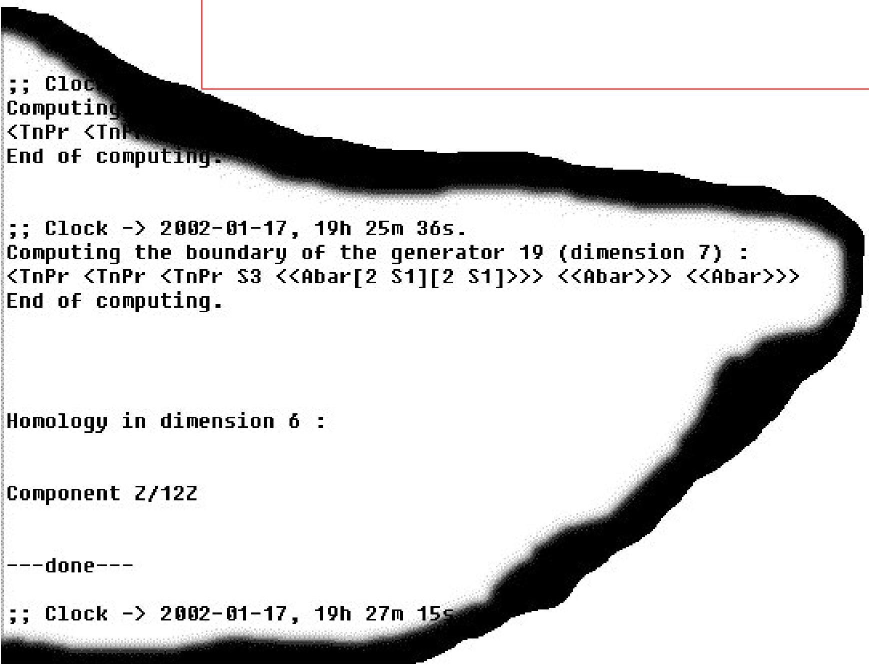
Constructive Homological Algebra VI.

Constructive Spectral Sequences



Francis Sergeraert, Institut Fourier, Grenoble, France
Genova Summer School, 2006

$\left[F \hookrightarrow F \times_\tau B \to B\right]$

Key tools for effective spectral sequences in topology:

- Eilenberg-Zilber theorem.

$$C_*(F \times B) \overset{\boxed{???}}{\longleftrightarrow} C_*(F) \otimes C_*(B)$$

- Twisted Eilenberg-Zilber theorem.

$$C_*(F \times_\tau B) \overset{\boxed{???}}{\longleftrightarrow} C_*(F) \otimes_{\boxed{???}} C_*(B)$$

Eilenberg-Zilber theorem.

$\exists$ (almost) canonical reduction:

$$EZ_{F,B} = (f, g, h) : C_*(F \times B) \Rrightarrow C_*(F) \otimes C_*(B)$$

- **f = Alexander-Whitney map (linear complexity).**
- **g = Eilenberg-MacLane map**
  = Decomposition $\Delta^p \times \Delta^q$ in simplices
  (exponential complexity).
- **h = Shih Weishu map (exponential complexity).**

Particular case $F = B = \Delta^7$.

Eilenberg-Zilber reduction

$$C_*(\Delta^7 \times \Delta^7) \Rrightarrow C_*^N(\Delta^7) \otimes C_*^N(\Delta^7)$$

n	$\times$	$\otimes$	n	$\times$	$\otimes$	n	$\times$	$\otimes$
0	64	64	5	759,752	11,424	10	1,475,208	1,820
1	1,232	448	6	1,549,936	12,868	11	673,134	560
2	11,872	1,680	7	2,360,501	11,440	12	208,824	120
3	69,524	4,256	8	2,703,512	8,008	13	39,468	16
4	272,944	9,527	9	2,322,180	4,368	14	3,432	1

Comparison between $C_*(F \times B)$ and $C_*(F \times_\tau B)$?

