# A note on the semi-global controllability of the semilinear wave equation

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**Abstract**

We study the internal controllability of the semilinear wave equation

$$v_{tt}(x,t) - \Delta v(x,t) + f(x, v(x,t)) = \mathbb{1}_\omega u(x,t)$$

for some nonlinearities $f$ which can produce several non-trivial steady states.

One of the usual hypotheses to get semi-global controllability, is to assume that $f(x,v)v \geq 0$. In this case, a stabilisation term $u = \gamma(x)v_t$ makes any solution converging to zero. The semi-global controllability then follows from a theorem of local controllability and the time reversibility of the equation.

In this paper, the nonlinearity $f$ can be more general, so that the solutions of the damped equation may converge to another equilibrium than 0. To prove semi-global controllability, we study the controllability inside a compact attractor and show that it is possible to travel from one equilibrium point to another by using the heteroclinic orbits.

Key words: semi-global controllability, wave equation, internal control, compact global attractor, heteroclinic orbits.
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# 1 Introduction

Let $\Omega$ be a smooth connected riemannian manifold of dimension $d$ with boundary and let $\omega$ be an open subset of $\Omega$. Let $X = H_0^1(\Omega) \times L^2(\Omega)$, let $\Delta$ be the Laplacian operator with Dirichlet boundary conditions and let $f \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. We consider the internal control of the wave equation

$$\left\{\begin{array}{ll} v_{tt}(x,t) - \Delta v(x,t) + f(x, v(x,t)) = \mathbb{1}_\omega u(x,t) & (x,t) \in \Omega \times (0,T) \\ v(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \\ (v, \partial_t v)(x,0) = V_0 \in X & \end{array}\right. \tag{1.1}$$

where the function $u \in L^1((0,T), L^2(\Omega))$ is the control and $V = (v, v_t) \in \mathcal{C}^0([0,T], X)$ is the state of the system.

This paper concerns the semi-global controllability of the semilinear wave equation, that is that we are interested in proving the following property:

**(SGC)** For any bounded subset $\mathcal{B}$ of $X$, there exists a time $T(\mathcal{B}) > 0$ such that, for any $V_0$ and $V_1$ in $\mathcal{B}$, there exist $T \leq T(\mathcal{B})$ and $u \in L^1((0,T), L^2(\Omega))$ such that the solution of (1.1) satisfies $V(T) = (v, v_t)(T) = V_1$.

Other articles consider the boundary control problem by replacing (1.1) with a control from the boundary for the same semilinear equation. We will not study these boundary conditions while our methods could apply, but it would require some further analysis of the non-homogeneous boundary value problem for the nonlinear equation.

The control theory for the linear case $f \equiv 0$ is now well known. The almost necessary and sufficient condition for global controllability is that $\omega$ satisfies the geometric control condition: there exists $L > 0$ such that any generalized geodesic of $\Omega$ of length $L$ meets the set $\omega$ where the control is effective. Note that the condition is always sufficient, see the works of Bardos, Lebeau and Rauch [4] and [5]. However, it is necessary only if we replace $1_\omega(x)$ by a smooth function $\gamma_\omega(x)$ satisfying $\omega = \{x \,|\, \gamma_\omega(x) \neq 0\}$, see Burq-Gérard [2] for a proof of the necessity for the closely related boundary control. In the case of a control with $1_\omega$, some subtleties can happen if one ray stays close to $\partial\omega$, as for example, for the control from a half hemisphere of $S^2$, see [23] p.174.

The study of the semilinear case $f \neq 0$ is still in progress (see [8] and [30]). To our knowledge, the control problem is solved mainly in three different cases:

- *Local control.* We assume $f(x,0) = 0$, that is that $v \equiv 0$ is a steady state of (1.1). The problem of local controllability states as property (SGC), except that $\mathcal{B}$ is not any bounded set but is a small neighborhood of 0. The local controllability is known for both internal and boundary control, see [7] and [32].

  The result of Coron and Trélat [9] is quite different but could be described roughly as local controllability near a whole connected component of the set of steady states.

The authors study local controllability by boundary control near a path of steady states, in dimension $d = 1$. Notice that their method using quasi-static deformations, enables to compute effectively the control. Remark also that an important part of our proof consists in proving that we can go from an equilibrium to another using the global compact attractor and is therefore similar, in spirit, to their result.

