Temperature-enhanced squeezing in cavity QED

D Spehner and M Orszag

1 Universit"at Essen, Fachbereich Physik, D-45117 Essen, Germany
2 Pontificia Universidad Catolica de Chile, Facultad de Fisica, Casilla 306, Santiago 22, Chile

Received 25 April 2002, in final form 1 August 2002
Published 29 August 2002
Online at stacks.iop.org/JOptB/4/326

Abstract

We study the time evolution of the quantum field inside a cavity coupled to a beam of two-level atoms of temperature $T$, given that each atom, after having crossed the cavity, interacts with a classical field $\mathcal{E}$ and finally with a detector measuring its state. It is found that, if the coupling between the atoms and the quantum field is weak and $\mathcal{E}$ is not too small, for any given realization of the measurements, an arbitrary initial state of the field localizes after some time into squeezed states. The centre $\alpha$ of the squeezed state moves randomly in time in the complex plane, but the squeezing amplitude $r$ and phase $\phi$ show very small fluctuations. Their mean values $\bar{r}$ and $\bar{\phi}$ are independent of the random results of the measurements, of the initial state and of the atom–field coupling constant $\lambda$. The time evolution of $r$ and $\phi$ is determined analytically by deriving and solving the quantum state diffusion equation describing the field dynamics in the limit of small $\lambda$, keeping $\mathcal{E}$ finite. It is shown that $\bar{r}$ increases with $T$, i.e., the squeezing is enhanced by increasing the temperature of the atomic beam.

Keywords: Open quantum systems, quantum trajectories, micromasers

1. Introduction

Dissipation has played a central role in quantum optics. A typical example of this is spontaneous emission, where an individual atom is coupled to an ensemble of modes of the electromagnetic field, giving, as a final result, a finite lifetime to every atomic excited state. Traditionally, the dynamics of a dissipative quantum system is described through a master equation for the reduced density matrix, obtained by tracing out the degrees of freedom of the reservoir (the electromagnetic field in the above example) and making the Markov approximation. Also, a great deal of work has been done in quantum optics on continuously monitored systems with dissipation, referred to as ‘quantum jumps’ or ‘Monte Carlo wavefunction’ schemes, which are examples of a wider class of techniques concerned with ‘quantum trajectories’. In these approaches, the master equation is replaced by a stochastic differential equation for a pure state. If one averages over the realizations of the dynamical noise, the master equation is reproduced. Such a stochastic equation is referred to as the ‘unraveling’ of the master equation. This is not a unique process and there can be several stochastic equations that will average to the same master equation [1–12]. This technique has become important, because the trend in modern optics has been towards isolating and manipulating individual quantum systems. Examples include cavity QED with single atoms and photons [13], micromasers [14, 15], microlasers [16], trapped ions cooled to the motional zero point [17], Coulomb blockade [18] and Bose–Einstein condensates in electromagnetic traps [19]. Wide interest in such systems has been stimulated by possible applications to quantum computers (manipulation and storage of quantum states).

The aim of this paper is to investigate the localization properties of quantum trajectories for the electromagnetic field inside a high-Q cavity interacting with a beam of two-level atoms, which form the reservoir of temperature $T$. The states of the atoms leaving the cavity are measured by a detector. A laser field $\mathcal{E}$ with frequency close to the resonance with the atomic transition is placed between the cavity and the detector. The same system has been considered in [20] in the reverse situation where one knows exactly the state of each atom before it crosses the cavity and no measurement is performed on it at its exit (its final state being thus unknown). It has been shown in this reference that the cavity field evolves at large
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3. Stochastic dynamics of the quantum field

Let us determine the evolution of the state of the field in the first cavity when one atom, initially in state $|i\rangle$, $i = g$ or $e$, crosses the two cavities and the detector. At the time $t$ just prior the entrance of the atom in the first cavity, the wavefunction $|\Psi(t)\rangle$ of the total system ‘atom + quantum field’ is a tensor product state $|\Psi(t)\rangle = |i\rangle|\psi(t)\rangle$. Since the two fields are in separated cavities, the atom interacts with the quantum field before interacting with the classical field $\mathcal{E}$. The total wavefunction at the exit of the second cavity (before the measurement) is thus in the interaction picture

$$|\Psi_{\text{out}}\rangle = e^{-i\tau\mathcal{H}_i}e^{-\int_{0}^{\tau}\mathcal{H}_e}e^{-\int_{0}^{\tau}\mathcal{H}_\mathcal{E}}|\psi(t)\rangle.$$ (3)

The interaction leads to an entanglement between the quantum field and the atom, i.e., $|\Psi_{\text{out}}\rangle$ is no longer a tensor product state. After the measurement on the atom has been performed, the field and the atom become again disentangled and the wavefunction of the total system is

$$|\psi(t + \delta\tau)\rangle = |j\rangle|\psi(t + \delta\tau)\rangle,$$ (4)

with $j = g$ if the atom is detected in state $|g\rangle$ and $j = e$ if it is detected in state $|e\rangle$. It is convenient to introduce the complex numbers

$$\eta = \lambda\tau, \quad \epsilon = \frac{u_{ge}}{\lambda\tau n_{ge}} = -\frac{u_{ge}^*}{\lambda\tau n_{ge}},$$ (5)

and the operator

$$\hat{a} = a\sin(\eta)n\hat{a}^*,$$ (6)

where $u_{ij} = \langle i|e^{-i\tau\mathcal{H}_i}j\rangle$, $n^{1/2} = (\alpha^\dagger\alpha)^{1/2}$ is the square root of the photon number operator and $\sin(x) = \sin(x)/x$. If the atom–laser coupling is given by (2) with $\delta\tau_L \ll 1$, one finds $\eta e \simeq \Omega\tan(\Omega\tau_L/2)/\Omega$. An easy computation using (1) leads to [25]

$$e^{-\int_{0}^{\tau}\mathcal{H}_u} = |g\rangle\langle g|\cos(\eta n\hat{a}^* + \xi e\epsilon^*\hat{a}^*) + |e\rangle\langle e|\cos(\eta(n + 1)\hat{a}^* + \xi e\epsilon^*\hat{a}^*) + |g\rangle\langle g|\eta e - n``\hat{a}^*\rangle|\epsilon^*\hat{a}^*\rangle.$$ (7)

Let $|\psi(t + \delta\tau)\rangle = (j|\Psi_{\text{out}}\rangle$ be the unnormalized wavefunction of the quantum field after the measurement. Since the initial and final atomic states $i$ and $j$ can take two values $g$ or $e$, we must distinguish four cases.

