\tilde{a}^\dagger-\tfrac{1}{2}\{\tilde{a}^\dagger\tilde{a},\rho(t)\})+\gamma_+(\tilde{a}^\dagger\rho(t)\tilde{a}-\tfrac{1}{2}\{\tilde{a}\tilde{a}^\dagger,\rho(t)\}). \tag{48}$$

Not surprisingly, this is the equation of the damped harmonic oscillator with finite temperature T. Here T is the temperature of the atomic beam:

$$\exp\left(-\frac{\omega}{k_BT}\right)=\frac{r_e}{r_g}=\frac{\gamma_+}{\gamma_-}, \tag{49}$$

where k_B is the Boltzmann constant. Equation (48) has the general form (1), with two Lindblad operators $L_-=\sqrt{\gamma_-}\tilde{a}$ and $L_+=\sqrt{\gamma_+}\tilde{a}^\dagger$. Recall that $\tilde{a}$ coincides with the usual annihilation operator a only if $k=0$ (perturbative regime).

In conclusion, if $|\lambda|\tau\ll1$ and condition (28) holds, i.e., if each atom modifies weakly the state of the field, the average density matrix (40) satisfies the master equation (48). Although the quantum trajectories of the mode depend on $\mathcal{E}$, the corresponding master equation is the same for all laser fields, in accordance with the general results of Sec. II.

IV. SINGLE QUANTUM TRAJECTORIES OF THE FIELD MODE

We focus in this section on single quantum trajectories of the field mode, corresponding to specific realizations of the measurements. The main question addressed in the following concerns the localization properties of the random dynamics at large times, for different laser fields $\mathcal{E}$. In the

case $\mathcal{E}=0$, i.e, for the ''standard'' Monte Carlo wave function dynamics, it has been shown in Ref. 27 that the field wave function evolves at large times toward Fock states $|n(t)\rangle$ [with a time fluctuating number of photons $n(t)$]. Note that it is straightforward to prove that Fock states form an invariant family of states under the stochastic dynamics. In fact, if $\epsilon=0$, the jump operators $W_{\pm}$ transform $|n\rangle$ into $|n\pm1\rangle$ (up to multiplicative constants), and $|n\rangle$ is an eigenvector of $H+K_0=\omega(n+1/2)-i(\gamma_-\sin^2(|\eta|n^{1/2})+\gamma_+\sin^2(|\eta|(n+1)^{1/2}))/(2|\eta|^2)$. The localization property is much more difficult to prove. For nonzero laser fields, Fock states do no longer form an invariant family and the localization is of different nature. In the case of zero temperature ($\gamma_+=0$), it has been shown in Ref. 8 that the field mode has a diffusive dynamics in the limit $\epsilon\to\infty$, given by a stochastic Schrödinger equation with real Wiener processes; we will prove in Sec. V that this remains true at any temperature. Such kind of dynamics has been widely studied in the literature.[11,12,15,26,28] It is expected that localization occurs toward coherent or squeezed states if the Lindblad operators are linear combinations of a and $a^\dagger$. However, this has only been proved in the case $L\propto(a+a^\dagger)$ as far as we are aware.[28] Here we present numerical results indicating that for $T>0$ and large enough ϵ, the mode wave function $|\psi(t)\rangle$ evolves toward squeezed states after some time $\Delta\tau$ of the order of the thermalization time in the absence of measurements. At large times, the squeezing amplitude $r(t)$ of the squeezed states is found to fluctuate slightly around a mean value $\overline{r}$, and the squeezing angle $\phi(t)$ evolves linearly in time, $\phi(t)=\overline{\phi_0}-\omega t$. Interestingly, $\overline{r}$ and $\overline{\phi_0}$ are independent of the realization and of the initial state, being, respectively, functions of the temperature T and the laser field $\mathcal{E}$ only. By letting $\epsilon\to\infty$, writing the corresponding quantum state diffusion equation (which is derived in the general case in Sec. V), and using a result due to Rigo and Gisin,[26] we obtain in Sec. IV B analytic expressions for $\overline{r}$ and $\overline{\phi_0}$ in good agreement with the numerical simulations.

A. Numerical results

Let us first look at the Husimi Q-function for the mode wave function in the interaction picture,

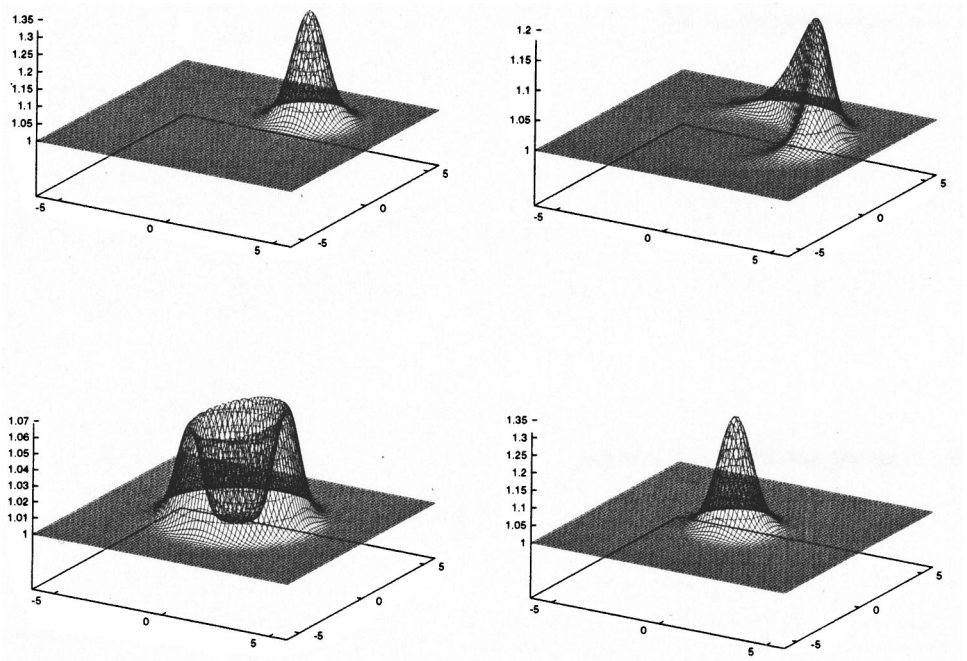

FIG. 2. $Q(\alpha,t)+1$ for $\epsilon=0$ and $\gamma_+/\gamma_-=3/4$, at four different times: $\gamma_- t=0$ (top left), $\gamma_- t=5$ (top right), $\gamma_- t=20$ (bottom left), and $\gamma_- t=60$ (bottom right).

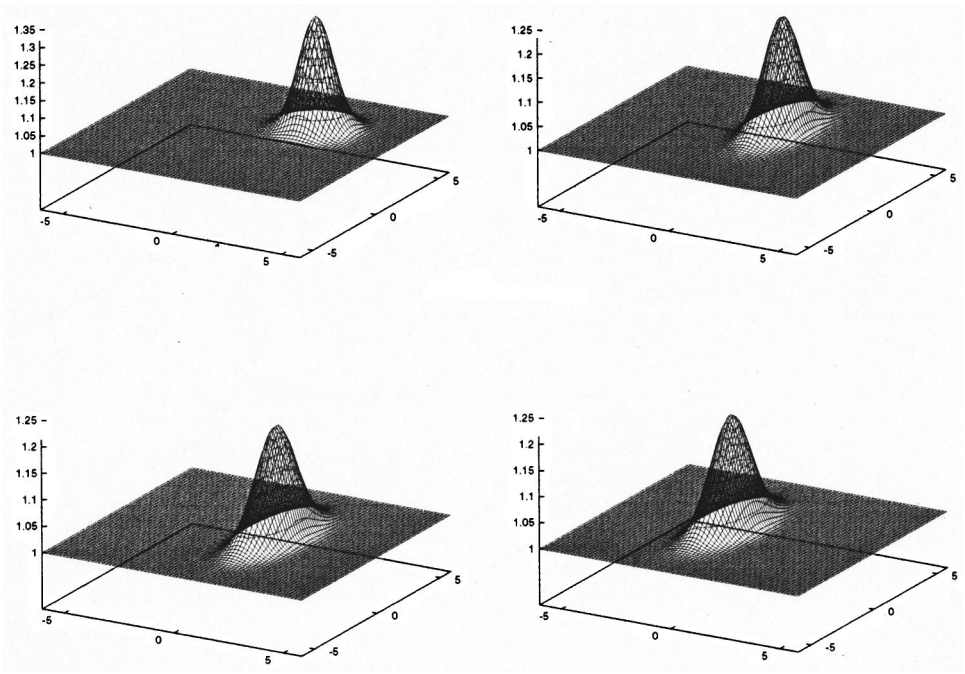

FIG. 3. $Q(\alpha,t)+1$ for $\epsilon=50$ and $\gamma_+/\gamma_-=3/4$, at the same times as in Fig. 2.