- *Quasilinear nonlinearity.* Another class of problems for which (SGC) holds, is the case where $f$ is almost linear, for instance globally lipschitzian or super-linear but with a growth of the type $s \ln^\beta s$, see [11], [20], [24], [33], [8] and the references therein. Notice that these papers prove in fact the global controllability, that is that the time $T$ in (SGC) can be chosen independently of $\mathcal{B}$.

- *Sign condition $f(x, s)s \geq 0$.* In [31], [10] and [18], stabilization results are proved, that is that, if $f(x, s)s \geq 0$ and if $f$ satisfies some additional conditions, all the solutions of the damped wave equation

$$v_{tt}(x,t) - \Delta v(x,t) + f(x, v(x,t)) = -\mathbb{1}_\omega v_t(x,t) \tag{1.2}$$

  goes to 0 when $t$ goes to $+\infty$. Adding a local control result near 0, one easily gets a global control result by using $u = -v_t$ as control in (1.1). Therefore, [31], [10] and [18] also yield the semi-global controllability of semilinear wave equations, under a sign assumption for $f$. Notice that this sign condition cannot be removed carelessly if $f$ is super-linear, due to the counter-example of [33].

The purpose of this paper is to release the sign condition $f(x, s)s \geq 0$ to an asymptotic sign condition and to allow more complicated dynamics for the damped wave equation (1.2). In this case, the solutions of (1.2) may converge to another equilibrium than 0. Therefore, one has to explain how to move from one equilibrium to another in the control problem (1.1).

At this point, we would like to make a remark about the dependence of $f$ on the space variable $x$. In fact, in the references given above, $f$ is assumed to be independent of $x$, in order to simplify the calculations and because it seems to be in the habit to do so in control theory. On the opposite, from the dynamical point of view, richer dynamics are welcome and it is in the habit to allow $f$ to depend on $x$. We will allow this $x-$dependence in this paper, however, we underline that:

– the results of the references cited above should also hold for $f \equiv f(x, v)$, as soon as the $x-$dependence does not change the important properties of $f$, as growth estimates, sign conditions...

– the result and the discussions of this paper are also meaningful for $f \equiv f(v)$, since functions as $f(v) = \lambda v(v - 1)(v + 1)$ can also generate non-trivial dynamics for (1.2).

## 2 Main result

The main idea of this paper is very general and may be applied to several kind of problems. However, to avoid too abstract formalisms, we state our main result in the following particular cases, corresponding to some already known results, that we will use.

We assume that one of these sets of assumptions is satisfied:

**Case A.** $\Omega$ is a smooth bounded open domain of $\mathbb{R}^d$ and there exists $x_0 \in \mathbb{R}^d$ such that

$$\{x \in \partial\Omega \ / \ (x - x_0).\nu > 0\} \subset \overline{\omega} \ . \tag{2.1}$$

Moreover, if $d \geq 2$, we assume that there exists $0 \leq p < d/(d-2)$ and $C > 0$ such that

$$|f'_x(x,s)| + |f(x,s)| \leq C(1+|s|)^p \quad \text{and} \quad |f'_s(x,s)| \leq C(1+|s|)^{p-1} \ . \tag{2.2}$$

$$\inf_{x\in\overline{\Omega}} \liminf_{|s|\to\infty} \frac{f(x,s)}{s} > \lambda_1 \ , \tag{2.3}$$

where $\lambda_1$ is the first (non-positive) eigenvalue of $\Delta$.

**Case B.** $\Omega = \mathbb{R}^3$ and $\omega$ is the exterior of a ball. Moreover, $f$ satisfies (2.2) with $p \in [0,5)$, and for any $(x,s) \in \Omega \times \mathbb{R}$,

$$(x \notin B(0,R) \text{ or } |s| \geq R) \quad \Longrightarrow \quad f(x,s)s \geq cs^2 \ . \tag{2.4}$$

for a positive constant $c > 0$.