(1) $i = j = g$. Then, by (3) and (7),

$$|\psi(t + \delta\tau)\rangle = u_{gg}W_{g\rightarrow g}|\psi(t)\rangle,$$

$$W_{g\rightarrow g} = \cos(\eta n\hat{a}^*) - |\eta|^2e^*\hat{a}^*.$$ (8)

(2) $i = j = e$. Then, by (3) and (7),

$$|\psi(t + \delta\tau)\rangle = u_{ee}W_{e\rightarrow e}|\psi(t)\rangle,$$

$$W_{e\rightarrow e} = \cos(\eta(n + 1)\hat{a}^*) - |\eta|^2e^*\hat{a}^*.$$ (9)

(3) $i = g$, $j = e$. Then, by (3) and (7),

$$|\psi(t + \delta\tau)\rangle = u_{ge}W_{g\rightarrow e}|\psi(t)\rangle,$$

$$W_{g\rightarrow e} = \cos(\eta(n\hat{a}^* + \xi e\epsilon^*\hat{a}^*) + |\eta|^2e^*\hat{a}^*.$$ (10)

(4) $i = e$, $j = g$. Then, by (3) and (7),

$$|\psi(t + \delta\tau)\rangle = u_{eg}W_{e\rightarrow g}|\psi(t)\rangle,$$

$$W_{e\rightarrow g} = \cos(\eta n\hat{a}^*) - |\eta|^2e^*\hat{a}^*.$$ (11)
(2) \( i = g, j = e \). Then,
\[
|\psi(t + \delta t)| = \eta u_{ge} W_{e\rightarrow g}|\psi(t)|, \\
W_{e\rightarrow g} = W_e = \tilde{a} + \epsilon \cos(\eta|n|^2).
\] (9)

(3) \( i = e, j = g \). Then,
\[
|\psi(t + \delta t)| = -\eta^* u_{eg} W_{e\rightarrow g}|\psi(t)|, \\
W_{e\rightarrow g} = W_e = \tilde{a}^+ + \epsilon^* \cos(\eta|n + 1|^2).
\] (10)

(4) \( i = j = e \). Then,
\[
|\psi(t + \delta t)| = u_{ee} W_{e\rightarrow e}|\psi(t)|, \\
W_{e\rightarrow e} = \cos(\eta|n + 1|^2) - |\eta|^2 \epsilon \tilde{a}^+.
\] (11)

Thus, the crossing by the atom of the two cavities and the detector modifies the normalized wavefunction \( |\psi(t)| \) of the field in the interaction picture as follows:
\[
|\psi(t + \delta t)| = \frac{W_{e\rightarrow g}|\psi(t)|}{\|W_{e\rightarrow g}|\psi(t)|\|}.
\] (12)

The cases (2) and (3) correspond, respectively, to the absorption and the emission of a photon of the quantum field or of the laser field by the atom. We then say that the quantum field suffers a ‘quantum jump’ – or + (this denomination comes from the limit \( |\eta| \rightarrow 0, \epsilon \) fixed, in which these jumps are separated by Hamiltonian-like evolutions [21]). The probability that the atom is detected in state \( |j\rangle \), given that it enters the first cavity in state \( |i\rangle \), is \( p_{i\rightarrow j} = \|\langle i|\Psi_{eq}\rangle\|^2 = \|\psi(t + \delta t)\|^2 \). The probability \( \delta p_{-\epsilon}(t) \) of a jump – is equal to \( p_{i\rightarrow g} \delta t \times \) the probability \( r_{\epsilon} \delta t \) that the atom is initially in state \( |g\rangle \). Similarly, \( \delta p_{\epsilon}(t) = p_{e\rightarrow g} r_{\epsilon} \delta t \). Introducing the damping rates
\[
y_- = r_g |\eta|^2, \\
y_+ = r_e |\eta|^2, \\
y = y_- + y_+ = \frac{|\eta|^2}{\delta t}
\] (13)

and using the unitarity of the matrix \( (u_{ij}) \), one arrives at
\[
\delta p_{\epsilon}(t) = \frac{y_+ \delta t}{1 + |\eta|^2} \|W_{e\rightarrow g}|\psi(t)|\|^2.
\] (14)

The probabilities of cases (1) and (4) are respectively \( r_e \delta t - \delta p_{-\epsilon}(t) \) and \( r_e \delta t - \delta p_{\epsilon}(t) \). The stochastic dynamics of the field depends only on the dimensionless parameters \( y, \epsilon \) and on the damping rates \( y_+ \). The ratio \( y_+ / y_- \) defines the temperature \( T \) of the atomic beam:
\[
y_+ / y_- = \frac{r_e}{r_g} = \exp\left(-\frac{\omega}{k_B T}\right),
\] (15)

where \( \omega \) is the atomic transition frequency and \( k_B \) the Boltzmann constant.

4. Localization into squeezed states

Let us follow the evolution of the state \( |\psi(t)\rangle \) of the quantum field when many atoms cross the cavities, for a given result (realization) of the measurements. Such a time evolution defines a quantum trajectory [3]. We computed \( |\psi(t)\rangle \) numerically, taking a particular initial state \( |\psi(0)\rangle \) and modifying \( |\psi(t)\rangle \) according to (12) at each time step \( \delta t \), with the above probabilities.

For big \( |\epsilon| \) and small \( |\eta| \), we find that, for any choice of \( |\psi(0)\rangle, |\psi(t)\rangle \) evolves to a squeezed state \( |\alpha(t), \xi(t)\rangle \), up to small fluctuating errors. In order to illustrate this result, let us study the mean square deviations (MSD) \( \Delta x_\phi^2(t) \) and \( \Delta y_\phi^2(t) \) of the field quadratures \( X_\phi = (ae^{-i\phi} + a^* e^{i\phi})/2 \) and \( Y_\phi = (ae^{-i\phi} - a^* e^{i\phi})/2i \). Denote by \( \phi_{\text{init}}(t) \) the angle for which \( \Delta x_\phi^2(t) = \langle \psi(t)|X^2_{\text{tot}}|\psi(t)\rangle = \langle \psi(t)|X_{\phi_{\text{tot}}}|\psi(t)\rangle^2 \) is minimum (the phase \( \omega t \) must be added to \( \phi \) since we are
working in the interaction picture). Let \( \Delta x^2(t) = \Delta x^2_{\text{msw}(t)}(t) \) and \( \Delta y^2(t) = \Delta y^2_{\text{msw}(t)}(t) \) be the minimal and maximal MSD. The time evolutions of \( \Delta x^2(t) \) and of the product \( \Delta x^2(t) \Delta y^2(t) \) are shown in figures 2 and 3 for different quantum trajectories. Figure 2 corresponds to \( \epsilon = 20 \) and figure 3 to \( \epsilon = 100 \). One sees in both figures that \( \Delta x^2(t) \) begins to fluctuate around a mean value \( \Delta x^2 \) after some transient time \( \Delta t \). The fluctuations are considerably reduced in the case \( \epsilon = 100 \) (figure 3) with respect to the case \( \epsilon = 20 \) (figure 2). Moreover, the product \( \Delta x^2(t) \Delta y^2(t) \) is much closer in figure 3 to the minimum value 1/16 allowed by Heisenberg’s uncertainty principle. The trajectories (a) and (b) start from the same initial coherent state but correspond to different values of \( \eta \) and \( t_\text{f} \). The trajectories (c) and (d) start from different initial states (the Fock state in figure 2 and a state chosen by Heisenberg’s uncertainty principle. The trajectories (a) (b) and (c) start from different initial states (the Fock state in figure 2 and a state chosen by Heisenberg’s uncertainty principle. The trajectories (a) (b) and (c) start from different initial states (the Fock state in figure 2 and a state chosen by 