$$Q(\alpha,t)=\frac{1}{\pi}|\langle\alpha|\psi(t)\rangle|^2, \quad \alpha\in\mathbb{C}.$$

The simulations of single quantum trajectories for $\epsilon=0$ and $\epsilon=50$ give the results plotted in Figs. 2 and 3, respectively. Recall that for Fock states $|\psi\rangle=|n\rangle$, $Q(\alpha)$ is equal to $|\alpha|^{2n}\exp(-|\alpha|^2)/\pi n!$, and for squeezed states $|\psi\rangle=|\alpha_0,\xi\rangle$ with $\xi\in\mathbb{R}$, $|\xi|=r$,

$$Q(\alpha)=\frac{1}{\pi\cosh(r)}\exp\left(-\frac{2\Re(\alpha-\alpha_0)^2}{1+e^{-2r}}-\frac{2\Im(\alpha-\alpha_0)^2}{1+e^{2r}}\right)$$

(see Ref. 29). This asymmetric paraboloid becomes symmetric for a coherent state $|\psi\rangle=|\alpha_0\rangle$, which corresponds to $r=0$ in the above-mentioned formula. It is seen in Fig. 2 that in the case $\epsilon=0$, the initial coherent state $|\alpha=2+2i\rangle$ is transformed into states which are close to Fock states at times $t_3=20/\gamma_-$ $(n\neq0)$ and $t_4=60/\gamma_-$ $(n=0)$. This is in agreement with the results of Ref. 27. On the other hand, for $\epsilon=50$, Fig. 3 shows that the same initial coherent state is transformed into states which are close to squeezed states $|\alpha,re^{i\phi}\rangle$. The squeezed states at t_3 and t_4 have approximately the same squeezing parameters r and $\phi=0$, but they have different centers α.

In order to study more precisely the localization toward squeezed states at large values of ϵ, let us follow the time evolution of the mean square deviations for a given quantum trajectory $t\mapsto|\psi(t)\rangle$:

$$\begin{aligned}\Delta x_\phi^2(t)&=\langle\psi_S(t)|X_\phi^2|\psi_S(t)\rangle-\langle\psi_S(t)|X_\phi|\psi_S(t)\rangle^2,\\ \Delta y_\phi^2(t)&=\langle\psi_S(t)|Y_\phi^2|\psi_S(t)\rangle-\langle\psi_S(t)|Y_\phi|\psi_S(t)\rangle^2.\end{aligned} \tag{50}$$

X_ϕ and Y_ϕ are the usual field quadrature operators rotated by an angle ϕ:

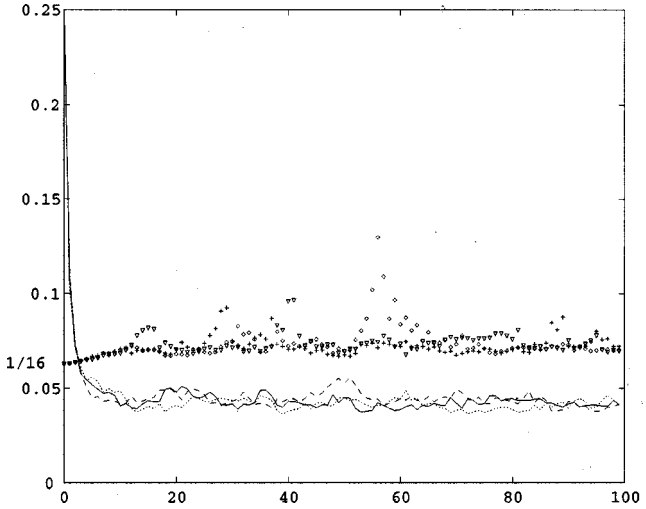

FIG. 4. $\Delta x^2(t)$ (solid, dashed, and dotted lines) and $\Delta x^2(t)\Delta y^2(t)$ [(+), ($\diamond$), and ($\triangledown$)] vs $\gamma_- t$ for three different quantum trajectories with initial coherent state. The values of the parameters are $\epsilon=20$ and $\gamma_+/\gamma_-=3/4$. The dotted line and the triangles correspond to the exact dynamics using the nonperturbative formulas (23)–(26) for K_g, $W_\pm$, and K_e, with $|\lambda|\tau=0.93\times10^{-2}$.

$$X_\phi=\frac{ae^{-i\phi}+a^\dagger e^{i\phi}}{2},\quad Y_\phi=X_{\phi+\pi/2}=\frac{ae^{-i\phi}-a^\dagger e^{i\phi}}{2i}. \tag{51}$$

We denote by $\phi_{\min}(t)$ the angle for which $\Delta x^2_\phi(t)$ is minimum. The minimum mean square deviation $\Delta x^2_{\phi_{\min}(t)}(t)$ is denoted by $\Delta x^2(t)$ and, similarly, $\Delta y^2_{\phi_{\min}(t)}(t)$ is denoted by $\Delta y^2(t)$. Obviously, $\Delta x^2(t)$ and $\Delta y^2(t)$ remain unchanged if the wave function $|\psi_S(t)\rangle$ in the Schrödinger picture is replaced in (50) by the wave function $|\psi(t)\rangle$ in the interaction picture. This replacement leads to an increase of $\phi_{\min}(t)$ by ωt.

The time evolutions of $\Delta x^2(t)$ and $\Delta x^2(t)\Delta y^2(t)$ are shown in Figs. 4 and 5, for different quantum trajectories starting from the same coherent state $|\psi(0)\rangle=|\alpha\rangle$. Note that the time scale is of order of the thermalization time of the density matrix $\rho(t)$. Figure 4 corresponds to $\epsilon=20$ and Fig. 5 to $\epsilon=100$. One sees in both Figs. 4 and 5 that $\Delta x^2(t)$ begins to fluctuate around a mean

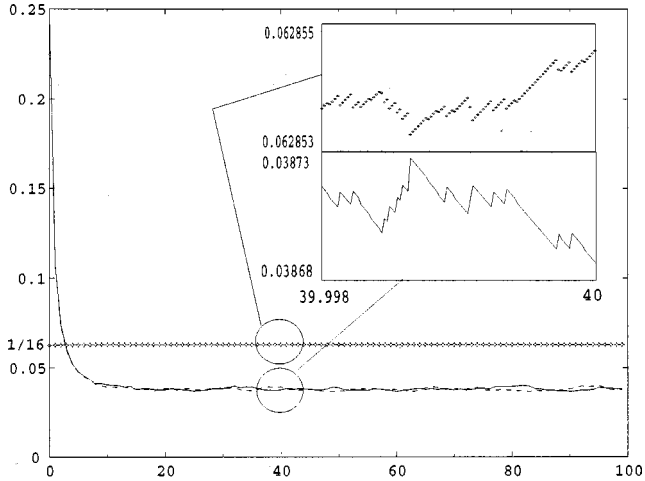

FIG. 5. Same as in Fig. 4 but with $\epsilon=100$ (the nonperturbative results are not shown but look similar). In the upper square, $\Delta x^2(t)$ and $\Delta x^2(t)\Delta y^2(t)$ are shown on a finer time scale, on which discontinuous quantum jumps can be seen separately.

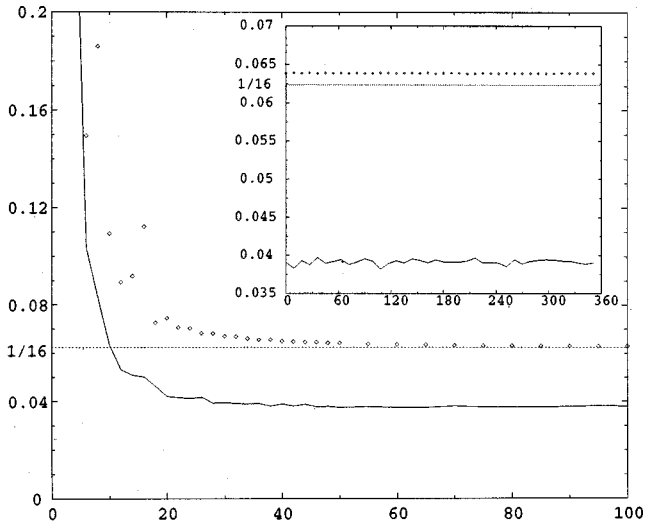

FIG. 6. Time averages $\overline{\Delta x^2}$ (plain line) and $\overline{\Delta x^2 \Delta y^2}$ ($\diamond$) for different values of $\Re\epsilon$ (shown in the horizontal axis) and $\Im\epsilon=0$. Inset: same for different values of arg ϵ (shown in degrees in the horizontal axis) and fixed modulus $|\epsilon|=50$. The time average is taken on the interval $[10/\gamma_-,100/\gamma_-]$ and $\gamma_+/\gamma_-=3/4$.

value $\overline{\Delta x^2}$ after some transient time $\Delta\tau \simeq 10/\gamma_-$. The comparison of the numerical results for the different trajectories shows that $\overline{\Delta x^2}$ does not depend upon the specific realization. The fluctuations are considerably reduced in the case $\epsilon=100$ (Fig. 5) with respect to the case $\epsilon=20$ (Fig. 4). Moreover, in the former case, the product $\Delta x^2(t)\Delta y^2(t)$ is much closer to the minimum value 1/16 allowed by the Heisenberg uncertainty principle. The variation of $\overline{\Delta x^2}$ and $\overline{\Delta x^2 \Delta y^2}$ with ϵ for fixed temperature $T \simeq 0.288\omega/k_B$ is presented in Fig. 6. One observes that $\overline{\Delta x^2}$ is almost constant for $50 \leqslant \epsilon \leqslant 100$. Furthermore, the time average of $\Delta x^2(t)\Delta y^2(t)$ goes closer and closer to 1/16 as ϵ increases. The result presented in the inset shows moreover that $\overline{\Delta x^2}$ and $\overline{\Delta x^2 \Delta y^2}$ are independent of arg(ϵ), up to small fluctuations.