**Case C.** $\Omega$ is a smooth compact manifold with boundary of dimension $d = 3$. We assume that the geodesics of $\overline{\Omega}$ do not have contact of infinite order with $\partial\Omega$ and that $\omega$ satisfies the geometric control condition of [5]. Moreover, $f(x,s)$ is of class $\mathcal{C}^\infty$, analytic with respect to $s$, and satisfies (2.2) with $p \in [0,5)$. We also ask

$$|s| \geq R \quad \Longrightarrow \quad f(x,s)s > 0 \ . \tag{2.5}$$

**Theorem 1.** *Assume that the above hypotheses, in one of the cases A, B or C, hold. Then, the wave equation* (1.1) *is semi-globally controllable in the sense that, for any bounded subset $\mathcal{B}$ of $X$, there exists $T(\mathcal{B}) > 0$ such that, for any $V_0$ and $V_1$ in $\mathcal{B}$, there exist $T \leq T(\mathcal{B})$ and $u \in L^1((0,T), L^2(\Omega))$ such that the solution of* (1.1) *satisfies $V(T) = (v, v_t)(T) = V_1$.*

As said in Section 1, Theorem 1 was already known when (2.3) is replaced by the more restricted sign condition $f(x,s)s > 0$ for all $s \in \mathbb{R}^*$. In this simpler case, all the solutions of the associated damped wave equation

$$v_{tt} + \mathbb{1}_\omega v_t = \Delta v - f(x,v) \tag{2.6}$$

converge to zero. When releasing the sign assumption to the asymptotic one (2.3), (2.4) or (2.5), the dynamics of (2.6) become more complicated. The main new idea of this paper is to show how one can use the heteroclinic connections of (2.6) to travel from one equilibrium to another and still obtain global controllability.

The three cases A, B and C may seem restrictive, but were chosen to fit to the already known results on existence of global compact attractors and local control:

- Case A is the classical case where the local control can be proved by multiplier methods and the Cauchy problem solved by Sobolev embedding.
- Case B is the euclidean case with a subcritical nonlinearity which requires Strichartz estimates. The corresponding stabilisation problem has been studied for instance by Dehman, Lebeau and Zuazua in [10].
- Case C assumes the optimal Geometric Control Condition. Using the global strategy of [10], we have recently proved the stabilisation and the existence of compact global attractor in this case (see [18]). The Unique Continuation Property was quite tricky and required some dynamical system tools associated with a unique continuation theorem requiring partial analyticity. That is why we also require $f$ to be analytic with respect to $s$ in this case. Notice that we do not have to assume that $f(x, 0) = 0$ on the boundary as explained in the revised version of [18] available online (see [16] p. 1111 for the original idea).

Of course, similar results should be true in different geometric situations. Also, the results in case B and C could certainly be applied in any dimension. The subcritical exponents are then $p \in [0, (d + 2)/(d - 2))$ for $d \geq 3$ and any finite $p \geq 0$ for $d = 1, 2$. In fact, the method for proving Theorem 1 may apply in a larger framework, including for example boundary control. This framework is written more explicitly in the next section.

# 3 The qualitative framework of Theorem 1

As already said, the framework behind the proof of Theorem 1 is more general than Cases A, B and C. In this section, we describe this framework through qualitative assumptions and we show that it holds in particular for Cases A, B and C.

## 3.1 Cauchy problem

The assumptions of Theorem 1 are such that the wave equation (1.1) satisfies the following properties. The Cauchy problem is locally well posed and if the solution $V$ of (1.1) exists and is bounded for all $t \in [0, T]$, then there is a neighborhood of $V_0$ such that all the solutions starting in this neighborhood are well defined for all $t \in [0, T]$ and the Cauchy problem is continuous with respect to the initial data. Notice that (1.1) is reversible in time and, therefore, that all these properties also hold for the Cauchy problem backward in time.

More precisely, in cases A, B and C, we define $X_T$ the functional spaces where the equation will be well posed:

- Case A: $X_T = C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$.
- Case B and C: $X_T = C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap L^{\frac{2p}{p-3}}([0,T], L^{2p}(\Omega)$

In case B and C, we need an additional Strichartz space which is chosen here to ensure $f(x,v) \in L^1([0,T], L^2)$ when $f$ satisfies (2.2) with $p$ in the specified range (we can also assume without loss of generality that $p > 3$).