\[
\text{Figure 4.} \quad \text{(a) (full curve) and Im}(\psi_0) \text{ (broken curve) versus } \gamma t \text{ for a quantum trajectory starting from a coherent state } |\psi(0)\rangle = |\alpha\rangle \text{ with } \gamma_\text{c}/\gamma_\text{r} = 0.892, \quad \gamma = 3.5, \quad \eta = 0.0118, \quad \epsilon = 84.5 \approx \eta^{-1}. \quad N_\Delta = 1.25 \times 10^6 \text{ atoms and } a = \sqrt{3/2}(1 + i).
\]

\[
\text{Figure 5.} \quad \Delta x^2(t) \text{ versus } \gamma t \text{ for trajectories with different temperatures: (a) } \gamma_\text{c}/\gamma_\text{r} = 0; \text{ (b) } \gamma_\text{c}/\gamma_\text{r} = 1/4; \text{ (c) } \gamma_\text{c}/\gamma_\text{r} = 3/4; \text{ (d) } \gamma_\text{c}/\gamma_\text{r} = 0.892; \text{ (e) } \gamma_\text{c}/\gamma_\text{r} = 0.987, +. \text{ In (a), (b) and (c), } \eta = 0.0132, \quad N_\Delta = 10^6 \text{ atoms; in (d), } \eta = 0.0118, \quad N_\Delta = 1.25 \times 10^6 \text{ atoms; in (e), } \eta = 0.0013, \quad N_\Delta = 10^3 \text{ atoms. For all trajectories, } \epsilon = \eta^{-1}, \gamma = 3.5 \text{ and the initial coherent state is as in figure 4. The full curves are the theoretical result (39).}
\]

5. Photon number statistics

In the absence of measurements, i.e., in the same experimental set-up as in figure 1 but without the detector, the quantum field thermalizes with the atomic beam at temperature \( T > 0 \). The state of the field is described by a density matrix \( \rho(t) = \text{tr}_A \sigma(t) \), obtained by tracing out the atomic degrees of freedom in the density matrix \( \sigma(t) \) of the total system ‘atoms + field’. The probability \( \langle n | \rho(t) | n \rangle \) of finding \( n \) photons in the first cavity converges at large times to the Bose–Einstein distribution \( \rho_{\text{BE}}(n) = \left(1 - e^{-\omega/\kappa_B T}\right) e^{-\omega n/\kappa_B T} \). Such a thermalization does not occur if the measurements on the atoms are performed: then the field is constantly maintained out of equilibrium. One should thus keep in mind when confronted with the above-mentioned numerical results that \( T \) is the temperature of the atomic beam, not that of the field!

Let us denote by \( M \) the mean over all results of the measurements. Since averaging the projector \( |\psi(t)| \langle \psi(t)| \) over all results of a measurement is the same as not performing the measurement,

\[
\rho(t) = \text{tr}_A (\sigma(t)) = M |\psi(t)| \langle \psi(t)| .
\] (16)

One may verify this formula explicitly by comparing the evolution of the second and third members when one atom
crosses the cavities [21]. It is expected from ergodicity that the
time average of the quantum probability \( P_n(t) = |\langle n|\psi(t)\rangle|^2 \)
of finding \( n \) photons coincides with the equilibrium value \( \rho_{eq}^{(n)} \):

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{ds}{t} P_n(t) = \lim_{t \to \infty} M P_n(t) = \lim_{t \to \infty} (n|\rho(t)|n) = \rho_{eq}^{(n)}.
\]

(17)

In order to check the ergodic hypothesis, we computed numerically the first and fourth members. The corresponding values, represented in figure 6 by the circles and the full curve, agree reasonably well. \( P_n(t) \) is shown in the same figure for a fixed time \( t \). One sees the well-known oscillations exhibited by squeezed states, with an overall exponential decay [26]. For squeezed states \( |\alpha, \xi\rangle \) with real \( \alpha \) and \( \xi \), \( P_n \) has the following asymptotic behaviour as \( n \to \infty \):

\[
P_n \sim \frac{2}{\sqrt{\pi}} \left( \frac{2n+1}{2n} \cosh^2 r - \frac{e^{2r} \alpha^2}{4\tanh r} \right)^{-\frac{1}{2}} \times \exp \left( \frac{\alpha^2}{2\sinh(2r)} \right) (\tanh r)^n \cos^n(\Phi_n),
\]

with \( \Phi_n = \int_{\mu(1-|\alpha|^2)}^{\alpha^2+1} d\nu \sqrt{2n+1 - \nu^2 - \pi/4} \) and \( r = |\xi| \) (see [26]). It has been seen in section 4 that the squeezing amplitude \( r(t) \approx \ln(4|\Delta x|^2)/2 \) is almost time and realization independent for sufficiently large \( t \). This must also be the case for the rate \( \ln(\tanh r)^{-1} \) of the exponential decay of \( P_n(t) \) as \( n \to \infty \). By (17), this rate must coincide with the decay rate \( k_B T/\omega \) of the Bose–Einstein distribution \( \rho_{eq}^{(n)} \). This gives \( \tanh r = e^{-\omega/k_B T} = \gamma_0/\gamma \). As a result,

\[
\Delta x^2 \approx \frac{\gamma_0 - \gamma}{4\gamma} = \frac{1 - e^{-\omega/k_B T}}{4(1 + e^{-\omega/k_B T})}.
\]

The values (19) are the large-time limits of the exponentially
decaying functions shown in full curves in figure 5. A good agreement is found with the numerical data. Another consequence of (17) is worth noting. In order to reproduce the Bose–Einstein distribution, the oscillations of \( P_n(t) \) as a function of \( n \) must disappear after a time (or realization) averaging. Since \( |\xi(t)| < \infty \) for \( t \gg \Delta t \), this implies that the centre \( \alpha(t) \) of the squeezed state varies randomly in time. The factor \( e^{-\omega/2\sinh(2r)} \) in (18) shifts the whole distribution to the right or to the left as \( \alpha \) varies, and the amplitude and phase of the oscillations are also changed. These large random time fluctuations are observed for \( \text{Re} \alpha \) in figure 4.

