Ana Romero’s thesis is devoted to constructive (effective) versions of the Serre, Eilenberg-Moore and Bousfield-Kan spectral sequences.

Commutative Algebra, Algebraic Geometry, Arithmetic, and many other mathematical domains have now an important component consisting in implementing on computers various tools of these domains. To achieve such an ambition, it is sometimes necessary to slightly or even strongly modify the general organisation of the relevant theory. From a mathematical point of view, the matter at issue is the following: how to make constructive the corresponding theory?

It so happens the main tools of Algebraic Topology, and more generally of Homological Algebra, namely the numerous exact and spectral sequences produced in various situations, are not constructive. These exact and spectral sequences are frequently presented as computing tools; yet, except in particular cases, in fact they cannot be elementarily transformed into algorithms allowing a computer to determine for example some unknown homology or homotopy group.

The so-called Effective Homology [3] is a constructive presentation of Homological Algebra. Following the constructive requirement, the usual existence statements of homological algebra must be transformed into algorithms producing (computing) an explicit expression of the object the existence of which is claimed. If some homology group $H$ is claimed “to be” $\mathbb{Z}/2\mathbb{Z}$, an explicit isomorphism between $H$ and $\mathbb{Z}/2\mathbb{Z}$ must be produced. An explicit homology class is the homology class of some explicit cycle, and claiming that two cycles $z$ and $z'$ are homologous requires an explicit boundary preimage, that is, an explicit chain $c$ satisfying $dc = z - z'$, and so on.

This point of view quickly leads to install as the main tool of Homological Algebra the Homological Perturbation Lemma, curiously rarely used in standard Homological Algebra. It is not then difficult to transform the simplest exact and spectral sequences into algorithms computing some groups or others; not only computing but giving also constructive versions of these groups, with explicit representatives for homology classes, with an explicit boundary preimage when a cycle is claimed homologous to 0, etc.

This is interesting but, as usual, solving a research problem produces other new ones. For example, the algorithms which are procured in Effective Homology work as “black boxes”: some homology group for example is obtained, the corresponding
effective homology algorithm has used some input and has produced the result as its output; but this algorithm did not give any hint about the “reasons” which could explain such a result. The only resulting “reasons” are then a long and meaningless chain of elementary computation steps, which in fact do not give any interesting information about some underlying structure able to describe the nature of the result.

Searching for such “reasons” arouses an interesting feedback phenomenon. For example, unfortunately, some spectral sequence is not enough to produce an algorithm computing some desired homology group. On the contrary, effective homology is able to compute this group; but effective homology can also be used to compute as a by-product the corresponding spectral sequence. In fact an extra-algorithm can be added to the effective homology computation to produce all the components of this spectral sequence: the groups \( E^r_{p,q} \) for arbitrary triples \((p, q, r)\), the differentials \( d^r_{p,q} \), arbitrary higher differentials included, the limit groups \( E^\infty_{p,q} \), the structure of the filtration of the total homology groups \( H_{p+q} \) by the \( E^\infty_{p,q} \)'s. Ana Romero not only designed the appropriate theoretical algorithms to achieve this goal, but she wrote down herself an important extension\(^1\) to the Kenzo program implementing these algorithms. It is the first time some general spectral sequences can actually, theoretically and concretely, be computed.

Designing algorithms computing these spectral sequences is the first half of Ana Romero’s thesis. The process works in general for spectral sequences coming from filtered chain complexes. This is the reason why the main applications given in the memoir are the computation of Serre and Eilenberg-Moore spectral sequences, which come from appropriate filtrations of chain complexes.

Let us consider for example the computation of the homology groups of the loop space \( \Omega(\Omega S^3 \cup 2 e^3) \). No algorithm was known for the homology groups of this loop space before effective homology. The (ordinary) Serre spectral sequence (1950) immediately gives \( H_*(\Omega S^3 \cup 2 e^3) \). One of the first applications of effective homology in 1990 was the computation of the homology groups \( H_*(\Omega(\Omega S^3 \cup 2 e^3)) \). But Eilenberg and Moore proved a spectral sequence connects all these groups. Thanks to Ana Romero’s work, this Eilenberg-Moore spectral sequence was entirely computed fifteen years later (2005), of course taking account of the obvious limitations in memory space and computing time.

These results were so satisfactory that it was tempting to extend them to other situations. The next “obvious” spectral sequence to be considered is the Adams spectral sequence. This spectral sequence connects the homology groups and the homotopy groups of a space and remains with all its other satellite spectral sequences the main tool to “compute” the homotopy groups of an arbitrary simply connected space and to study the rich underlying structure. The verb compute is enclosed between quotes because there is no known elementary method transforming this spectral sequence into an algorithm.

Jean-Pierre Serre designed his famous spectral sequence to obtain a number of homotopy groups of spheres. The Steenrod algebra already had an important

\(^1\)About 2500 Lisp lines.
role in these computations, and the description of the exact role of the Steenrod
algebra in this matter was given by Frank Adams thanks to his spectral sequence:
its $E_2$ terms can be expressed as Ext-groups of cohomology groups with respect
to the Steenrod algebra. To be complete, note also Edgar Brown in 1956 proved
the theoretical computability of homotopy groups in the simply connected case,
but through a method which does not seem concretely usable, even with the most
powerful modern computers. On the contrary, effective homology allowed us to
very simply reprove the computability of homotopy groups, this time through
elementary methods, Postnikov and Whitehead towers, methods which have been
concretely implemented and produced some homotopy groups so far unreachable.