**Theorem 2** (Cauchy problem)**.**
*For any $u \in L^1([0,T], L^2)$ and $V_0 \in X = H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique solution $v \in X_T$ of the controlled equation* (1.1)*. Moreover, the flow map*

$$\begin{array}{rccc} \Phi: & X \times L^1([0,T], L^2) & \longmapsto & X_T \\ & (V_0, u) & \longrightarrow & v \end{array}$$

*is locally Lipschitz.*

This local Cauchy theory is well known and the dependence of $f$ on $x$ does not change the proof. In Case A, $f$ generates a lipschitz map in the bounded sets of $X$ and thus all these properties of the Cauchy problem follow from the classical semigroup theory (see [25]). In Cases B and C, $f$ has a higher growth rate and Strichartz estimates are required for the local existence. For case B, Strichartz estimates are true in their full range and can be found for instance in Ginibre-Velo [14]. For case C, Strichartz estimates are more recent and we refer to Burq-Lebeau-Planchon [3] for the first result and Blair-Smith-Sogge [1] for some wider range of exponents. We refer to the books of Sogge [28], Tao [29] or Cazenave [6] (for the closely related nonlinear Schrödinger equation) for references on the Cauchy problems using Strichartz estimates. See also the articles of Ginibre and Velo [12, 13].

The globalisation of the solution is obtained as follows. The natural energy of the nonlinear wave equation is defined by

$$E(V) = \int_\Omega \frac{1}{2}|v_t|^2 + \frac{1}{2}|\nabla v|^2 + F(x,v)\ dx \tag{3.1}$$

where $F(x,v) = \int_0^v f(x,\zeta)d\zeta$. Notice that, in any case A, B or C, the energy $E$ is well defined because of (2.2). Bounding the variation of the energy for a solution $V(t)$ of (1.1) gives

$$E(V(s)) \leq E(V(t)) + C\,\|u\|_{L^1([s,t],L^2)} \sup_{\tau\in[s,t]} E(V(\tau)). \tag{3.2}$$

This shows that the energy remains bounded on bounded intervals. We then use the different assumptions on the sign of $f$ to infer similar estimate for the $H^1 \times L^2$ norm. Indeed, (2.3), (2.4) or (2.5) implies that $F(x,v) - \lambda_1 |v|^2$ is bounded from below everywhere and even non-negative for $x$ outside a ball $B(0,R)$. Thus, using Poincaré inequality $|\lambda_1| \, \|v\|^2_{L^2} \leq \|\nabla v\|^2_{L^2}$, we obtain the existence of $\eta > 0$ such that

$$\forall V \in X \ , \quad E(V) \geq \eta \|V\|^2_X + vol(B(x_0, R)) \inf F \ .$$

Thus, the control of the energy given by (3.2) implies a bound on the norm of the solution of (3.4). Moreover, the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ shows that the bounded sets of $X$ have a bounded energy.

## 3.2 Local controllability near equilibrium points

We will use as a black-box the local controllability near equilibrium points. It is precisely stated as follows.

**Proposition 3.** *For any equilibrium $e \in H^1_0(\Omega)$ of (1.1), there exists a neighborhood $\mathcal{N}(e)$ of $(e,0)$ in $X$ such that (1.1) is controllable in $\mathcal{N}(e)$. In other word, there exists a time $T(e)$ such that for any $V_0$ and $V_1$ in $\mathcal{N}(e)$, there exist $T \leq T(e)$ and $u \in L^\infty((0,T), L^2(\Omega))$ such that the solution of (1.1) satisfies $V(T) = V_1$.*

Actually, in cases A, B or C, $T(e)$ can be chosen uniformly equal to the time of the geometric control condition. The proof of the local controllability consists in writing the problem as a perturbation of the linear controllability

$$h_{tt} - \Delta h + f'_v(x, e(x))h = \mathbb{1}_\omega u(x,t)$$

and to apply a fixed point theorem. The proof is mutatis mutandis given in [32]. See also [8], Theorem 3 of [10], Theorem 3.2 of [21] for control close to 0. See also [22] where local control near trajectories are constructed for the nonlinear Schrödinger equation. The only difference is that we are close to a non trivial solution and we apply similar local control method to $r(t,x) = v(t,x) - e(x)$ solution of

$$\left\{ \begin{array}{ll} r_{tt}(x,t) - \Delta r(x,t) + f'_v(x,e(x))r(x,t) + g(x,r) = \mathbb{1}_\omega u(x,t) & \quad (x,t) \in \Omega \times (0,T) \\ r(x,t) = 0 & \quad (x,t) \in \partial\Omega \times (0,T) \\ (r, \partial_t r)(x,0) = R_0 \in X & \end{array} \right. \tag{3.3}$$

where $g(x,r) = r^2 \int_0^1 (1-s) f''_v(x, r(t,x)s + e(x)) \, ds$ is of order 2 in $r$.