Most numerical simulations were done using the approximations (31) and (38) for the jump operators $W_\pm$ and for K, together with $\tilde{a}=a$. This is justified if the maximum number of photons $n_{\max}$ is small with respect to $|\eta|^{-2}=|\lambda|^{-2}\tau^{-2}$, and $|\eta|^{-2}=|\lambda|^{-2}\tau^{-2}$, and $n_{\max}^{3/2} \ll |\eta|^{-4}|\epsilon|^{-1}$, $n_{\max}^{3/2} \ll |\eta|^{-2}|\epsilon|$ (see Sec. III B). Since we worked in the Fock states basis, $n_{\max}$ was bounded by the dimension of the Hilbert space (which was taken between 75 and 150). One may be worried, however, that, despite the smallness of the error made at each time step δt, the error made after many steps might be large and could invalidate our results in the long time limit. We checked that this is actually not the case by integrating numerically the exact dynamics, using formulas (23)–(26) for the evolution on each time step δt, and the exact possibilities of occurrence of the four cases. The curve in the dotted line and the triangles in Fig. 4 are the values of $\Delta x^2(t)$ and $\Delta x^2(t)\Delta y^2(t)$ obtained from a simulation of this nonperturbative dynamics. This means using the exact formulas (23)–(26) for the evolution on each time step δt, and the exact probabilities of occurrence of the four cases. The total number of atoms crossing the cavity in the whole time interval is 2×10^6, and $|\eta|=\sqrt{(\gamma_-+\gamma_+)\delta t} \simeq 0.0093$. The exact results show very similar fluctuations, around the same values $\overline{\Delta x^2}$ and $\overline{\Delta x^2 \Delta y^2}$, as the two trajectories obtained using the perturbative scheme. No systematic deviation increasing with time is seen. This result, together with other simulations for different values of ϵ and η,[31] shows errors in the considered time range. Note however that if $|\eta|$ is too large (for values in the range 0.06–0.09 or larger in Fig. 4, and 0.02–0.04 or larger in Fig. 5), large time fluctuations of $\Delta x^2(t)$ and $\Delta x^2(t)\Delta y^2(t)$ are observed and therefore the localization is of different nature.

In all our simulations, we found that $\overline{\Delta x^2}$ and $\overline{\Delta x^2 \Delta y^2}$ are insensitive to the initial state $|\psi(0)\rangle$ of the field mode. This is illustrated in Fig. 7 for $\epsilon=20$ and $\epsilon=20+10i$ (similar results are

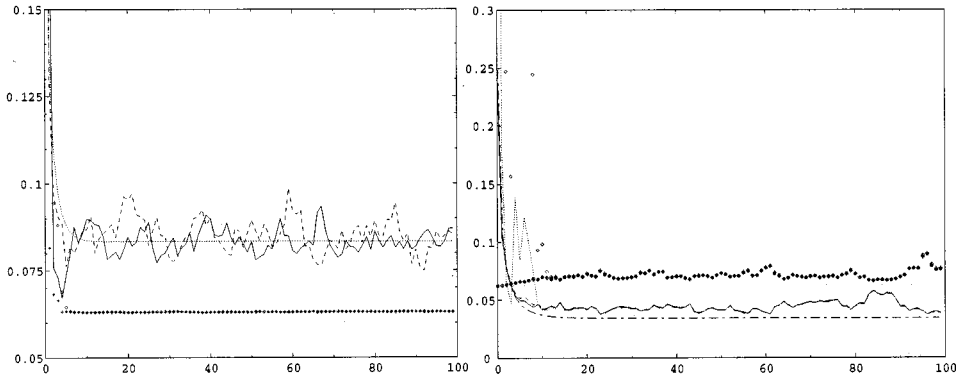

FIG. 7. $\Delta x^2(t)$ (lines) and $\Delta x^2(t)\Delta y^2(t)$ [(+), ($\diamond$), and ($\bigcirc$)] vs $\gamma_- t$ for trajectories with different initial states $|\psi(0)\rangle$. Left-hand side: in the two shown trajectories, $\langle n|\psi(0)\rangle$ is chosen randomly for $0 \leqslant n \leqslant 20$ and vanishes for $n>20$; $|\psi(0)\rangle$ and the realization of the measurements are different in each trajectory; $\epsilon = 20+10i$ and $\gamma_+/\gamma_- = 1/2$. Right-hand side: (1) $|\psi(0)\rangle = |\alpha\rangle$ with $\alpha = \sqrt{3/2}(1+i)$ [plain line, ($\bigcirc$);] (2) id. with $\alpha = -1+5i$ [dashed line, (+);] (3) $|\psi(0)\rangle = |n=5\rangle$ [dotted line, ($\diamond$)]; in all cases, $\epsilon = 20$ and $\gamma_+/\gamma_- = 3/4$. The dotted line on the left-hand side and dot-dashed line on the right-hand side correspond to the theoretical result (64) for $\epsilon \to \infty$.

obtained for $\epsilon = 100$). On the right-hand side of the Fig. 7, it is seen that $\Delta x^2(t)$ differs noticeably at small t's if $|\psi(0)\rangle$ is a coherent state [cases (1) and (2)] and if it is a Fock state [case (3); the first small time values are outside the range of the figure: $\Delta x^2(0) = 2.75$]. However, the three curves are hard to distinguish for $\gamma_- t \gtrsim 10$. In the two trajectories shown on the left-hand side, the two distinct initial states are chosen by picking randomly the 20 first components $\langle n|\psi(0)\rangle$ in the Fock states basis (the realization of the measurements is also different in the two cases). For times bigger than $10/\gamma_-$, one observes that $\Delta x^2(t)$ fluctuates around the same value for both trajectories.

For the quantum trajectories studied in Figs. 3–5, the angle $\phi_{\min}(t)+\omega t$ is found to be zero at times $t \gtrsim \Delta\tau$. Nonvanishing angles are obtained if one considers non real ϵ's. The time dependence of $\phi_{\min}(t)+\omega t$ for $\epsilon = 20+10i$ is presented in Fig. 8 for different initial states. It is seen that it evolves toward a constant value $\overline{\phi_0}$ which neither depends on $|\psi(0)\rangle$ nor on the ratio γ_+/γ_-. As shown in the inset, we obtain $\overline{\phi_0} = \arg(\epsilon)$.

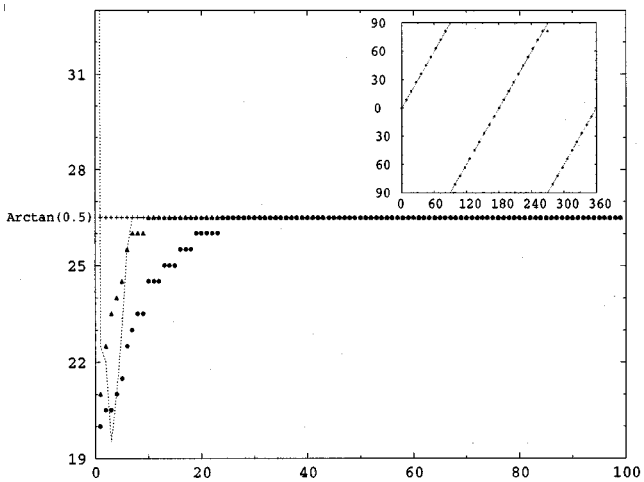

FIG. 8. Angle $\phi_{\min}(t)+\omega t$ (in degrees) vs $\gamma_- t$ for $\epsilon = 20+10i$ and different initial states and temperatures: (1) $|\psi(0)\rangle = |\alpha\rangle$ with $\alpha = 2-2i$; $\gamma_+/\gamma_- = 1/4$ ($\triangle$); (2) $|\psi(0)\rangle = (2+2e^{-9})^{-1/2}(|\alpha\rangle + |\beta\rangle)$ with $\alpha = -3i$ and $\beta = 3$; $\gamma_+/\gamma_- = 3/4$ ($\bullet$); (3) $|\psi(0)\rangle = |n\rangle$ with $n = 10$; $\gamma_+/\gamma_- = 3/4$ (+); (4) $|\psi(0)\rangle$ is chosen randomly as in Fig. 7; $\gamma_+/\gamma_- = 1/2$ (dashed line). Inset: time average of $\phi_0(t) = \phi_{\min}(t)+\omega t$ as function of $\arg\epsilon$ (both in degrees) for $|\epsilon| = 50$ and $\gamma_+/\gamma_- = 3/4$ (time average as in Fig. 6). The result is well fitted by the broken lines $\overline{\phi_0} = \arg(\epsilon)$ mod. 180.