Formula (19) shows that \( \Delta x^2 \) decreases on increasing \( T \). A perfect squeezing \( (\Delta x^2 = 0) \) is predicted at \( T = \infty \). However, one finds numerically that the localization into squeezed states only holds for \( |\eta| \) smaller than some value \( \eta_0 \) and for \( |\epsilon| \) bigger than \( \epsilon_0 \), where \( \eta_0 \) decreases to zero and \( \epsilon_0 \) increases to infinity as \( k_B T/\omega \to \infty \). Thus perfect squeezing cannot be reached in a practical numerical or real experiment with finite \( \eta \) and \( \epsilon \). We shall come back below to this limit of validity of (19).

6. Quantum state diffusion

The argument of section 5 does not explain why the squeezed states form an invariant family under the stochastic dynamics and why the squeezing amplitudes evolve exponentially to a temperature-dependent limiting value at large times, as observed in the numerical simulations. In order to understand these points, we determine in this section the coarse-grained evolution of the field state \( |\psi(t)\rangle \) for timescales on which many atoms cross the cavities.

The numerical data suggest considering the limit

\[
|\epsilon| \to \infty, \quad |\eta| \to 0, \quad |\eta\epsilon| = \text{const.}
\]

(20)

Let us assume moreover that the moments \( \langle n^q \rangle \) are much smaller than \( |\eta|^{-3q} \); more precisely,

\[
|\eta|^{2q} \langle n^q \rangle = \mathcal{O}(\eta^q), \quad q = 1, 2, \ldots, t \geq 0,
\]

(21)

where \( \langle \cdot \rangle = (\psi(t)|\langle \cdot \rangle|\psi(t)) \) is the quantum expectation in state \( |\psi(t)\rangle \). Under this condition, the crossing of the cavities and the detector by one atom weakly perturbs the quantum field in all cases (1)–(4) (perturbative regime). In fact, since \( |\psi(t)\rangle \) is renormalized at each time step \( \delta t \), \( |\psi(t+\delta t)\rangle \) is defined up to an arbitrary multiplicative constant in equations (8)–(11). Hence one may divide the right-hand sides of these equations by \( u_{g\delta}, \eta u_{a\epsilon}, -\eta^* u_{g\delta} \epsilon^* \) and \( u_{a\epsilon} \), respectively. This yields

\[
|\psi(t+\delta t)| = (1 + \delta W_{\epsilon \to f})|\psi(t)| + \mathcal{O}(\eta^{1/2})
\]

(22)

with

\[
\delta W_{\epsilon \to f} = -\frac{|\eta|^2}{2}(a^\dagger a + 2\epsilon^* a)
\]

\[
\delta W_{\epsilon \to a} = \delta W_{a \to f} = \frac{a}{\epsilon} - \frac{|\eta|^2}{2} a^\dagger a
\]

(23)

\[
\delta W_{\epsilon \to a} = \delta W_{a \to \epsilon} = \frac{|\eta|^2}{2} (aa^\dagger + 2\epsilon a^\dagger a).
\]

In all cases (1)–(4), the wavefunction \( |\psi(t)\rangle \) is modified by a small amount, of order \( \eta^{1/2} \). The equality \( \tilde{a} = a + \mathcal{O}(\eta^{1/2}) \), which follows from (6) and (21), has been used.
Let us consider a time interval \([t, t + \Delta t]\) such that many atoms cross the cavities between \(t\) and \(t + \Delta t\) but \(|\psi(t + \Delta t)\rangle\) does not differ much from \(|\psi(t)\rangle\). This is the case if
\[
1 \ll \frac{\Delta t}{\delta t} \ll |\eta^2|^{\frac{1}{2}}.
\] (24)

Let us denote by \(\Delta N_a(t)\) the number of jumps \((\pm 1)\) and by \(\Delta N(t)\) the number of atoms entering the first cavity in state \(i\) \((i = g \text{ or } e)\) between \(t\) and \(t + \Delta t\). Then,
\[
|\psi(t + \Delta t)\rangle = [1 + \Delta N_a(t)(\delta W_a - \delta W_{g-e}) + \Delta N_g(t)(\delta W_{g-e} + \delta W_{e-g})]|\psi(t)\rangle + \mathcal{O}\left(\frac{\Delta t}{\delta t}|\eta|^2\right).
\] (25)