Note that for the controllability of the linear system, the geometric control condition satisfied by $\omega$ and the fact that $f'_v(x,e(x))$ is smooth and independent on the time are crucial for the unique continuation, see Theorem 3.8 and Corollary 4.10 of [5]. The smoothness of $e$, and thus the one of the potential $f'_v(x,e(x))$, follows from elliptic estimates and the subcriticality of $f$.

## 3.3 The damped wave equation is a gradient dissipative dynamical system

We set $\gamma = \mathbb{1}_\omega$ and we consider the damped wave equation associated to (1.1)

$$\left\{ \begin{array}{ll} v_{tt} + \gamma(x) v_t = \Delta v - f(x,v) & \quad (x,t) \in \Omega \times \mathbb{R}_+ \ . \\ (v, v_t)(0) = V_0 \in X & \end{array} \right. \tag{3.4}$$

We are interested in the dynamical properties of Equation (3.4), which are recalled below. We do not give detailed proofs, since these results are classical or straightforward adaptations of already existing results.

**Theorem 4** (Cauchy problem)**.**
*Let the assumptions of Case A, B or C be fulfilled. Then, for any $V_0 \in X = H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique solution $v \in X_T$ of the subcritical damped wave equation* (3.4)*. Moreover, this solution is defined for all $t \in \mathbb{R}$.*

***Proof:*** The local Cauchy theory follows from Theorem 2. Estimating the variation of the energy $E$, defined by (3.1), gives

$$E(V(s)) \leq E(V(t)) + C \int_s^t E(V(\tau)) d\tau.$$

When combined with Gronwall inequality, it gives backward and forward global well-posedness, as already discussed after Theorem 2. $\square$

Since (3.4) is globally well posed, it generates a global dynamical system $S(t)$ on $X$, defined by $S(t)V_0 = V(t)$. An important dynamical feature is that $S(t)$ is a gradient dynamical system.

**Theorem 5** (Gradient property)**.**
*The energy $E$ defined by* (3.1) *is a strict Lyapounov functional for the dynamical system $S(t)$ generated by the damped wave equation* (3.4)*, that is that:*

1. *the energy $E(V(t))$ is non-increasing in time for all solutions $V(t)$ of* (3.4)*,*
2. *if $E(V(t))$ is constant for any $t \geq 0$, then $V(t)$ is an equilibrium point of* (3.4)*.*

***Proof:*** The non-increase of the energy comes from the direct computation $\frac{d}{dt} E(V(t)) = -\int_\omega |v_t|^2$. To show the second property, the classical argument consists in noticing that, if $E(V(t))$ is constant, $w = v_t$ satisfies

$$w_{tt} = \Delta w - f'_v(x,v) w \qquad \text{and} \qquad w_{|\omega} \equiv 0 \ . \tag{3.5}$$

Then one uses a unique continuation result to show that $w$ vanishes identically. In Cases A and B, using (2.1), this last property is a straightforward consequence of a unique continuation property of [19]. As shown in [18], Case C is more involved since the geometric control condition of [5] is more general than (2.1). The goal is to use the unique continuation result of [27] to (3.5) to show that $E$ is a strict Lyapounov function. This unique continuation result is optimal in the geometric point of view, but it requires that $t \mapsto f'(\cdot, v(\cdot, t))$ is analytic, which can be shown for analytic $f$ via an asymptotic regularization result of [16]. $\square$

The main result of this section is the following. We refer to [15] and [26] for introductions to the notion of attractor and gradient dynamical system.

**Theorem 6** (Existence of a compact global attractor)**.**
*The dynamical system $S(t)$ generated by the damped wave equation* (3.4) *admits a compact global attractor $\mathcal{A}$, that is a connected compact invariant subset of $X$, which attracts all the bounded sets of $X$. The attractor $\mathcal{A}$ consists in all the trajectories of* (3.4)*, which are globally defined and bounded for all $t \in \mathbb{R}$.*

*Moreover, when $t$ goes to $+\infty$, any trajectory of $S(t)$ converges to the set of equilibrium points. If the trajectory is also defined and uniformly bounded for all $t \leq 0$, then the convergence toward the set of equilibrium points also holds for $t$ going to $-\infty$. As a consequence, the compact global attractor consists exactly in the set of equilibrium points and trajectories connecting two parts of this set.*