The usual Serre and Eilenberg-Moore spectral sequences are not constructive
and the Adams spectral sequence is not constructive either, which did not prevent
from using computers for auxiliary computations, a striking application of this
sort being for example the computations of $\Lambda$-algebras by Martin Tangora [4].
Nevertheless, it is a little amazing that forty-five years after its invention, the
question of making constructive an object as important as the Adams spectral
sequence has not yet been considered. Please note our challenge is a constructive
version for the unstable Adams spectral sequence for an arbitrary simply connected
space with effective homology.

It is true this problem is terribly more sophisticated than the same problem for
the Serre and Eilenberg-Moore spectral sequences. These difficulties are mainly
concentrated around two points. It is clear the modern presentation of the Adams
spectral sequence through the more general Bousfield-Kan spectral sequence, which
has also a more algebraic flavour, is to be used.

On the one hand the Bousfield-Kan spectral sequence does not come from a
filtered chain complex, but from an exact couple produced by a tower of fibrations.
This difficulty is real but not so complex. After all the relevant tower of fibrations
is nothing but a cofiltration of its limit space, and the standard convergence prop-
ties of the Bousfield-Kan spectral sequence in the simply connected case allows
the constructive topologist to easily overcome this obstacle. Easy in theory: pro-
gramming the solution is postponed to future work and is by itself an interesting
programming challenge in a totally new domain.

On the other hand, and still more new and interesting, the Bousfield-Kan spec-
tral sequence is a spectral sequence of homotopy groups, not of homology groups as
in Serre and Eilenberg-Moore spectral sequences. Examining carefully the subject
with the usual constructive requirements leads to the following conclusion: effec-
tive homology is not enough, effective homotopy is now required. It is classical
to present homotopy groups as homology groups thanks to John Moore’s trick,
but this is possible only for simplicial groups. Unfortunately, the stages of the
relevant tower of fibrations are not simplicial groups, they are only Kan spaces,
this is unavoidable, so that an effective version of the usual Kan theory is required
to make constructive the Bousfield-Kan spectral sequence.

In a sense the results which are obtained in Ana Romero’s thesis are the most
advanced results which can be obtained toward a constructive Bousfield-Kan spec-
tral sequence without effective homotopy. The Bousfield-Kan spectral sequence is based upon a simplicial-cosimplicial set $R_\infty X$ repetitively using the magic $R$-construction, the object $R$ being some ring, for example $\mathbb{Z}$: given some simplicial set $X$, a simplicial group $RX$ is obtained by replacing every simplex $\sigma$ of $X$ by the “line of simplices” $R\sigma$, and there is an obvious way to assemble these lines as a simplicial group. John Moore taught us that $\pi_*(RX) = H_*(X)$ and Bousfield and Kan systematically used this observation to connect the homology and the homotopy groups of $X$ by their famous spectral sequence.

So that one of the unavoidable requirements to make constructive the Bousfield-Kan spectral sequence is the following: if the simplicial set $X$ is given with effective homology, is it possible to present $RX$ also with effective homology? The answer is positive, and is certainly the main contribution of Ana Romero’s work toward a constructive version of the Bousfield-Kan spectral sequence. Ana Romero proves this problem can be reduced to the computability of the effective homology of Eilenberg-MacLane spaces $K(R, n)$’s ($\mathbb{Z}$ only if $R = \mathbb{Z}$!). And this computability is a consequence of the effective version of the Eilenberg-Moore spectral sequence, in fact already underlying in the fantastic work by Eilenberg, MacLane and Cartan about the $H_*(K(\pi, n))$’s in the fifties. Iterating this result, Ana Romero proves all the “vertical” components of the simplicial-cosimplicial Bousfield-Kan space $R_\infty X$ are objects with effective homology, of course when the initial object $X$ is so too.

Once this result is obtained, Ana Romero shows how this gives an algorithm computing the $E^1_{p,q}$ and $E^2_{p,q}$ groups of the Bousfield-Kan spectral sequence of an arbitrary simply connected space $X$ with effective homology. For example the standard simplicial presentation of $\Omega S^3 \cup e^3$ is infinite, but it is a space with effective homology, so that the $E^1_{p,q}$ and $E^2_{p,q}$ groups of the corresponding Bousfield-Kan spectral sequence are now computable. Ana Romero gives also a few indications in her memoir to explain how the method of the so-called unique spectral relations of Bousfield and Kan could be used to go further toward the next $E^r_{p,q}$’s and the corresponding differentials, but it is clear an available general effective homotopy theory would produce more comfortable and synthetic results.

Effective Homotopy is by itself at least another thesis subject; the general style of effective homology must be followed, but the framework of Kan’s theory, much less algebraic, makes the work significantly more technical. Typically the exact Serre homotopy sequence connects the homotopy groups of the ingredients of a fibration, so that the effective homotopy version must produce an algorithm computing for example the effective homotopy of the total space when the effective homotopy of the base and fiber spaces is given. The Bousfield-Kan spectral sequence with respect to this exact homotopy sequence is then “nothing but” the same as an ordinary homological spectral sequence with respect to collections of exact sequences.

The professional topologists will be a little amazed not to see the various Steenrod algebras in Ana Romero’s thesis. In fact the role of these algebras is here postponed to the rich structure connecting the groups $H_* K(\mathbb{Z}, n)$, which gives also other ideas. The main other theoretical solution for effective homology is operadic, cf. for example [2]. It happens the most algebraic versions of the unavoidable
$E_\infty$-operad are also made of $K(Z,n)$’s [1] and it is clear all these points of view are connected to each other in a rich way where most is still to be discovered. As usual, studying some right research subject opens other new research domains.

References


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