Since the variation of \(|\psi(t)\rangle\) between \(t\) and \(t + \Delta t\) is small, the jump probability \(\delta p_{ab}(t_m)\) can be approximated by \(\delta p_a(t)\) for any \(t_m = t + m \delta t, m = 0, \ldots, \Delta t/\delta t\). From (9), (10) and (14),
\[
\mathbb{M}\Delta N_{ae} = \sum_{m=0}^{\Delta t/\delta t} \delta p_a(t_m) \approx \frac{|\eta|^2 |\gamma_a|^2 \Delta t}{1 + |\eta|^2} \left[1 + 2|\eta| \text{Re}(e^{-\theta_2} a_t)\right].
\]
with \(\epsilon = |\eta|e^{i\theta}\). Replacing this formula into (25), one obtains the mean value over the measurements of \(|\Delta \psi(t)\rangle = |\psi(t + \Delta t)\rangle - |\psi(t)\rangle\):
\[
\mathbb{M}|\Delta \psi(t)\rangle \approx \frac{2|\eta| \text{Re}(e^{-\theta_2} a_t) |\gamma_a|^2 e^{i\theta} a_t + \gamma_a e^{-\theta_2} a_{t}}{1 + |\eta|^2} \delta \Delta t
\]
\[
\frac{\rho}{\delta t} = -\frac{1}{2} \mathbb{M}\Delta \psi(t)\rangle = \frac{1}{2} \text{Re}(e^{-\theta_2} a_t) |\gamma_a|^2 e^{i\theta} a_t + \gamma_a e^{-\theta_2} a_{t}}{1 + |\eta|^2} \delta \Delta t
\]
\[
The identity \(\Delta N_a(t) = \delta 1\) has been used.
\]
It remains to evaluate the fluctuation of \(|\Delta \psi(t)\rangle\). Consider the random variable \(\delta N_{ae}(t_m)\) equal to 1 if the \(m\)th atom enters in the first cavity at time \(t_m = t + m \delta t\) in state \(|i\rangle\), and equal to 0 otherwise. The random variable \(\delta \bar{N}_{ae}(t_m) = (\delta N_{ae}(t_m))\) is equal to 1 if a jump + (-) occurs when the \(m\)th atom is sent into the cavity in state \(|e\rangle\) (\(|g\rangle\)). The probability that \(\delta \bar{N}_{ae}(t_m) = 1\) is equal to the conditional probability \(p_{g-e}\) of occurrence of a jump +, given that the \(m\)th atom is initially in its upper state \(|e\rangle\). Similarly, \(\delta \bar{N}_{ge}(t_m) = 1\) has probability \(p_{e-g}\). Let \(i_e = e\) and \(i_g = g\). With these definitions, \(\delta \bar{N}_{ae}(t_m)\) and \(\delta \bar{N}_{ge}(t_m)\) are independent random variables. Variables corresponding to different times \(t_m\) are also independent. The fluctuation of \(\Delta \bar{N}_{ae}(t_m) = \sum_{m=0}^{\Delta t/\delta t} \delta \bar{N}_{ae}(t_m) \delta \bar{N}_{ae}(t_m)\) can be written as a sum of two terms:
\[
\Delta \bar{N}_{ae}(t_m) = \Delta \bar{N}_{ae}(t) = \sum_{m=0}^{\Delta t/\delta t} \delta \bar{N}_{ae}(t_m) \Delta \bar{N}_{ae}(t_m)
\]
\[
+ \sum_{m=0}^{\Delta t/\delta t} \delta \bar{N}_{ae}(t_m) \delta \bar{N}_{ge}(t_m)\]
where \(\delta \bar{N}_{ae}(t_m)\) and \(\delta \bar{N}_{ge}(t_m)\) are the fluctuations of \(\delta \bar{N}_{ae}(t_m)\) and \(\delta \bar{N}_{ge}(t_m)\), respectively. The square variance of the first term is found to be approximately \((\gamma_a \Delta t/\delta t|\eta|^2)(1 + |\theta|^2)^{-2}\). By the central limit theorem,
\[
\Delta w_{ae}(t) = \sqrt{\frac{\gamma_a \Delta t}{2} 1 + |\eta|^2 |\gamma_a|^2 \sum_{m=0}^{\Delta t/\delta t} \delta \bar{N}_{ae}(t_m) \delta \bar{N}_{ae}(t_m)}
\] (27)
tends in the limit \(\Delta t \gg \delta t\) to a Gaussian random variable of zero mean and variance \(\Delta t\). Shifting \(t\) by \(\Delta t\) leads to an independent Gaussian variable \(\Delta w_{ae}(t + \Delta t)\). Hence \(\Delta w_{ae}(t)\) are the increments of two independent Wiener processes \((w_{ae}(t))_{t \geq 0}\). With the help of (25), we find
\[
|\Delta \psi(t)\rangle - M|\Delta \psi(t)\rangle \approx \frac{\sqrt{\gamma_a \epsilon_{ae}} a_t \Delta w_{ae}(t) + \sqrt{\gamma_a \epsilon_{ae}} a \Delta w_{ae}(t) + 2|\eta|^2 \text{Re}(e^{-\theta_2} a_t)} + \text{Re}(e^{-\theta_2} a \Delta M_e(t) + e^{-\theta_2} a \Delta M_g(t))}
\]
\[
- |\eta|^2 (a a^* \Delta M_e(t) + a^* a \Delta M_g(t)) \rangle \langle |\psi(t)\rangle.\]
(28)
The third and fourth terms, proportional to the fluctuations \(\Delta M_e(t)\) of the numbers \(\Delta N_{ae}(t)\) of atoms entering the cavity in state \(i\) between \(t\) and \(t + \Delta t\), are of order \((\Delta t/\delta t)^{1/2} |\eta|^2\). They can be neglected with respect to the first and second terms, proportional to \(\Delta w_{ae}(t)\), which are larger by an amount \(|\eta|^2\).

Since we are interested in the coarse-grained dynamics of the field with a ‘time resolution’ \(\Delta t\), the Gaussian increments \(\Delta w_{ae}(t)\) will be treated as infinitesimal Itô differentials, denoted by \(dw_{ae}(t)\), and \(\Delta t\) will be denoted by a time differential \(dt\).

The last step consists in normalizing \(|\psi(t + dt)\rangle\). Its inverse norm, assumed to be one at time \(t\), can be computed by using the Itô formalism of stochastic differentials [27]:
\[
\|\psi(t + dt)\|^2 = 1 + \frac{1}{2} \langle d\|\psi(t)\|^2 \rangle = 1 + \frac{1}{2} \frac{d\|\psi(t)\|^2}{dt}.
\]
(29)
The first differential in the right-hand side is \(d\|\psi(t)\|^2 = 2 \text{Re}(\psi(t)\langle \psi(t)\rangle + \langle \psi(t)\psi(t)\rangle)\). The last term is non-zero since \(dw_{ae}dt = dw_{ae}dt\). Collecting (26), (28) and (29) and using the other Itô rules \(dw_{ae}dw_{ae} = dw_{ae}dt = 0\), we arrive at our final result:
\[
|d\psi(t)\rangle = \left[\sqrt{\gamma_a e^{i\theta_2} a_t - \text{Re}(e^{i\theta_2} a_t)}\right] dw_{ae}(t) + \sqrt{\gamma_a e^{i\theta_2} a - \text{Re}(e^{i\theta_2} a)} \Delta M_e(t)\] + \text{Re}(e^{i\theta_2} a)\rangle \langle \gamma_a e^{i\theta_2} a + \gamma_a e^{i\theta_2} a \Delta M_e(t)\rangle dt
\]
(30)
Note that \(|d\psi\rangle\) is independent of \(|\eta|\), i.e., the coarse-grained dynamics depends only on \(\theta = \arg \epsilon\) and on the damping constants \(\gamma_a\) and \(\gamma_g\).

Equation (30) pertains to a known class of stochastic Schrödinger equations with real Wiener processes, which has been studied in [8, 10]; related equations with complex Wiener processes have been discussed in [9]. Both kinds of equation unravel the same Lindblad master equation for the density matrix (16). In our case, the master equation is easy to determine directly. The change of \(\rho(t)\) when one atom, initially in state \(|i\rangle\), crosses the cavity is
\[
\rho(t + \Delta t) = \text{tr}_a(e^{-i\Delta H_a} |i\rangle\langle i| \rho(t) e^{i\Delta H_a}).
\]
(31)
\(\rho(t)\) describes the evolution of the quantum field without the measurements on the atoms. Hence it does not depend upon the classical field \(\epsilon\) in the second cavity (this is seen
mathematically by using the cyclicity of the trace to get rid in (31) of the two exponentials $e^{i\pi t H t}$ multiplying $e^{i\pi t H a}$. The coarse-grained average dynamics on the timescale $\Delta t \gg \delta t$ is obtained by replacing (7) into (31), using similar approximations to those above. Not surprisingly, one finds the equation of the damped harmonic oscillator:

$$\frac{d^2\rho}{dt^2} = \gamma_+ (a \rho(t) a^\dagger - \frac{1}{2} [a^\dagger a, \rho(t)]) + \gamma_- (a^\dagger \rho(t) a - \frac{1}{2} [aa^\dagger, \rho(t)]).$$

(32)

The curly brackets denote the anticommutators. Although the quantum trajectories depend on $\arg(e)$, the corresponding master equation is the same for all $e$. The equilibrium is the Bose–Einstein matrix $\rho_e$ with temperature $T$ ($T \geq 0$).