***Proof:*** The existence of a compact global attractor for $S(t)$ follows from the classical arguments of Theorem 3.8.5 of Hale's book [15] (see also Theorem 4.6 of [26]). The complete proofs can be found in [26] (Theorem 4.38 and the discussion above) for Case A and in Theorem 1.4 of [18] for Case C. The proof is not explicitly written in case B, but one can simply follow the arguments given in [18], using the results of [10]. The three important properties are: $S(t)$ is gradient, it is asymptotically smooth and the set of equilibrium points is bounded. The asymptotic smoothness comes from compactness properties of the nonlinearity $f$: in Case A, $V \mapsto (0, f(x, v))$ is compact in $X$ due to (2.2) and in Case B and C, Hypothesis (2.2) also implies some compactness property for the nonlinearity $f$ (see Theorem 8 of [5] and Proposition 4.3 of [18]).

Once the existence of the global attractor $\mathcal{A}$ is proved, its properties stated in Theorem 6 are the classical properties of the attractor of a gradient dynamical system. The convergence of the trajectories of the attractor to the set of equilibrium points, when $t$ goes to $+\infty$ and when $t$ goes to $-\infty$, is a consequence of Lasalle's principle (see Lemma 3.8.2 of [15] or Proposition 4.2 of [26]). The fact that the compact global attractor is connected is a general fact and can be found for example in Lemma 2.4.1 of [15] of Proposition 2.19 of [26]. $\square$

**Remarks:**

• For a good and simple insight into the dynamics of the attractor $\mathcal{A}$, one may think of

$\mathcal{A}$ as a finite number of isolated equilibrium points connected by heteroclinic orbits, that is trajectories $u(t)$ converging, when $t$ goes to $+\infty$ and $-\infty$, to two different equilibria $e_+$ and $e_-$. Indeed, this structure is generic and typical of gradient dynamical systems (see Theorem 3.8.5 of [15]). However, one should be aware that more complicated structure may occur in general, as a continuum of equilibria and trajectories limiting to this continuum.
• The classical case where the sign condition $f(x,s)s \geq 0$ is assumed for all $s \in \mathbb{R}$, as in [31, 10, 18] for example, corresponds to the case where the compact global attractor $\mathcal{A}$ is reduced to $\{0\}$. Hence, in this paper, we show how to deal with asymptotic dynamics, which are more involved than a single equilibrium point. The main idea is to use the heteroclinic connections to travel from an equilibrium point to another in the attractor $\mathcal{A}$.

# 4 Proof of Theorem 1

As already mentioned, the main ideas of this paper are more general than the framework of Theorem 1. To underline this fact, we prove our theorem by using only the qualitative properties stated in the previous section. Hence, our method easily extends to any control problem for which these properties hold.

In what follows, we will often write: "*we go from $V_0$ to $V_1$*" or "*there exists a control bringing $V_0$ to $V_1$*" or similar formulations. In more precise way, this will mean "*there exists $T \geq 0$ and $u \in L^1([0,T], L^2)$ such that the unique solution $v$ of (1.1) satisfies $(v, \partial_t v)(T) = V_1$*". In particular, the time is not fixed and can be eventually large. The dependence of this time $T$ on the data will be evaluated after. We call reachable state from $V_0$, and denote $\mathcal{R}(V_0)$, the set of $V_1$ such that we can go from $V_0$ to $V_1$ (without precision on the time).

The proof of our main result can be split into several steps. The first ones are basic facts. Step 4 consists in the key "double U-turn" argument. Steps 5-7 contain typical arguments coming from the study of dynamical systems. To lighten the arguments of the proof, we will only consider the question of the uniformity of the time of control in the end this section.

**Basic fact 1:** by Proposition 3, (1.1) is locally controllable in a neighborhood $\mathcal{N}(e)$ of an equilibrium point $(e, 0)$. In the following, we assume without loss of generality that $\mathcal{N}(e)$ is a ball of $X$.

**Basic fact 2:** if $V(t) = (v, v_t)(t)$ is a solution of the damped wave equation (3.4), then, by using the control $u(x,t) = -\gamma(x)v_t(x,t) \in L^1([0,T], L^2)$ on the time interval $[0,T]$, we can go from $V(t_0)$ to $V(t_0 + T)$ in the control problem (1.1).