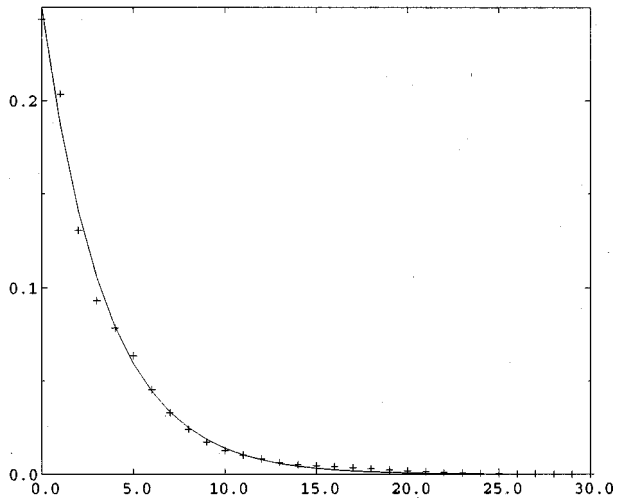

FIG. 9. Time average of the probability $|\langle n|\psi(t)\rangle|^2$ to find n photons for $\epsilon=50$, $\gamma_+/\gamma_-=3/4$ (+). The average is taken over 5000 discrete times in the interval $[0,100/\gamma_-]$. The solid line corresponds to the Bose–Einstein distribution.

Bringing together the above-mentioned results, we are led to the following conclusions. For large $|\epsilon|$ and small $|\eta|$ smaller than $\simeq 0.01$, the wave function $|\psi_S(t)\rangle$ of the field mode evolves, whatever its initial state $|\psi_S(0)\rangle$, to some almost minimum uncertainty states (MUS) at times $t \geqslant \Delta\tau$, where $\Delta\tau$ is a transient time. Moreover, if $T>0$,

$$\Delta x^2(t) \simeq \overline{\Delta x^2} < 1/4, \quad \phi_{\min}(t) \simeq \overline{\phi_0} - \omega t \tag{52}$$

at large times where both $\overline{\Delta x^2}$ and $\overline{\phi_0} = \arg(\epsilon)$ are independent of $|\psi_S(0)\rangle$ and of the realization of the measurements. Therefore, provided $|\epsilon|$ is big enough and $|\eta|$ small enough, the state of the field mode at times $t \geqslant \Delta\tau$ is close to a squeezed state:

$$|\psi_S(t)\rangle \simeq |\alpha(t), \xi(t)\rangle. \tag{53}$$

The squeezing amplitude $r(t)=|\xi(t)|$ fluctuates slightly around a time-independent value $r \simeq -\ln(4\overline{\Delta x^2})/2$ and $\arg(\xi(t)) \simeq 2(\overline{\phi_0} - \omega t)$. The center $\alpha(t)$ of the squeezed state moves around in the complex plane. Actually, by ergodicity, the time average of $|\psi_S(t)\rangle\langle\psi_S(t)|$ must coincide with the equilibrium density matrix $\rho^{(\mathrm{eq})} = Z^{-1}\exp(-\omega(a^\dagger a + 1/2)/k_B T)$. To check ergodicity, we have computed numerically the time average of $|\langle n|\psi(t)\rangle|^2$ on the interval $[0,100/\gamma_-]$ for a *single* quantum trajectory. It is indeed seen in Fig. 9 that it reproduces well the Bose–Einstein exponential distribution $\rho_{nn}^{(\mathrm{eq})}$.

B. Analytical results

The above-given numerical results suggest that the dynamics has the localization property toward squeezed states in the limit of large laser fields $|\epsilon| \to \infty$. In particular, the squeezed states should form an invariant family of states under the stochastic dynamics in this limit. The second statement can be shown analytically as follows. We restrict our analysis here to the perturbative regime where $|\eta| \ll 1$, $|\eta\epsilon| \ll 1$ and (28) holds with $k=0$, so that $\tilde{a} \simeq a$. As said previously, one can describe the mode's dynamics in the limit $|\epsilon| \to \infty$, $|\eta| \to 0$, $|\eta\epsilon| \to 0$ by a stochastic Schrödinger equation with real Wiener processes (quantum state diffusion). This is because the probability of occurrence of jumps grows like $|\epsilon|^2$ (for instance, in Figs. 4 and 5, the total number of jumps in the whole time interval $[0,100/\gamma_-]$ is close to 7×10^4 and 1.75×10^6, respectively). On the other hand, as is clear from (31) and (39), the change of the wave function during a jump is of order $1/\epsilon$. Hence there are infinitely many jumps with an infinitesimal impact on the wave function in the

limit $|\epsilon|\to\infty$. On a time scale much bigger than the inverse frequencies of jumps [but small enough so that $|\psi(t)\rangle$ does not change much on such a time scale], $|\psi(t)\rangle$ satisfies the following Itô stochastic differential equation in the limit $\epsilon\to\infty$, $\arg\epsilon=\theta$:

$$|\mathrm{d}\psi\rangle=\Big[\sqrt{\gamma_-}(e^{-i\theta}a-\Re\langle e^{-i\theta}a\rangle_t)\mathrm{d}w_-(t)+\sqrt{\gamma_+}(e^{i\theta}a^\dagger-\Re\langle e^{i\theta}a^\dagger\rangle_t)\mathrm{d}w_+(t)$$
$$+\Re\langle e^{i\theta}a^\dagger\rangle_t\Big(\gamma_-e^{-i\theta}a+\gamma_+e^{i\theta}a^\dagger-\frac{\gamma_-+\gamma_+}{2}\Re\langle e^{i\theta}a^\dagger\rangle_t\Big)\mathrm{d}t-iK_0\,\mathrm{d}t\Big]|\psi(t)\rangle. \quad (54)$$

The notation is as follows: $K_0=(\gamma_-a^\dagger a+\gamma_+aa^\dagger)/2i$, $\langle O\rangle_t=\langle\psi(t)|O|\psi(t)\rangle$ is the quantum expectation at time t and $\mathrm{d}w_\pm(t)$ are the stochastic infinitesimal increments of two independent real Wiener processes, which have zero mean and satisfy the Itô rules:[30]

$$\mathrm{d}w_\pm(t)\mathrm{d}w_\pm(t)=\mathrm{d}t,\quad \mathrm{d}w_-(t)\mathrm{d}w_+(t)=0,\quad \mathrm{d}w_\pm(t)\mathrm{d}t=0. \quad (55)$$

Equation (54) will be derived in Sec. V in the general case. It belongs to the class of stochastic Schrödinger equations studied in Refs. 11 and 12 and has been derived from the QJ dynamics by Wiseman and Milburn[8] (see also Refs. 5 and 9) in the case of a mode in a decaying cavity (corresponding here to $\gamma_+=0$). It can actually be derived directly from the stochastic dynamics of Sec. III B, Eqs. (23)–(26), with the probabilities (27), in the limit $|\epsilon|\to\infty$, $|\eta|\to0$, $|\epsilon\eta|=$const., under assumption (28) with $k=0$.[31] A related equation with complex Wiener processes has been studied in Ref. 10.