- **Same simplices ⇒ Same underlying graded modules.**
- **Different incidence relations ⇒ Differential perturbation.**

⇒ Basic perturbation lemma can be applied.

Theorem (**Edgar Brown + Shih Weishu**):

$$\exists(f, g, h) : C_*(F \times_{\boxed{\tau}} B) \Rrightarrow C_*(F) \otimes_{\boxed{t}} C_*(B)$$

for t = some algebraic tensor product twist.

Example 1. **Effective homology version of the Serre spectral sequence.**

$$F = (F,\ C_*(F),\ EC_*^F,\ \varepsilon^F)$$
$$+\ \ B = (B,\ C_*(B),\ EC_*^B,\ \varepsilon^B)$$
$$+\ \ \tau : B \rightarrow F$$
$$\Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \text{Serre}_{EH}$$
$$E = F \times_\tau B = (E,\ C_*(E),\ EC^E,\ \varepsilon^E)$$

(Serre + G. Hirsch + H. Cartan + Shih W. + Szczarba + Ronnie Brown + J. Rubio + FS)

Proof.

$$\boxed{\begin{array}{l} C_*(F \times B) \overset{\text{id}}{\Lleftarrow\!\!\!\Lleftarrow} C_*(F \times B) \overset{EZ}{\Rrightarrow\!\!\!\Rrightarrow} C_*F \otimes C_*B \\[2ex] C_*F \otimes C_*B \overset{\otimes}{\Lleftarrow\!\!\!\Lleftarrow} \widehat{C}^F \otimes \widehat{C}^B \overset{\otimes}{\Rrightarrow\!\!\!\Rrightarrow} EC^F \otimes EC^B \end{array}}$$

$\Downarrow\Downarrow\Downarrow\Downarrow\Downarrow$ Serre$_{EH}$

$$\boxed{\begin{array}{l} C_*(F \times_\tau B) \overset{\text{id}}{\Lleftarrow\!\!\!\Lleftarrow} C_*(F \times_\tau B) \overset{\text{Shih}}{\Rrightarrow\!\!\!\Rrightarrow} C_*F \otimes_t C_*B \\[2ex] C_*F \otimes_t C_*B \overset{EPL}{\Lleftarrow\!\!\!\Lleftarrow} \widehat{C}^F \otimes_{t'} \widehat{C}^B \overset{BPL}{\Rrightarrow\!\!\!\Rrightarrow} EC^F \otimes_{t''} EC^B \end{array}}$$

$+$ Composition of equivalences $\Longrightarrow$ O.K.

Serre's canonical loop space fibration.

$I := [0, 1]$

$P(X, *) := \mathcal{C}([I, 0]; [X, *]) =: PX$

$\Omega(X, *) := \mathcal{C}([I, 0, 1]; [X, *, *]) =: \Omega X$

$\Rightarrow$ Canonical "fibration":

$$\Omega X \hookrightarrow PX \longrightarrow X$$

Combinatorial Kan version $\Rightarrow$ Genuine principal fibration:

$$GX \hookrightarrow [E_{PX} = GX \times_\tau X] \longrightarrow X$$

Similar algebraic fibration.

$[C = \text{coalgebra}] + [M + N = C\text{-comodules}]$

$$\Rightarrow \text{Cobar}^C(M, N).$$

Particular case: $\text{Cobar}^C(C, \mathbb{Z}) \ggg \mathbb{Z}$.