Let us end this section by making two remarks. Firstly, the above analysis can be carried out under the following hypothesis, which is more general than (21):

$$\langle |\eta|^2, k\pi \rangle_{\gamma^q} = \mathcal{O}(\gamma^{q}), \quad q = 1, 2, \ldots, t \geq 0,$$

(33)

with $k$ a non-negative integer. For $k \geq 1$, this means that the typical number of photons is close to $(k\pi/|\eta|)^2 \gg 1$, with relative fluctuations at most of order $|\eta|$. Then, most atoms crossing the first cavity make approximately $k/2$ Rabi oscillations and leave it in the same state as when they entered it. The coarse-grained stochastic and average dynamics are thus given by (30) and (32) with $a$ replaced by $\bar{a}$ and $\bar{\theta}$ replaced by $\bar{\theta} + k\pi$. If $(k\pi/|\eta|)^2$ is an integer $n_k$, $\bar{a} |n_k - 1\rangle = 0$ and the atoms cannot bring more than $n_k - 1$ photons into the cavity (trapping states) [15]. Secondly, let us mention that the stochastic dynamics of section 3 has been studied in [21] in the limit $|\eta| \to 0$, keeping $\epsilon$ finite and assuming that (21) holds. This yields a quantum jump (or Monte Carlo wavefunction) model similar to the model introduced in [2, 3]. This is because for small $|\eta|e$ and $|\eta|$, the jump probabilities $bp_s(t)$ are very small (see (14)) and the (coarse-grained) evolution between these jumps is Hamiltonian-like, with an effective non-self-adjoint Hamiltonian $K = \gamma_+ (a^\dagger a + 2e^2 a^\dagger a + [\psi, \psi^\dagger])/2t + \gamma_- (a^\dagger a + 2e^2 a^\dagger a + [\psi, \psi^\dagger])/2t$. The jump operators are $W_s = a^\dagger e^\epsilon$. Both $K$ and $W_s$ depend on $\epsilon$. The limit $|\eta| \to \infty$ of the quantum jump dynamics of course gives the stochastic Schrödinger equation (30) again [6, 21].

7. Evolution of the squeezing parameters

As shown by Rigó and Gisin [28], the stochastic Schrödinger equation (30) preserves squeezed states. More precisely, $|\psi(t)\rangle = |\alpha(t), \xi(t)\rangle$ is a solution of (30) if $\alpha(t)$ and $\xi(t) = r(t)e^{2i\theta(t)}$ satisfy the Itô stochastic differential equations:

$$d\Gamma = -e^{2i\theta} (1 + e^{-2i\theta} \Gamma(t)) (\gamma_+ + e^{-2i\theta} \gamma_- \Gamma(t)) \, dt$$

$$d\beta = -\left(\frac{\gamma_+}{2} + e^{-2i\theta} \gamma_- \Gamma(t)\right) \beta(t) \, dt + 2\gamma_+ \gamma_- e^{i\theta} \Gamma(t) Re(e^{-i\theta} \alpha(t)) \, dt + \gamma_- e^{-i\theta} \Gamma(t) dw_s(t) + \sqrt{\gamma_+} \gamma_- e^{i\theta} \Gamma(t) dw_{-s}(t),$$

(34)

with $\Gamma(t) = -e^{2i\theta(t)} \Gamma(0) \Gamma(t)$ and $\beta(t) = \alpha(t) - \Gamma(t) |\alpha(t)|^2$. These equations are derived in [21, 28], so we only quote here the result (note that $dw_s$ are Itô stochastic differentials, whereas Stratanovich differentials are used in [28]).

The squeezing parameters $r(t)$ and $\phi(t)$ are given by the solution of the first equation, which is deterministic:

$$-\Gamma(t) = e^{2i\theta(t)} \gamma_+ \gamma_- e^{-i\theta} \gamma_- + c \Gamma(t) |\alpha(t)|^2 + e^{2i\theta(t)} \gamma_+ \gamma_- e^{-i\theta} \gamma_- + c,$$

(35)

where $c$ is an arbitrary complex constant. Replacing (35) into the second equation in (34), $x_0(t) = Re(e^{-i\theta} \alpha(t))$ and $y_0(t) = Im(e^{-i\theta} \alpha(t))$ are found to satisfy

$$dx_0 = \frac{\gamma_+ - \gamma_-}{2} x_0(t) \, dt + \gamma_+ e^{2i\theta} \gamma_- e^{-i\theta} \Gamma(t) \, dt$$

$$- \gamma_+ e^{2i\theta} \gamma_- e^{-i\theta} \Gamma(t) \, dt + 2\gamma_+ \gamma_- e^{2i\theta} \Gamma(t) \, dt + 2\gamma_+ e^{2i\theta} \Gamma(t) \, dw_s(t)$$

(36)

$$dy_0 = \frac{\gamma_+ - \gamma_-}{2} y_0(t) \, dt.$$

The second equation is again deterministic and gives

$$y_0(t) = Im(e^{-i\theta} \alpha(t)) = e^{-i\theta} \gamma_- e^{-i\theta} \gamma_- y_0(0).$$

(37)

In figure 4, we indeed observe an exponential decay of $y_0(t) = Im(\alpha(t))/\gamma_-$, with the rate $(\gamma_- - \gamma_+)/2$.

In the large-time limit $t \gg (\gamma_- - \gamma_+)^{-1}$, the centre $\alpha(t)$ moves randomly on the line $arg(\alpha) = \theta$, and

$$\tanh(r(t)) \to \frac{\gamma_+}{\gamma_-} = \exp(-\omega/k_B T),$$

$$\phi(t) \to \theta.$$

Since for squeezed states $\Delta x^2 = e^{-2\gamma} / 4$, one recovers the expression (19) for the time average (or, equivalently, the infinite-time limit) of $\Delta x^2(t)$. In the Schrödinger picture, the field wavefunction is $|\psi(t)\rangle = |\alpha(t)e^{-i\theta}, \xi(t)e^{2i\theta/2}\rangle$. The centre of this squeezed state now also rotates in the complex plane. The squeezing phase is $\phi(t) \approx \theta - o(t)$ at large times.