Rigo and Gisin[26] have shown that the stochastic Schrödinger equation (54) preserves squeezed states. Since (54) actually differs from the equation considered by these authors by some additional phase factors $e^{\pm i\theta}$, and the explicit solution of the evolution equations for the squeezing parameters is not given in Ref. 26, we briefly recall here their derivation. We use the Itô formalism of stochastic differential equations,[30] whereas Stratanovich formalism was used in Ref. 26. It is convenient to characterize squeezed states by the following criterion:

$$|\psi\rangle=\|\psi\||\alpha,\xi=re^{2i\phi}\rangle\Leftrightarrow(a-\Gamma a^\dagger-\beta)|\psi\rangle=0,\quad \Gamma=-e^{2i\phi}\tanh(r),\quad \beta=\alpha-\Gamma\alpha^*. \quad (56)$$

The family of the squeezed states is invariant under (54) if $|\psi\rangle+|d\psi\rangle$ remains a squeezed state for any $|\psi\rangle=|\alpha,\xi\rangle$, i.e., $(a-\Gamma a^\dagger-\mathrm{d}\Gamma\,a^\dagger-\beta-\mathrm{d}\beta)(|\psi\rangle+|\mathrm{d}\psi\rangle)=0$. This is equivalent to[26]

$$[a-\Gamma a^\dagger,D(\psi)]|\psi\rangle-(\mathrm{d}\Gamma\,a^\dagger+\mathrm{d}\beta)D(\psi)|\psi\rangle=(\mathrm{d}\Gamma\,a^\dagger+\mathrm{d}\beta)|\psi\rangle, \quad (57)$$

where $D(\psi)$ is the operator inside the square brackets in (54). The left-hand side is found to be

$$\Big(-\frac{\gamma_++\gamma_-}{2}(a+\Gamma\,a^\dagger)\mathrm{d}t+\Re\langle e^{i\theta}a^\dagger\rangle(\gamma_+e^{i\theta}\mathrm{d}t+\gamma_-e^{-i\theta}\Gamma\,\mathrm{d}t+\sqrt{\gamma_+}\mathrm{d}\beta\,\mathrm{d}w_++\sqrt{\gamma_-}\mathrm{d}\beta\,\mathrm{d}w_-)$$
$$+\sqrt{\gamma_+}\,e^{i\theta}(-a^\dagger\mathrm{d}\beta\,\mathrm{d}w_++\mathrm{d}w_+)+\sqrt{\gamma_-}e^{-i\theta}(-a\,\mathrm{d}\beta\,\mathrm{d}w_-+\Gamma\,\mathrm{d}w_-)\Big)|\psi\rangle. \quad (58)$$

We have thrown away all terms $\mathrm{d}\Gamma\,\mathrm{d}w_\pm$ since $\mathrm{d}\Gamma$ is proportional to $\mathrm{d}t$. This is because, by inspection of (58), the terms containing the noises $\mathrm{d}w_\pm$ on the left-hand side of (57) are proportional to $|\psi\rangle$. Multiplying both members of (57) by $\mathrm{d}w_\pm$ and using (55) and (58) gives

$$\mathrm{d}\beta\,\mathrm{d}w_+=\sqrt{\gamma_+}e^{i\theta}\mathrm{d}t,\quad \mathrm{d}\beta\,\mathrm{d}w_-=\sqrt{\gamma_-}e^{-i\theta}\Gamma\,\mathrm{d}t. \quad (59)$$

Now we use the well-known identity $S(-\xi)D(-\alpha)aD(\alpha)S(\xi)=a\cosh r+\Gamma a^\dagger\cosh r+\alpha$, where $S(\xi)=\exp(\xi^*a^2/2-\xi a^{\dagger2}/2)$ and $D(\alpha)=\exp(\alpha a^\dagger-\alpha^*a)$ are, respectively, the squeezing and the displacement operators.[2] The squeezed state $|\psi\rangle$ is equal to $D(\alpha)S(\xi)|0\rangle$. Let us multiply the two members of (57) by $S(-\xi)D(-\alpha)$ and substitute (58) and (59) into this equation. This leads to the two coupled stochastic differential equations:

$$
\begin{aligned}
d\Gamma &= -e^{2i\theta}(1+e^{-2i\theta}\Gamma)(\gamma_+ + \gamma_- e^{-2i\theta}\Gamma)dt, \\
d\beta &= -\left(\frac{\gamma_- + \gamma_+}{2} + \gamma_- e^{-2i\theta}\Gamma\right)\beta\, dt + 2\Re(e^{i\theta}a^*)(\gamma_+ e^{i\theta} + \gamma_- e^{-i\theta}\Gamma)dt \\
&\quad + \sqrt{\gamma_+}e^{i\theta}dw_+ + \sqrt{\gamma_-}e^{-i\theta}\Gamma\, dw_- .
\end{aligned}
\tag{60}
$$

These equations are equivalent to (57) and have a solution. Hence the squeezed states $|\alpha,\xi\rangle$ form an invariant family under the quantum state diffusion dynamics, and α and ξ evolve in time according to (60).

We now obtain the squeezing amplitudes $r(t)$ and angles $\phi(t)$ of the squeezed states by solving (60). They are determined by the first equation:

$$
\Gamma(t) = -e^{2i\theta}\frac{\gamma_+ e^{(\gamma_- - \gamma_+)t} + c}{\gamma_- e^{(\gamma_- - \gamma_+)t} + c} = -e^{2i\phi(t)}\tanh(r(t)), \tag{61}
$$

where c is an arbitrary complex constant. Thus $r(t)$ and $\phi(t)$ are deterministic, unlike $\alpha(t)$, which is given by the second equation. Going back to the Schrödinger picture, one has for $t(\gamma_- - \gamma_+) \gg 1$:

$$
\begin{aligned}
\tanh(r(t)) &\simeq \frac{\gamma_+}{\gamma_-} = \exp\left(-\frac{\omega}{k_B T}\right), \\
\phi(t) &\simeq \theta - \omega t.
\end{aligned}
\tag{62}
$$

It is worth noticing that perfect squeezing $[r(t)\to\infty]$ is obtained in the *high* temperature limit $k_B T \gg \omega$. The rate $(\gamma_- - \gamma_+)^{-1}$ of convergence of $\tanh r(t)$ to $\tanh \bar{r} = \exp(-\omega/k_B T)$ also tends to infinity in this limit. Since the number of photons in the cavity becomes very large at very high temperatures and long times, the perturbative approximation (28) should however break down at some point. This can put a limitation on the attainment of very large $r(t)$. A more detailed study of this apparently surprising result is the object in a separate work.[31]

As stated previously, the second equation in (62) agrees quite well with the results of the numerical simulations of Fig. 8. Using the minimum mean square deviation $\Delta x^2 = e^{-2r}/4$ of squeezed states[2] yields the time average $\overline{\Delta x^2}$:

$$
\overline{\Delta x^2} \simeq \frac{\gamma_- - \gamma_+}{4(\gamma_- + \gamma_+)}. \tag{63}
$$

In the particular case of an initial coherent state $|\psi(0)\rangle = |\alpha\rangle$, the constant c is equal to $-\gamma_+$, so that (62) is exact at all times $t \geqslant 0$ and

$$
\Delta x^2(t) \simeq \frac{\gamma_- - \gamma_+}{4\gamma_- + 4\gamma_+(1 - 2e^{-(\gamma_- - \gamma_+)t})}. \tag{64}
$$

This solution is compared in Fig. 7 with the numerical results for $|\epsilon| \sim 20$. The exact value of $\overline{\Delta x^2}$ is close to the approximated value 1/12 obtained from (63) for $\gamma_+/\gamma_- = 1/2$ (left-hand side of Fig. 7). It is a bit higher than the theoretical prediction $1/28 \simeq 0.0357$ for $\gamma_+/\gamma_- = 3/4$ (right-hand side). For $50 \leqslant \epsilon \leqslant 100$, one has a better agreement, as seen in Fig. 6. Hence the dynamics of the mode is well described by the quantum state diffusion equation (54) for $\epsilon \geqslant 50$, at this temperature.

V. STOCHASTIC SCHRÖDINGER EQUATIONS

In this section we discuss the mathematical link between the quantum jump dynamics of Sec. II and various stochastic Schrödinger equations found in the literature. The analysis is done for arbitrary open quantum systems having a Lindblad-type dynamics.

A. Linear quantum jump dynamics

A linear version of the QJ dynamics of Sec. II, in which the random wave function is not normalized at each time step δt, has been introduced in Ref. 18. The unormalized wave function $|\varphi(t)\rangle$ of the open system S satisfies the following Itô stochastic differential equation:

$$|\mathrm{d}\varphi\rangle=\left[-i(H+K)\mathrm{d}t+\sum_m\ (W_m-1)\mathrm{d}N_m(t)\right]|\varphi(t)\rangle, \tag{65}$$

where H is the Hamiltonian of S, the jump operators W_m are related to the Lindblad operators L_m by

$$L_m=\sqrt{\gamma_m}(W_m-1), \tag{66}$$

and $\mathrm{d}N_m(t)=0,1$ are the stochastic increments of independent Poisson processes N_m with parameters γ_m. These increments have mean $\mathsf{M}\,\mathrm{d}N_m(t)=\gamma_m\,\mathrm{d}t$ and satisfy the Itô rules:

$$\mathrm{d}N_m(t)\mathrm{d}N_n(t)=\delta_{n,m}\mathrm{d}N_m(t),\quad \mathrm{d}N_m(t)\mathrm{d}t=0. \tag{67}$$