$\Rightarrow$ **Algebraic fibration:**

$$\begin{array}{ccc} \text{Cobar}^C(\mathbb{Z}, \mathbb{Z}) \hookrightarrow & \text{Cobar}^C(C, \mathbb{Z}) & \rightarrow C \\ & \Vert & \\ & \text{Cobar}^C(\mathbb{Z}, \mathbb{Z}) \otimes_t C & \end{array}$$

$=$ **Algebraic translation of:**

$$\begin{array}{ccccccc} GX & \hookrightarrow & GX \times_\tau X & \rightarrow & X & & \text{(simplicial)} \\ \Omega X & \hookrightarrow & PX & \rightarrow & X & & \text{(topological)} \end{array}$$

Example 2:

Julio Rubio's solution of Adams' problem.

$$\boxed{\boldsymbol{X} = (\boldsymbol{X},\ C_*(\boldsymbol{X}),\ \boldsymbol{EC}_*^{\boldsymbol{X}},\ \varepsilon^{\boldsymbol{X}})}$$

$$\Downarrow\Downarrow\Downarrow \qquad \text{Eil.-Moore}_{EH}$$

$$\boxed{\boldsymbol{\Omega X} = (\boldsymbol{\Omega X},\ C_*(\boldsymbol{\Omega X}),\ \boldsymbol{EC}_*^{\boldsymbol{\Omega X}},\ \varepsilon^{\boldsymbol{\Omega X}})}$$

$\Longrightarrow$ **Trivial iteration now available.**

Proof (Step 0):

Three algebraic versions for the path space Serre fibration:

$\Omega X \hookrightarrow PX \to X$.

$C_*(GX)$	$C_*(GX)$	$\mathrm{Cobar}^{C_*(X)}(\mathbb{Z},\mathbb{Z})$
$\downarrow$	$\downarrow$	$\downarrow$
$C_*(GX \times_\tau X)$	$\boxed{C_*(GX) \otimes_t C_*(X)}$	$\mathrm{Cobar}^{C_*(X)}(C_*(X),\mathbb{Z})$
$\downarrow$	$\downarrow$	$\downarrow$
$C_*(X)$	$C_*(X)$	$C_*(X)$
Simplicial	Mixed	Algebraic

where GX = Kan model for the loop space ΩX.

Proof (Step 1):

$$C_*(GX \times X) \overset{EZ}{\Rrightarrow} C_*(GX) \otimes C_*(X)$$

$\mathbf{BPL_1} \Rightarrow$

$$\clubsuit_1 \qquad C_*(GX \times_\tau X) \overset{\text{Shih}}{\Rrightarrow} C_*(GX) \otimes_t C_*(X)$$

$GX \times_\tau X$ **contractible** $\Rightarrow$

$$\clubsuit_2 \qquad C_*(GX \times_\tau X) \Rrightarrow \mathbb{Z}$$

$\clubsuit_1 + \clubsuit_2 \Rightarrow$

$$C_*(GX) \otimes_t C_*(X) \overset{H}{\Rrightarrow} \mathbb{Z}$$

Proof (Step 2):

$$\mathrm{Cobar}^{C_*(X)}(C_*(X), \mathbb{Z}) \Rrightarrow \mathbb{Z}$$

$[C_*(GX) \otimes -] \Rightarrow$

$$\mathrm{Cobar}^{C_*(X)}(C_*(GX) \otimes C_*(X), \mathbb{Z}) \Rrightarrow C_*(GX)$$

$\mathbf{BPL_2} \Rightarrow$

$$\underbrace{\mathrm{Cobar}^{C_*(X)}(C_*(GX) \otimes_t C_*(X), \mathbb{Z})} \Rrightarrow C_*(GX)$$

↑ → Key object

Proof (Step 3):

Step 1 $\Rightarrow$ $C_*(GX) \otimes_t C_*(X) \overset{H}{\Rrightarrow} \mathbb{Z} \quad \Rightarrow$

$$\mathrm{Cobar}^{C_*(X)}(C_*(GX) \otimes_t C_*(X), \mathbb{Z}) \Rrightarrow \mathrm{Cobar}^{C_*(X)}(\mathbb{Z}, \mathbb{Z})$$

for the *trivial* $C_*(X)$-comodule structure, but

BPL$_3$ $\Rightarrow$

$$\mathrm{Cobar}^{C_*(X)}(C_*(GX) \otimes_t C_*(X), \mathbb{Z}) \Rrightarrow \mathrm{Cobar}^{C_*(X)}(\mathbb{Z}, \mathbb{Z})$$

for the *canonical* $C_*(X)$-comodule structure.