**Basic fact 3:** if $(v, v_t)(t)$ is a solution of the damped wave equation (3.4), then, by using the control $u(x,t) = \gamma(x)v_t(x, t_0 - t)$ on the time interval $[0,T]$, we can go from $(v, -v_t)(t_0)$ to $(v, -v_t)(t_0 - T)$ in the control problem (1.1).

**Step 4: the "double U-turn" argument.**
This step consists in proving the following property.

**Proposition 7.** *If $V(t) = (v, v_t)(t)$ is a globally bounded solution of the damped wave equation* (3.4) *(i.e. $V(t)$ belongs to the attractor $\mathcal{A}$), then for any points $V(t_0)$ and $V(t_1)$ of the trajectory, there exists a control $u$ on a time interval $[0, T]$ bringing $V(t_0)$ to $V(t_1)$.*

***Proof:*** If $t_1 \geq t_0$, this is simply Basic Fact 2. Assume that $t_1 < t_0$, we proceed as follows. By Theorem 6, the damped wave equation (3.4) generates a gradient dynamical system. Thus, there exist two equilibrium points $e_\pm$ and two times $t_\pm$ with $t_- < t_1 < t_0 < t_+$ such that $V(t_\pm)$ belongs to $\mathcal{N}(e_\pm)$. We start from $V(t_0)$ and using Basic Fact 2 with the control $u = -\gamma v_t(t_0+t)$, we reach $V(t_+) = (v, v_t)(t_+)$ which belongs to $\mathcal{N}(e_+)$. Notice that $\mathcal{N}(e_+)$ is assumed to be a ball centered in $(e_+, 0)$, and thus that $(v, -v_t)(t_+)$ also belongs to $\mathcal{N}(e_+)$. Due to Basic Fact 1, we can find a control to go from $(v, v_t)(t_+)$ to $(v, -v_t)(t_+)$. Then, Basic Fact 3 shows that we can travel backward to reach $(v, -v_t)(t_-)$. Using Basic Fact 1 again, we go from $(v, -v_t)(t_-)$ to $V(t_-) = (v, v_t)(t_-)$. Finally, we simply follow the trajectory as described in Basic Fact 2 to reach $V(t_1)$. $\square$

**Step 5: the reachable set $\mathcal{R}(e, 0)$ of an equilibrium $(e, 0)$ is open in $X$.** Let $e$ be an equilibrium point of the wave equation. Assume that there exists a control $u$ on a time interval $[0, T]$ such that the solution of (1.1) satisfies $V(0) = (e, 0)$ and $V(T) = V_1$. By Theorem 2 and the reversibility of the equation, we know that if a control $u \in L^1([0, T], L^2)$ is fixed, the flow map $\Phi$ which sends an initial data $V_0$ to a final data $V(T)$ at time $T$ for a solution of (1.1) is an homeomorphism of $X$. In particular, there exists a neighborhood $\mathcal{U}$ of $V_1$ such that the backward Cauchy problem (1.1) starting at $t = T$ in $\mathcal{U}$ with control $u$ is defined in $[0, T]$ and arrives in the neighborhood $\mathcal{N}(e)$ of $(e, 0)$. In other words, applying the control $u$ in (1.1), we can reach any point of $\mathcal{U}$ by starting from some $V_0 \in \mathcal{N}(e)$ and applying the control $u$. On the other hand, due to the local control hypothesis (Basic Fact 1), there exist controls $\tilde{u}$ bringing $(e, 0)$ to any point of the ball $\mathcal{N}(e)$. Therefore, applying successively the controls $\tilde{u}$ and $u$, we can reach any point of $\mathcal{U}$ from $(e, 0)$.

**Step 6: the reachable set $\mathcal{R}(e, 0)$ of an equilibrium $(e, 0)$ is closed in the attractor $\mathcal{A}$.** Let $V_1^n$ be points of the compact global attractor $\mathcal{A}$ such that there exist controls $u^n$ on $[0, T^n]$ bringing $(e, 0)$ to $V_1^n$ and such that $(V_1^n)$ converges to $V_1 \in \mathcal{A}$. We denote by $V(t)$ the solution of the damped wave equation (3.4) with $V(0) = V_1$. Since the flow $S(t)$ of (3.4) is assumed to be gradient, there exists $T > 0$ and $\widetilde{e}$ an equilibrium point such that $V(T)$ belongs to $\mathcal{N}(\widetilde{e})$. By continuity of the flow of (3.4), $S(T)V_1^n$ also belongs to $\mathcal{N}(\widetilde{e})$ for $n$ large enough. Using successively Basic Fact 2, Basic Fact 1 and Step 4, we can go from $V_1^n$ to $V(0) = V_1$ via $S(T)V_1^n$ and $V(T)$.