Equation (65) has been first considered by Belavkin[14] with some jump operators W_m proportional to L_m [the relation with the jump operators (66) is given by the transformation (8)]. It has also been considered independently in Ref. 16. It is easy to show that $\rho(t)=\mathsf{M}|\varphi(t)\rangle\langle\varphi(t)|$ obeys the Lindblad equation (1) if K is appropriately chosen. In fact, by (65) and (67),

$$\begin{aligned}\mathrm{d}\rho&=\mathsf{M}(|\mathrm{d}\varphi\rangle\langle\varphi|+|\varphi\rangle\langle\mathrm{d}\varphi|+|\mathrm{d}\varphi\rangle\langle\mathrm{d}\varphi|)\\&=-i[H,\rho]\mathrm{d}t-iK\rho\,\mathrm{d}t+i\rho K^\dagger\,\mathrm{d}t+\sum_m\ \gamma_m((W_m-1)\rho+\rho(W_m^\dagger-1)+(W_m-1)\rho(W_m^\dagger-1))\mathrm{d}t.\end{aligned}$$

Replacing (66) into this equation yields

$$\mathrm{d}\rho=-i[H,\rho]\mathrm{d}t+\sum_m\ \left(L_m\rho L_m^\dagger-\frac{1}{2}\{L_m^\dagger L_m,\rho\}\right)\mathrm{d}t,$$

provided that

$$K=\frac{1}{2i}\sum_m\ \gamma_m(W_m^\dagger+1)(W_m-1)=\frac{1}{2i}\sum_m\ (L_m^\dagger L_m+2\sqrt{\gamma_m}L_m). \tag{68}$$

To give a physical meaning to the linear stochastic dynamics, one must choose the jump rates γ_m equal to the probability per unit time of the corresponding transitions. These are given by the Fermi golden rule to second order in perturbation theory. For instance, for the field mode considered in Sec. III, $\gamma_\pm$ are the damping rates (33) for the absorption and emission of a photon by the atoms.

Let us write (65) in the "dissipative interaction picture" $|\tilde{\varphi}(t)\rangle=U(t)|\varphi(t)\rangle$, where $U(t)=\exp(it(H+K))$. It reads $|\mathrm{d}\tilde{\varphi}\rangle=\Sigma_m(\tilde{W}_m(t)-1)\mathrm{d}N_m(t)|\tilde{\varphi}(t)\rangle$, with $\tilde{W}_m(t)=U(t)W_mU(-t)$. This implies:

$$|\tilde{\varphi}(t)\rangle=\tilde{W}_{m_p}(t_p)\cdots\tilde{W}_{m_1}(t_1)|\varphi(0)\rangle,$$

where $0\leqslant t_1\leqslant\cdots\leqslant t_p\leqslant\cdots$ are the jump times (times such that $\Sigma_m\,\mathrm{d}N_m(t)=1$), $t_p\leqslant t<t_{p+1}$, and m_q is the index of the jump occurring at time t_q ($\mathrm{d}N_{m_q}(t_q)=1$, $q=1,\ldots,p$). Hence the stochastic Schrödinger equation (65) admits the solution:

$$|\varphi(t)\rangle=e^{-i(t-t_p)(H+K)}W_{m_p}e^{-i(t_p-t_{p-1})(H+K)}\cdots W_{m_1}e^{-it_1(H+K)}|\varphi(0)\rangle,\quad t_p\leqslant t<t_{p+1}. \tag{69}$$

In other words, the evolution of the quantum state may be computed as follows.

(1) If there is a jump m between t and $t+\mathrm{d}t$, then

$$|\varphi(t+\mathrm{d}t)\rangle = W_m|\varphi(t)\rangle. \tag{70}$$

This occurs with a probability $\gamma_m \mathrm{d}t$ independent of $|\varphi(t)\rangle$.

(2) If no jump occurs between t and $t+\mathrm{d}t$, then

$$|\varphi(t+\mathrm{d}t)\rangle = (1 - i\,\mathrm{d}t(H+K))|\varphi(t)\rangle. \tag{71}$$

If $\gamma = \Sigma_m \gamma_m < \infty$, the time delays $t-t_p, t_p - t_{p-1}, \ldots, t_1$ and the jump indices $m_p, \ldots, m_1$ in (69) are independent random variables. The time delays are equally distributed, according to the exponential law $e^{-\gamma s}\,\mathrm{d}s$. Because of the independence of the Poisson processes N_m, the probability that $m_q = m$ is γ_m/γ.

It has been proven in Ref. 18 that, under appropriate hypothesis on W_m and γ_m, the wave function (69) is still well defined (as some limit of the right-hand side) if $\gamma = \infty$, in which case infinitely many jumps occur between 0 and t. This result is important for electrons in strongly disordered solids, where $m=(i,j)$ labels pairs of eigenstates $|i\rangle, |j\rangle$ of the electronic Hamiltonian, and L_{ij} is equal or "close" to $\sqrt{\gamma_{ij}}|j\rangle\langle i|$, i.e., $W_{ij} \simeq 1 + |j\rangle\langle i|$ (locality condition). Then, the number of jumps in a finite interval becomes infinite in the infinite volume limit, due to the divergence of the double sum $\Sigma_{i,j}\,\gamma_{ij}$. In this case, the presence of the identity operator inside the parentheses in (66) is necessary for the mathematical definiteness of $|\varphi(t)\rangle$.

B. Nonlinear quantum jump dynamics

The nonlinear QJ scheme of Sec. II can be deduced from the above-mentioned linear QJ dynamics in the following way. By comparing (9) and (4) with (70) and (71), it is clear that, for a given realization of the jumps, the normalized wave function

$$|\psi(t)\rangle = \frac{|\varphi(t)\rangle}{\|\varphi(t)\|} \tag{72}$$

evolves according to the nonlinear QJ scheme with $\lambda_m = \sqrt{\gamma_m}$. However, it was argued in Sec. II that the density matrix $\rho(t)$ is the mean value of $|\psi(t)\rangle\langle\psi(t)|$, whereas it has been seen earlier that $\rho(t) = \mathsf{M}|\varphi(t)\rangle\langle\varphi(t)|$. This means that the probability P' attached to $|\psi(t)\rangle$ is different from the probability P attached to $|\varphi(t)\rangle$, that is, to the Poisson processes N_m. This change of probability $P \to P'$ provides the link between the two random evolutions for $|\varphi(t)\rangle$ and $|\psi(t)\rangle$. We define it as follows. Let us denote by M' and M the mean values with respect to P' and P, respectively. Let F be an arbitrary (operator-valued) stochastic process such that $F(t)$ depends only upon the realizations of the jump times up to time t. Such F is said to be *adapted* to the filtration of the Poisson processes N_m.[33] We ask that

$$\mathsf{M}'(F(t)) = \mathsf{M}(F(t)\|\varphi(t)\|^2) \tag{73}$$

for any such process F. Taking $F(t) = |\psi(t)\rangle\langle\psi(t)|$, this implies in particular

$$\rho(t) = \mathsf{M}'|\psi(t)\rangle\langle\psi(t)| = \mathsf{M}|\varphi(t)\rangle\langle\varphi(t)|. \tag{74}$$

Let us compute the probability of occurrence of a jump m between times t and $t+\delta t$ for the new probability P':

$$\delta p_m(t) = \mathsf{M}'(\delta N_m(t)|\psi(t)). \tag{75}$$

The right-hand side is the conditional (mean) expectation of $\delta N_m(t)$ given $|\psi(t)\rangle$, for the probability P'. Indeed, the (P') probability of a jump between t and $t+\delta t$ depends upon the wave function $|\psi(t)\rangle$ at time t. Let $F(t)$ be an arbitrary stochastic force adapted to the filtration of the

N_m's. Then $F(t)\delta N_m(t)$ depends upon the realizations of the N_m's until time $t+\delta t$ [recall that $\delta N_m(t)=N_m(t+\delta t)-N_m(t)$]. Therefore, replacing $F(t)$ by $F(t)\delta N_m(t)$ in (73),

$$\mathsf{M}'(F(t)\delta N_m(t))=\mathsf{M}(F(t)\delta N_m(t)\|\varphi(t+\delta t)\|^2)=\mathsf{M}(F(t)\delta N_m(t)\|W_m|\varphi(t)\rangle\|^2).$$

Formula (70) together with the fact that $\delta N_m(t)=0,1$ have been used in the second line. By the independence of the forward increment $\delta N_m(t)$ and $F(t)\|W_m|\varphi(t)\rangle\|^2$, one gets

$$\mathsf{M}'(F(t)\delta N_m(t))=\mathsf{M}(\delta N_m(t))\mathsf{M}(F(t)\|W_m|\varphi(t)\rangle\|^2)=\gamma_m\delta t\mathsf{M}'(F(t)\|W_m|\varphi(t)\rangle\|^2\|\varphi(t)\|^{-2}). \tag{76}$$

But $F(t)$ is arbitrary, thus (76) implies the identity of the conditional expectations:

$$\delta p_m(t)=\mathsf{M}'(\delta N_m(t)|\psi(t))=\gamma_m\delta t\mathsf{M}'(\|W_m|\varphi(t)\rangle\|^2\||\varphi(t)\rangle\|^{-2}|\psi(t))=\gamma_m\delta t\|W_m|\psi(t)\rangle\|^2, \tag{77}$$

in accordance with (12). As a result, the stochastic evolution of the normalized wave function $|\psi(t)\rangle$ with probability P' coincides with that described in Sec. II for $\lambda_m=\sqrt{\gamma_m}$. It is moreover given by the nonlinear stochastic Schrödinger equation:[15]

$$|\mathrm{d}\psi\rangle=\left[-i\left(H+K+\frac{1}{2}\langle K^\dagger-K\rangle_t\right)\mathrm{d}t+\sum_m\left(\frac{W_m}{\sqrt{\langle W_m^\dagger W_m\rangle_t}}-1\right)\mathrm{d}N_m(t)\right]|\psi(t)\rangle, \tag{78}$$

with $\langle O\rangle_t=\langle\psi(t)|O|\psi(t)\rangle$. This equation is readily obtained by computing $|\psi(t+\mathrm{d}t)\rangle$ from (70) and (71).