Proof (Step 4):

Step 2 $\Rightarrow$

$$\mathbf{Cobar}^{C_*(X)}(C_*(GX) \otimes_t C_*(X), \mathbb{Z}) \Rrightarrow C_*(GX)$$

Step-3 $\Rightarrow$

$$\mathbf{Cobar}^{C_*(X)}(C_*(GX) \otimes_t C_*(X), \mathbb{Z}) \Rrightarrow$$

$$\Rrightarrow \mathbf{Cobar}^{C_*(X)}(\mathbb{Z}, \mathbb{Z})$$

Step-2 + **Step-3** $\Rightarrow$

$$C_*(GX) \Lleftarrow\!\!\Rrightarrow \mathbf{Cobar}^{C_*(X)}(\mathbb{Z}, \mathbb{Z})$$

Proof (Step-5):

$$C_*(X) \lll \widehat{C}_*^X \ggg EC_*^X$$

$\Rightarrow$

$$\mathrm{Cobar}^{C_*(X)}(\mathbb{Z},\mathbb{Z}) \lll \mathrm{Cobar}^{\widehat{C}_*^X}(\mathbb{Z},\mathbb{Z}) \ggg \mathrm{Cobar}^{EC_*^X}(\mathbb{Z},\mathbb{Z})$$

for the *trivial* coalgebra structures

$\Rightarrow$

$$\mathrm{Cobar}^{C_*(X)}(\mathbb{Z},\mathbb{Z}) \overset{EPL}{\lll} \mathrm{Cobar}^{\widehat{C}_*^X}(\mathbb{Z},\mathbb{Z}) \overset{BPL_4}{\ggg} \mathrm{Cobar}^{EC_*^X}(\mathbb{Z},\mathbb{Z})$$

for the *right* (A_∞-) coalgebra structure

$\Rightarrow$ $$\mathrm{Cobar}^{C_*(X)}(\mathbb{Z},\mathbb{Z}) \lll\!\!\Longleftrightarrow\!\!\ggg \mathrm{Cobar}^{EC_*^X}(\mathbb{Z},\mathbb{Z}).$$

Proof (Step-6):

Step-4 $\Rightarrow$

$$C_*(GX) \Longleftrightarrow \text{Cobar}^{C_*(X)}(\mathbb{Z}, \mathbb{Z})$$

Step-5 $\Rightarrow$

$$\text{Cobar}^{C_*(X)}(\mathbb{Z}, \mathbb{Z}) \Longleftrightarrow \text{Cobar}^{EC_*^X}(\mathbb{Z}, \mathbb{Z})$$

Step-4 + Step-5 + Composition of equivalences $\Rightarrow$

$$C_*(GX) \Longleftrightarrow \text{Cobar}^{EC_*^X}(\mathbb{Z}, \mathbb{Z})$$

Q.E.D.

$\Rightarrow$ **Very simple solution of Adam's problem :**

Indefinite iteration of the Cobar construction ???

$$\begin{aligned}
X &= (X, C_*(X), EC_*^X, \varepsilon^X) \\
&\Downarrow \; \Omega_{EH} \\
\Omega X &= (\Omega X, C_*(\Omega X), EC_*^{\Omega X}, \varepsilon^{\Omega X}) \\
&\Downarrow \; \Omega_{EH} \\
\Omega^2 X &= (\Omega^2 X, C_*(\Omega^2 X), EC_*^{\Omega^2 X}, \varepsilon^{\Omega^2 X}) \\
&\Downarrow \; \Omega_{EH} \\
\Omega^3 X &= (\Omega^3 X, C_*(\Omega^3 X), EC_*^{\Omega^3 X}, \varepsilon^{\Omega^3 X}) \\
&\Downarrow \; \Omega_{EH} \\
\Omega^4 X &= \ldots
\end{aligned}$$

"Cobar" $\boxed{3}$ (EC_*^X)

Example of CA-Spectral Sequence.