In the special case of an initial coherent state $|\psi(0)\rangle = |\alpha\rangle$, the constant $c$ is equal to $-\gamma_+$, so $\phi(t) = \theta$ at all times and

$$\Delta x^2(t) = \frac{\gamma_- - \gamma_+}{4\gamma_+} \frac{1}{1 - 2e^{2i\theta} \gamma_+ |\alpha|^2}.$$ 

(39)

In particular, $\Delta x^2(t) = 1/4$ at zero temperature ($\gamma_- = 0$). This means that coherent states $|\psi(t)\rangle = |\alpha(t)\rangle$ are preserved by the stochastic dynamics if all atoms are initially in their lower state. The result (39) is compared in figure 5 with the numerical data. One sees a very good agreement. Note that the rate $(\gamma_- - \gamma_+)$ of the exponential decay of $\Delta x^2(t)$ decreases to zero at high temperatures: one needs to wait longer and longer to reach the value $\Delta x^2$ given by (19) as $T$ becomes larger. In any real or numerical experiment, the quantum field is observed up to a finite time $t$, so this limits the maximal squeezing that can be reached: $\Delta x^2(t) \leq \Delta x^2$.

By ergodicity, the time average of $|\langle n |\rangle = |\alpha|^2$ is the Bose–Einstein average $1/(e^{\langle w_s/\gamma_+ T} - 1)$. For squeezed states, $|\langle n |^2 = \sinh^2 r + |\alpha|^2$ (see [22]). Using (38) gives the time average of $|\langle n |^2$:

$$|\langle n |^2 = X^0_0 = \frac{\gamma_+ \gamma_-}{\gamma_- - \gamma_+}.$$ 

(40)
8. Limitations on the squeezing

Let us discuss the limit of validity of (19) and (39). The first restriction comes from the fact that real cavities are not perfect. In experiments with micromasers [15], the time $\delta t$ separating consecutive atoms is smaller than the photon lifetime $\tau_{\text{cav}}$, and the interaction time $\tau$ is small compared with $\delta t$, in order to avoid events involving more than one atom in the cavity. It is legitimate under these conditions to neglect photon losses during the interaction. The dynamics thus splits into squeezing and damping cycles, with time durations $\tau$ and $\delta t - \tau$, respectively. In order to ensure that damping cycles do not destroy the squeezing effect, $\gamma \tau_{\text{cav}}$ must be large enough. The interesting problem of the determination of the large-time limit of the field state in the presence of photon losses is left to a future work. Note also that the presence of thermal photons in the cavity has not been taken into account in our analysis. These can be neglected if the cavity is cooled at very low temperature (much smaller than the temperature $T$ of the beam), as is generally the case in the experiments [15].

A second limitation is brought about by the fact that the computation of section 6 and the preservation of squeezed states by the dynamics is only valid in the limits (20) and (21). A closer look at (23) shows that the condition of validity is

$$|\eta|^2|\epsilon| \sqrt{(\alpha)} \ll 1 \quad \text{and} \quad |\epsilon|^{-1} \sqrt{(\eta)} \ll 1$$

(41) at all times. Given $\eta$ and $T$, the best choice of $|\epsilon|$ minimizing both quantities in (41) is clearly $|\epsilon| = |\eta|^{-1}$. Then the condition reduces to $|\eta|^2|\epsilon| \sqrt{(\eta)} \ll 1$. At low or intermediate temperatures, the mean number of photons in the cavity is not very large and this condition is met in the weak-coupling limit $|\eta| = |\eta| \tau \ll 1$.

However, since $\sqrt{(\eta)} = 1/(e^{\eta/2} - 1)$ increases to infinity at large $T$, the condition (41) is violated, for any value of $\eta$, above a certain temperature $T_0$ depending on $\eta$ and $\epsilon$. For $T \gtrsim T_0$ the localization of the quantum field into squeezed states no longer occurs. This reasoning explains qualitatively why we had to go to smaller values of $\eta$ and higher values of $\epsilon$ when increasing $T$ in the simulations of figure 5. Taking $|\epsilon| = |\eta|^{-1}$, $k_{\text{v}} T_0/\omega \gg \gamma_0 (\gamma_0 - \gamma_0)^{-1}$ must be small with respect to $|\eta|^{-2}$. This gives, however, an overestimation of $T_0$, since the fluctuations of $|\eta| = \sinh^2 r + |\alpha|^2$, which appear to be important in figure 4, have not been taken into account.

One may be concerned that, even if (41) is fulfilled, the error made after $N_A \gg 1$ atoms have crossed the cavity might be large, despite the smallness of the error made for each atom taken separately. We argue below that such an accumulation of errors at large times, if it exists, leads to much smaller errors than one might expect. It seems reasonable to assume that only the average part of the variation of the state $|\psi(t)\rangle$ when one atom crosses the cavities, denoted by $M_0 |\delta \psi(t)\rangle$, can lead to accumulating errors. For fixed $|\psi(t)\rangle = |\alpha, \xi\rangle$ and $|\epsilon| = |\eta|^{-1}$,

$$|\psi(t)\rangle + M_0 |\delta \psi(t)\rangle = |\alpha + \delta \alpha, \xi + \delta \xi\rangle + O(|\eta|^2(n_\eta)^2).$$

(42)

The proof of this formula is based on the following result: if $b, c$, and $d$ are complex numbers with $b = 1 + O(\eta)$, $c = O(\gamma_0 \sqrt{(\eta)}|\eta|)$ and $d = O(\eta^2)$, then there are complex numbers $a'$ and $\xi'$ and a normalization constant $A$ such that

$$b + c a' + c a + d n|\langle \alpha, \xi| = A|\alpha', \xi'\rangle + O(\gamma_0^2(n_\eta)^2).$$

(43)

Indeed, using the known properties of squeezed states, this is equivalent to the existence of an operator $a' - \Gamma a'^2 - \beta' b'$, with $\Gamma'$ and $\beta' \in C$, which, when applied to the left-hand member of (43), gives zero plus some terms of order $\eta^2(n_\eta)^2$. A simple computation shows that such an $\Gamma'$ and $\beta'$ exist. This proves (43). The mean variation $M_0 |\delta \psi(t)|$ of the unnormalized wavefunction $|\psi(t)\rangle$ is given, to lowest order, by (26) with $\Delta t = \delta t$. It can be further checked that it has the same form as the left-hand member of (43), with $b, c$, and $d$ satisfying the above hypothesis, up to corrections of order $\eta^2(n_\eta)^2$. Moreover, as is clear from (22) and (23), the fluctuation $|\delta \psi_f(t)| = (1 - M_0) |\delta \psi(t)|$ is also of the form (43), up to terms of order $\eta^2(n_\eta)^2$, but with $b, d = O(\eta^2)$ and $c = O(\eta)$. We now write

$$|\psi(t)\rangle + M_0 |\delta \psi_f(t)\rangle = C (|\psi(t)\rangle + M_0 |\delta \psi(t)|)$$

$$+ M_0 (|\psi(t + \delta t)|^{-1})^{-1} - 1 |\delta \psi_f(t)|,$$

with $C = M_0 |\psi(t + \delta t)|^{-1}$. Since $|\delta \psi_f(t)| = O(\eta^2(n_\eta)^2)$, this makes it clear that $|\psi(t + \delta t)|^{-1} = O(|\eta|^{-2})$ and $|\delta \psi_f(t)| = O(\eta^2 |\eta|^{-2})$. It follows that (42) holds true.