C. Linear quantum state diffusion

It has been shown by Carmichael and Wiseman and Milburn[5,8] that the nonlinear QJ dynamics of the quantum field considered in Sec. III, with $\gamma_+=0$, is well described in limit $|\epsilon|\to\infty$ by a quantum state diffusion (QSD) stochastic equation involving real Wiener processes (white noise). The linear version of QSD is obtained in this section in the more general setting of arbitrary Markovian quantum open systems, by taking the limit of infinite jump rates $\gamma_m\to\infty$ in the linear QJ dynamics of Sec. V A.

Following Wiseman and Milburn,[8] we introduce a small dimensionless parameter $\varepsilon>0$ that will tend to zero. Our goal is to increase up to infinity the rates γ_m of the jumps in the linear QJ dynamics, without modifying the master equation giving the average dynamics. Hence the Lindblad operators L_m are here considered as *fixed*, i.e., independent of ε. So are the damping rates contained in the master equation, given by some ε-independent rates $\overline{\gamma}_m>0$. The jump rates γ_m are assumed to go to infinity like ε^{-4}:

$$\gamma_m=\varepsilon^{-4}\overline{\gamma}_m. \tag{79}$$

The magnitude of the negative power of ε is chosen for future convenience. Let $|\varphi(t)\rangle$ be the solution of the linear QJ stochastic equation (65). We are interested in the variation of $|\varphi(t)\rangle$ on a time interval $\Delta t\gg\gamma_m^{-1}$ such that infinitely many jumps occur between t and $t+\Delta t$ in the limit $\varepsilon\to0$. On the other hand, we want Δt to be small enough so that the change $|\Delta\varphi\rangle=|\varphi(t+\Delta t)\rangle-|\varphi(t)\rangle$ of the wave function goes to zero as $\varepsilon\to0$. A possible Δt realizing these two conditions is

$$\Delta t=\varepsilon^3\overline{\gamma}^{-1}, \tag{80}$$

where $\overline{\gamma}=\Sigma_m\,\overline{\gamma}_m$ is the sum of the fixed damping rates. Indeed, from (66),

$$W_m=1+\varepsilon^2\overline{\gamma}_m^{-1/2}L_m. \tag{81}$$

Since the L_m are ε-independent, the impact of each jump on $|\varphi\rangle$ is of order ε^2. Moreover, the number of jumps m between t and $t+\Delta t$ is of order $\gamma_m \Delta t = \varepsilon^{-1}\overline{\gamma_m}/\overline{\gamma}$. This means that the impact of the jumps between t and $t+\Delta t$ is of order $\varepsilon^{-1}\times\varepsilon^2=\varepsilon$, which is indeed small for small ε.

Let $\Delta N_m(t)$ be the number of jumps m in the time interval $[t,t+\Delta t]$. By dividing this interval into smaller intervals $[\tau_n,\tau_{n+1}]$ of length γ_m^{-1}, $\Delta N_m(t)$ can be written as a sum of $\gamma_m \Delta t = \mathcal{O}(\varepsilon^{-1})$ independent random variables $N_m([\tau_n,\tau_{n+1}])$ (number of jumps between τ_n and τ_{n+1}), which have mean and variance 1. Therefore, by the central limit theorem,

$$\Delta z_n = \frac{\Delta N_m - \gamma_m \Delta t}{\sqrt{\gamma_m}} \tag{82}$$

can be approximated for small ε by a Gaussian random variable of zero mean and variance Δt [the convergence as $\varepsilon\to 0$ actually holds for a fixed ε-independent Δt; for Δt given by (80), one gets an infinitesimal increment dz_n]. Shifting t by Δt or changing m leads to independent increments ΔN_m and Δz_m. It follows that Δz_m are the infinitesimal increments of some independent real Wiener processes z_m in the limit $\varepsilon\to 0$. Indeed, a Wiener process z is by definition a stochastic process with independent increments $\Delta z = z(t+\Delta t)-z(t)$ distributed according to a Gaussian law of variance Δt. The convergence of $(N_m-\gamma_m t)/\sqrt{\gamma_m}$ to a Wiener process can be shown more rigorously by using a theorem proven in Ref. 32.

The next step consists in evaluating both the mean and the fluctuating parts of $|\Delta\varphi\rangle$ to leading order in ε. Let $p=\Sigma_m \Delta N_m(t)$ be the total number of jumps between t and $t+\Delta t$. We denote by $s_q = t_{q+1}-t_q$, $q=1,\ldots,p-1$, the time delays between consecutive jumps and set $s_0=t_1-t$ and $s_p = t+\Delta t - t_p$. By (79) and (80), p and s_q have mean values $\gamma\Delta t$ and γ^{-1} of order ε^{-1} and ε^4, respectively. The generalized Hamiltonian (68) can be decomposed into two parts:

$$K = K_0 - i\varepsilon^{-2}R, \tag{83}$$

where K_0 and R are ε-independent:

$$K_0 = \frac{1}{2i}\sum_m L_m^\dagger L_m, \quad R = \sum_m \sqrt{\overline{\gamma_m}} L_m. \tag{84}$$

Let us set $V_q = \overline{\gamma_{m_q}}^{-1/2} L_{m_q}$ if $q=1,\ldots,p$ and $V_0=0$. With the help of (69), (81), and (83), one obtains

$$|\Delta\varphi\rangle = \left\{ \prod_{q=p}^{0} \left[\left(1-\varepsilon^{-2}s_q R + \frac{1}{2}\varepsilon^{-4}s_q^2 R^2 - i s_q (H+K_0) + \mathcal{O}(\varepsilon^6) \right)(1+\varepsilon^2 V_q) \right] - 1 \right\} |\varphi(t)\rangle, \tag{85}$$

where the product is taken in decreasing order in q. The terms of order ε in the expansion of the product are

$$\begin{aligned} |\Delta\varphi\rangle^{(f)} &= \sum_{q=0}^{p} (-\varepsilon^{-2}s_q R + \varepsilon^2 V_q)|\varphi(t)\rangle \\ &= \sum_m (-\varepsilon^{-2}\Delta t\,\overline{\gamma_m}^{1/2} + \varepsilon^2 \Delta N_m(t)\,\overline{\gamma_m}^{-1/2}) L_m |\varphi(t)\rangle = \sum_m \Delta z_m(t) L_m |\varphi(t)\rangle. \end{aligned} \tag{86}$$

These are the leading order fluctuating forces (of zero mean). Since fluctuating terms of higher order should not contribute in the limit $\varepsilon\to 0$, we may replace the product in (85) by its mean value in computing the terms of order ε^2 and more. This simplifies greatly the computation, because the order of the operators in the product then becomes of no importance. In fact, by the remark following (71), the random variables $s_0,\ldots,s_p$, $m_1,\ldots,m_p$ are independent, so that the mean of the product is the product of the means. Moreover, $\mathrm{M}V_q = \Sigma_m(\gamma_m/\gamma)\overline{\gamma_m}^{-1/2} L_m = R/\overline{\gamma}$. Therefore, one

can commute all operators in the product when computing the mean value. It is easy to show that the terms of order ε^2 cancel on average. The terms of order ε^3 are found to be proportional to Δt:

$$|\Delta\varphi\rangle^{(d)} = \mathsf{M}|\Delta\varphi\rangle = -i(H+K_0)\Delta t(1+\mathcal{O}(\varepsilon)). \tag{87}$$

This is the drift contribution to $|\Delta\varphi\rangle$.