Computation of the homotopy groups of $S_2 P^\infty \mathbb{R} =$

$=$ Infinite real projective space stunted at dimension 2

$$:= P^\infty \mathbb{R} / P^1 \mathbb{R}.$$

$$\begin{array}{ccccccccc} S^0 & \subset & S^1 & \subset & S^2 & \subset & S^3 & \subset \cdots \subset & S^\infty \\ P^0\mathbb{R} & \subset & P^1\mathbb{R} & \subset & P^2\mathbb{R} & \subset & P^3\mathbb{R} & \subset \cdots \subset & P^\infty\mathbb{R} \end{array}$$

Elementary: $H_2 = \mathbb{Z} \Rightarrow \pi_2 = \mathbb{Z} \Rightarrow$ **consider the fibration:**

$$K(\mathbb{Z}, 1) \hookrightarrow [X_3 := K(\mathbb{Z}, 1) \times_\tau S_2 P^\infty \mathbb{R}] \rightarrow S_2 P^\infty \mathbb{R}$$

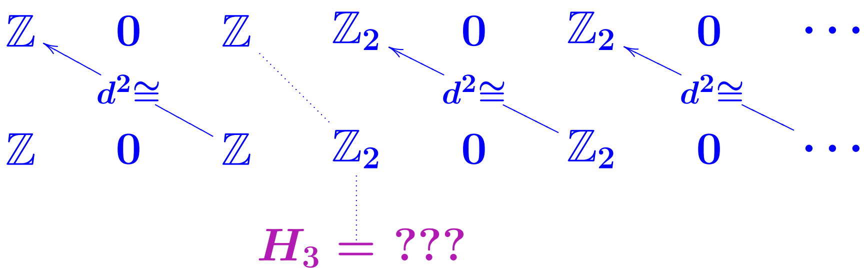
$$H_*(X_3) = ???$$

Beginning of the Serre spectral sequence.

$$H_*(K(\mathbb{Z},1)) = (\mathbb{Z}, \mathbb{Z}, 0, 0, \ldots \text{(1-periodic)})$$

$$H_*(S_2P^\infty\mathbb{R}) = (\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \ldots \text{(2-periodic)})$$

$\Rightarrow E^2_{*,*}$ **(page 2) :**

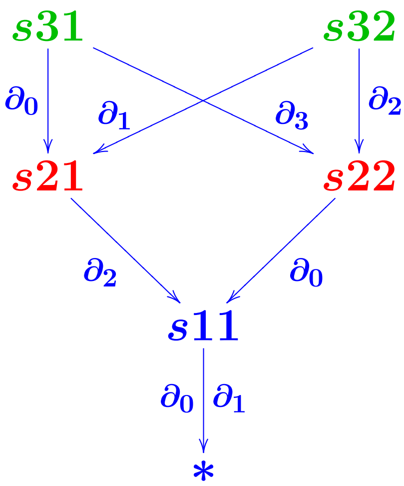


Note the non-trivial extension problem:

$$0 \rightarrow \mathbb{Z} \rightarrow \;???\; \rightarrow \mathbb{Z}_2 \rightarrow 0$$

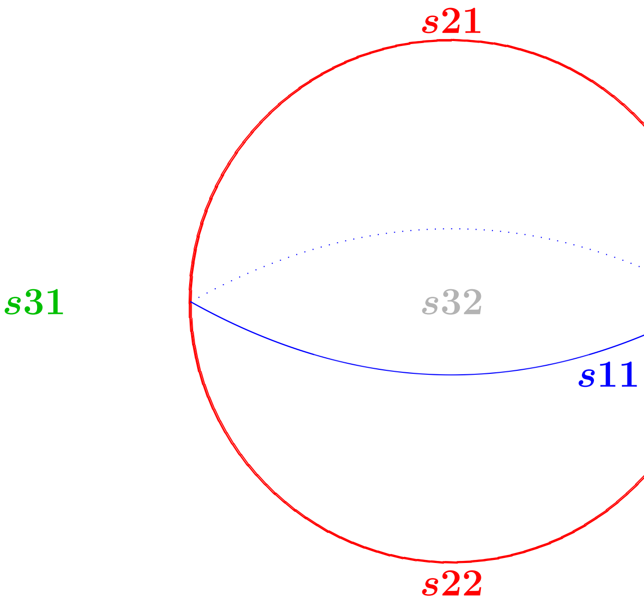
automatically solved by Kenzo.