Let us assume that the above errors accumulate. The total error after the crossing of a large number $N_A$ of atoms is then

$$\sum_{m=0}^{N_A} |\eta|^2|\epsilon|^\frac{1}{2} \leq N_A |\eta|^2|\epsilon|^{-2} \leq N_A |\eta|^2|\epsilon|^{-2}$$

(44)

where the last expression is the average of $\eta^2$ at thermal equilibrium. If $N_A$ is bigger than $(N_A)_{\text{th}} = |\eta|^{-2}(\gamma_0 - \gamma_0)^{-1}$, or, equivalently, if $\gamma_0$ is bigger than

$$\gamma_0 = (\gamma_0 - \gamma_0)^{-1} \gamma_0 \gamma_0|\eta|^{-2},$$

(45)

then the field state should differ significantly from a squeezed state. Replacing the values of $\gamma_0$ and $\gamma_0$ in the trajectories (d) and (e) in figure 5, one finds $\gamma_0 \simeq 50$. Nevertheless, a good agreement with (39) is observed at much bigger times! Numerically, we find that $\Delta x^2(t) \Delta y^2(t)$ cannot be distinguished from 1/16 over the whole time interval, using the scale of figure 5. The values of $\gamma_0$ for the trajectories (a), (b) and (c) are higher; other simulations not presented here also show no deviation of $\Delta x^2(t) \Delta y^2(t)$ from 1/16 for e.g. $\gamma_0 = 3/4$ and $\eta = e^{-1} = 0.03$, corresponding to $\gamma_0 \simeq 50$ (but the fluctuations of $\Delta x^2(t)$ around $\gamma_0 - \gamma_0$ are larger). Moreover, longer-time simulations, running up to $t = 1750/\gamma_0$, show no further deviations of the quantum field from a squeezed state, and the squeezing amplitudes continue to fluctuate around the value (39). Hence one can conclude that there are no accumulations of errors of the kind studied above.

9. Conclusions

We have presented a quantum trajectory model applied to a particularly simple optical system, that of a damped harmonic oscillator at temperature $T$. The model that we have considered is a beam of two-level atoms crossing one by one a lossless cavity containing the quantum field studied, coupled to a classical field and finally to a detector at the exit of the cavity.
The idealized case of two-level atoms interacting with a single mode of the electromagnetic field in a lossless cavity is nearly realized in micromasers [15]. In the recent experiment [29], a cavity with a quality factor as high as $Q = 4 \times 10^{10}$ has been employed, corresponding to a photon lifetime $t_{\text{cav}} = 0.3$ s. The main difference between the physics of the micromaser and our model is that, in the experiments known to us, the two-level atoms are pumped in their upper level before interacting with the quantum field, whereas we assumed in this work that the atomic beam has a positive temperature $T$. We have given numerical evidence that, for a non-zero classical field intensity and a small enough coupling strength $\eta = \lambda \tau$ ($\lambda$ is the atom-field coupling constant and $\tau$ the time spent by each atom in the cavity), the cavity field localizes into squeezed states. The corresponding squeezing amplitude $r(t)$ and phase $\phi(t)$ have been shown to be nearly deterministic and given by (38) in the large-time limit. The centre $c(t)$ of the squeezed state moves randomly in the complex plane, in such a way that the quantum probabilities for the photon numbers reproduce the thermal (Bose–Einstein) distribution after a time averaging. The main result is that the final degree of squeezing increases with the temperature of the atomic beam. A perfect squeezing can be obtained $\textit{a priori}$ in the limit of infinite temperature and of zero coupling strength $\eta$, keeping a finite laser field intensity. However, the time needed to obtain this perfect squeezing goes to infinity in this high-temperature limit, and corrections for finite $\eta$ and $\epsilon$ become important at high temperatures. This, together with the finite lifetime of photons in a real cavity, limits in practice the squeezing that can be achieved.

The squeezing effect found in the present work should be in principle observable in micromasers. Since the probability of field-induced transitions between adjacent atomic levels with principal quantum number $n$ scales as $n^4$, experiments with values of $n$ ranging from $10^{-3}$ to more than 1 can be found in the literature. According to our numerical simulations, the best value for observing a rapid localization of the cavity field into squeezed states is $\eta \approx 0.03$, and the localization time is typically $t \approx 10/\gamma$ with $\gamma = \eta^2 \delta t^{-1}$. The experiments described in [15] have higher values of $\eta$, due to the high degree of excitation of the rubidium atoms ($\lambda = 10^5$ s$^{-1}$ for the $63\text{P}_3/2 \rightarrow 61\text{D}_3/2$ transition and the shorter $\delta t = 25 \times 10^{-6}$ s, giving $\eta = 0.25$). The intensity of the beam must be small enough in order to ensure that the cavity field is coupled to at most one atom at any time. Using a flux of atoms $r = \delta t^{-1} = 0.1 \tau^{-1}$, one finds, assuming a Poissonian statistics for the number of atoms in the cavity, that 95% of the events involve only one atom. This flux is in the range of the experimental data for $\tau = 25 \times 10^{-6}$ s. The condition of observation of the predicted localization into squeezed states is met for $\gamma t_{\text{cav}} = \eta^2 t_{\text{cav}}(0.1 \tau^{-1}) \gtrsim 10$ and $\eta \lesssim 0.03$. One concludes that $\tau$ must be smaller than $3 \times 10^{-6}$ s. Again, this value is a bit too small compared with typical micromaser experimental data (relatively large cavities are used in order to ensure that the low mode is in resonance with the nearby Rydberg level transition). In order to observe the temperature-enhanced squeezing described in this work, atomic transitions corresponding to lower coupling constants could be more suitable, or faster atoms spending less time in the cavity could be used. According to [15], the squeezing of the cavity field could be measured by using probe atoms.

Acknowledgments

DS is grateful to Walter Strunz for fruitful discussions. DS acknowledges the financial support during the early years of this work of a Fondecyt postdoctoral grant 3000035. MO acknowledges the financial support of the Fondecyt project 1010777.

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