The previous computation shows that the QJ dynamics is transformed as $\varepsilon \to 0$, if one looks at it with the time resolution $\Delta t = \overline{\gamma}^{-1}\varepsilon^3$, to a diffusive dynamics given by the linear Itô stochastic differential equation:[12,15,17]

$$|\mathrm{d}\varphi\rangle = \left[-i(H+K_0)\mathrm{d}t + \sum_m L_m \mathrm{d}z_m(t)\right]|\varphi(t)\rangle. \tag{88}$$

Here $K_0 = \Sigma_m L_m^\dagger L_m/2i$ and $\mathrm{d}z_m(t)$ are the infinitesimal increments of independent *real* Wiener processes z_m, which have zero mean and satisfy the Itô rules:

$$\mathrm{d}z_m(t)\mathrm{d}z_n(t) = \delta_{n,m}\,\mathrm{d}t, \quad \mathrm{d}z_m(t)\mathrm{d}t = 0. \tag{89}$$

D. Nonlinear quantum state diffusion

We have so far determined the linear stochastic Schrödinger equation for $|\varphi(t)\rangle$. The corresponding nonlinear equation for the normalized wave function $|\psi(t)\rangle$ is obtained by means of the above-mentioned change of probability $P \to P'$. This derivation of the nonlinear QSD equation from the linear one is actually well-known.[12,28] It is slightly more complicated than for the QJ dynamics, because $|\mathrm{d}\psi\rangle$ is to be expressed in terms of Wiener processes for the new probability P'. This can be done by using Girsanov's theorem,[33] which states that a Wiener differential $\mathrm{d}w_m$ for P' is obtained by adding an appropriate drift differential to $\mathrm{d}z_m$. For the change of probability defined by (73), the conditional (mean) expectation of $\mathrm{d}P'/\mathrm{d}P$ given $|\varphi(t)\rangle$ is $\|\varphi(t)\|^2$. The drift differential is then $-\|\varphi(t)\|^{-2}\mathrm{d}\|\varphi\|^2\mathrm{d}z_m$ (for more details, see Refs. 12, 28). From (88) and (89), one gets

$$\mathrm{d}\|\varphi\|^2 = 2\|\varphi(t)\|^2 \sum_m \Re\langle L_m\rangle_t \mathrm{d}z_m, \tag{90}$$

which implies that $\|\varphi(t)\|^2$ is a local martingale. Thus,

$$\mathrm{d}w_m(t) = \mathrm{d}z_m(t) - 2\Re\langle L_m\rangle_t \mathrm{d}t. \tag{91}$$

According to Itô's formula,[30] one has

$$\frac{|\mathrm{d}\varphi\rangle}{\|\varphi\|} = \frac{\mathrm{d}(\sqrt{\|\varphi\|^2}|\psi\rangle)}{\|\varphi\|} = |\mathrm{d}\psi\rangle + \left(\frac{\mathrm{d}\|\varphi\|^2}{2\|\varphi\|} - \frac{\mathrm{d}\|\varphi\|^2\mathrm{d}\|\varphi\|^2}{8\|\varphi\|^3}\right)\frac{|\psi\rangle + |\mathrm{d}\psi\rangle}{\|\varphi\|}.$$

The multiplication of both members by $\mathrm{d}z_m$ leads, with the help of (88)–(90), to $\mathrm{d}z_m|\mathrm{d}\psi\rangle = (L_m - \Re\langle L_m\rangle_t)|\psi\rangle\mathrm{d}t$. Going back to the original equation, it follows:

$$|\mathrm{d}\psi\rangle = \left[-i(H+K_0)\mathrm{d}t + \sum_m \Re\langle L_m\rangle_t\left(-L_m + \frac{3}{2}\Re\langle L_m\rangle_t\right)\mathrm{d}t + \sum_m (L_m - \Re\langle L_m\rangle_t)\mathrm{d}z_m\right]|\psi\rangle.$$

The nonlinear QDS equation is obtained by replacing (91) into this equation:[11,12,15]

$$|\mathrm{d}\psi\rangle = \left[-i(H+K_0)\mathrm{d}t + \sum_m \Re\langle L_m\rangle_t\left(L_m - \frac{1}{2}\Re\langle L_m\rangle_t\right)\mathrm{d}t + \sum_m (L_m - \Re\langle L_m\rangle_t)\mathrm{d}w_m(t)\right]|\psi(t)\rangle. \tag{92}$$

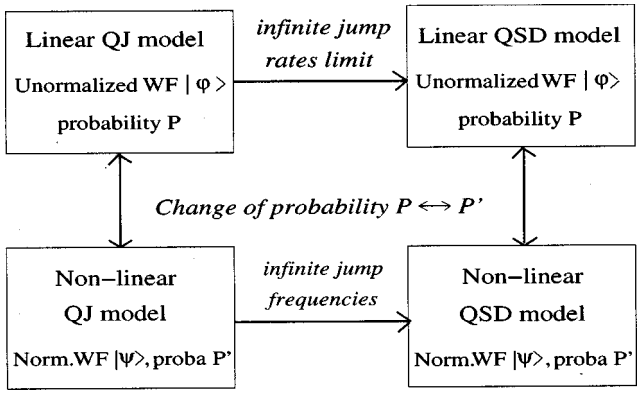

FIG. 10. The links between the different stochastic Schrödinger equations.

Let us come back to the mode dynamics of Sec. III. If one sticks to the norm-preserving QJ dynamics, the jump operators $W_{\pm}$ in (31) are defined up to a multiplicative constant. If $k=0$, $\tilde{a}\simeq a$, they can be obtained, up to such a constant, from the Lindblad operators $L_{+}=\sqrt{\overline{\gamma}_{+}}a^{\dagger}$ and $L_{-}=\sqrt{\overline{\gamma}_{-}}a$ by means of formula (66), provided that the transition rates are replaced by some effective rates $\gamma_{\pm}=\overline{\gamma}_{\pm}|\epsilon|^2$ and $L_{\pm}$ are multiplied by the phase factors $e^{\pm i\theta}$, $\theta=\arg\epsilon$. Thus the above-presented analysis can be used to compute the QSD equation for the normalized mode wave function $|\psi(t)\rangle$ in the limit of large laser fields $|\epsilon|\to\infty$. The introduction of the phase factors $e^{\pm i\theta}$ in (92) leads to the QSD equation (54).

E. Links between the stochastic Schrödinger equations

The summary of the results of this section is given in Fig. 10. The stochastic Schrödinger equations in Itô form for the linear and nonlinear QJ models are, respectively, Eqs. (65) and (78), and those for the linear and nonlinear QSD models are, respectively, Eqs. (88) and (92).

VI. CONCLUSION

We have shown that the nonlinear quantum jump (or Monte Carlo wave function) model applied to a simple optical system (damped harmonic oscillator at finite temperature T) can be generalized to describe the evolution of the quantum field in a cavity monitored by an atomic beam of two level atoms. These atoms cross one by one the cavity and interact at its exit first with a classical laser field $\mathcal{E}$, and then with a detector measuring their states. This kind of monitoring by measurements is similar to that obtained by homodyne measurement of the field in a decaying cavity,[5,8] the photon counting on the output field being replaced by the measurements on the atoms. Actually, if all atoms are sent in their ground state ($T=0$), the stochastic evolution of the wave function of the quantum field (quantum trajectories) is the same for the two kinds of monitoring. If the atoms form a beam of randomly prepared atoms with temperature $T>0$, they may also emit photons in the cavity and thus a new kind of quantum jump comes into play (creation of a photon). This has notable effects on the quantum trajectories. The effect of the laser field $\mathcal{E}$ is to modify the two jump operators. In fact, the measured atomic transitions can be driven by this field as well as by the interaction with the studied quantum field in the cavity. As a result, $\mathcal{E}$ also modifies the generalized Hamiltonian that rules the evolution between jumps. The average over all realizations of the measurements leads to an $\mathcal{E}$-independent dynamics, described by a density matrix satisfying the master equation of the damped harmonic oscillator with temperature T. Whereas the density matrix converges at large times to the Bose–Einstein equilibrium, the individual quantum trajectories for given realizations experience localization toward squeezed states at large $\mathcal{E}$. The squeezing parameters evolve to some almost constant values, up to small fluctuations going to zero in the infinite laser intensity limit. This localization occurs at large enough times, for any initial state of the quantum field. The centers of the squeezed states move

randomly in the complex plane, in such a way that the time averages of the quantum probabilities to find n-photons are distributed according to Bose–Einstein (ergodicity). The squeezing amplitude r and phase ϕ are controlled, respectively, by the temperature T and the laser field $\mathcal{E}$. r is found to increase with T, which means that the squeezing is enhanced by increasing the temperature of the atomic beam; however, the waiting time before r reaches its almost stationary value is also temperature increasing. On the other hand, no squeezing is obtained at $T=0$, and localization toward Fock states occurs if $\mathcal{E}=0$. As in the case of the homodyne measurement, the quantum trajectories are given in the infinite $\mathcal{E}$ limit by a so-called quantum state diffusion (QSD) stochastic Schrödinger equation, involving real white noise.[5,8,11,12] A precise mathematical derivation of this equation from the quantum jump dynamics was performed in Sec. V for arbitrary open quantum systems having a Lindblad-type dynamics. More precisely, this derivation starts from a linear version of the QJ dynamics proposed in Ref. 18, in which the wave function is not normalized at each step, which is proven to be related to the nonlinear QJ model by a simple change of probability.

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