Simplex diagram for the generator $s3?$ of $H_3(X_3)$.



⇒ $s3?$ = two 3-cells glued along their boundary = 3-sphere.

QED.


s21
s31
s32
s11
s22

Gunnar Carlsson + James Milgram,

in "Handbook of Algebraic Topology", 1995:

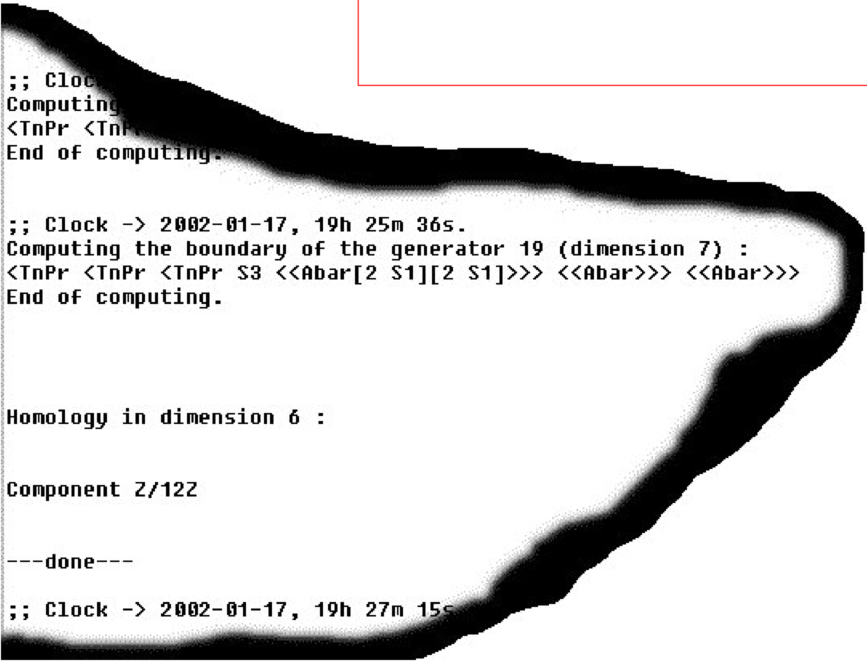
"Stable Homotopy and Iterated Loop Spaces"

In Section 5 we showed that for a connected CW complex with no one cells one may produce a CW complex, with cell complex given as the free monoid on generating cells, each one in one dimension less than the corresponding cell of X, which is homotopy equivalent to ΩX. To go further one should study similar models for double loop spaces, and more generally for iterated loop spaces. . . . / . . .

$\dots/\dots$

In principle this is direct. Assume X has no i-cells for $1 \leq i \leq n$ then we can iterate the Adams-Hilton construction of Section 5 and obtain a cell complex which represents $\Omega^n X$. However, the question of determining the boundaries of the cells is very difficult as we already saw with Adam's solution of the problem in the special case that X is a simplicial complex with $sk_1(X)$ collapsed to a point. It is possible to extend Adams' analysis to $\Omega^2 X$, but as we will see there will be severe difficulties with extending it to higher loop spaces except in the case where $X = \Sigma^n Y$.

The END


;; Cloc
Computing
<TnPr <Tn
End of computing.
;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.
Homology in dimension 6 :
Component Z/12Z
---done---
;; Clock -> 2002-01-17, 19h 27m 15s