**Step 7: conclusion.** The compact global attractor $\mathcal{A}$ of $S(t)$ is a connected set (see Theorem 6). Therefore, Steps 5 and 6 show that the reachable set of an equilibrium point is a neighborhood $\mathcal{N}(\mathcal{A})$ of the attractor $\mathcal{A}$. In particular, we can go from the neighborhood of any equilibrium point to the one of any other equilibrium point. Let $V_0$ and $V_1$ be two points of $X$. Let $V(t)$ and $\tilde{V}(t)$ be the trajectories of (3.4) satisfying $V(0) = V_0$ and $(\tilde{v}, -\tilde{v}_t)(0) = V_1$. For $T$ large enough, $V(T)$ and $\tilde{V}(T)$ are in neighborhoods of equilibrium points, and so is $(\tilde{v}, -\tilde{v}_t)(T)$. Combining the previous argument, we can go from $V_0$ to $V_1$ through the control problem (1.1), via $V(T)$ and $(\tilde{v}, -\tilde{v}_t)(T)$.

**Step 8: uniformity of the time of control.** By assumption, the time of local control is uniformly bounded by $T(e)$ in a neighborhood $\mathcal{N}(e)$ of an equilibrium point. Since the set of all the equilibria is a closed subset of the compact attractor $\mathcal{A}$ and is therefore compact, the time of control $T(e)$ can be chosen to be independent of $e$ and we can consider only a finite number of equilibrium points in the above arguments. For any neighborhood $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ and any ball $B_X(0, R)$, there is a time $T$ such that $S(T)B_X(0, R) \subset \mathcal{N}(\mathcal{A})$. Considering the paths yielded by the above arguments to link two given points of $B_X(0, R)$, there is only one last property to show: there is a time $T(\mathcal{A})$ and a neighborhood $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ such that, for any $V_0 \in \mathcal{N}(\mathcal{A})$, $S(t)V_0$ belongs to a neighborhood $\mathcal{N}(e)$ for some $t \in [0, T(\mathcal{A})]$. To show this last property, we argue by contradiction. Assume that there exist sequences $(V_0^n) \subset X$, with $d(V_0^n, \mathcal{A}) \leq 1/n$, and $(T^n) \to +\infty$ such that $S(t)V_0^n$ does not cross any neighborhood $\mathcal{N}(e)$ for $t \in [0, T^n]$. Since $\mathcal{A}$ is compact, we can assume that $(V_0^n)$ converges to $V_0 \in \mathcal{A}$. Because $S(t)$ is a gradient dynamical system, $S(T)V_0$ belongs to a neighborhood $\mathcal{N}(e)$ for $T$ large enough, and thus $S(T)V_0^n$ also belongs to $\mathcal{N}(e)$ for $n$ large enough. This contradicts the definition of $T^n$ and the fact that $T^n$ goes to $+\infty$.

# 5 Discussion

First notice that Theorem 1 is a result of semi-global controllability in the sense that the time of control depends on the sizes of the initial and final data. One may expect that, for the damped wave equation (3.4), the large balls converge, with a uniform exponential rate, to an absorbing ball $\mathcal{B}_0$, which contains the attractor $\mathcal{A}$. This would implies that the time of control in a large ball $\mathcal{B}$ of radius $R$ is of order $T(\mathcal{B}) \sim C \ln R + T(\mathcal{B}_0)$. However, proving such a uniform exponential rate of convergence to the absorbing ball is a difficult problem related to the uniform stabilization of the semilinear wave equation (see [31] and its application to the convergence to the absorbing ball in Proposition 3.6 of [17]).

The arguments of the proof of Theorem 1 are very related to the qualitative dynamics of the wave equation (1.1) with the feedback control $u = -\gamma(x)v_t$. In the simplest cases, one can follow the arguments to construct a simple control as illustrated in Figures 1 and 2.

Notice that the proof of Theorem 1 is somehow constructive: the way to travel between two equilibrium points is described in terms of the heteroclinic orbits and moreover the local control may be explicit if it is obtained via a Banach fixed point theorem. In this sense, our method of proof is more explicit than the Leray-Schauder fixed point argument used in

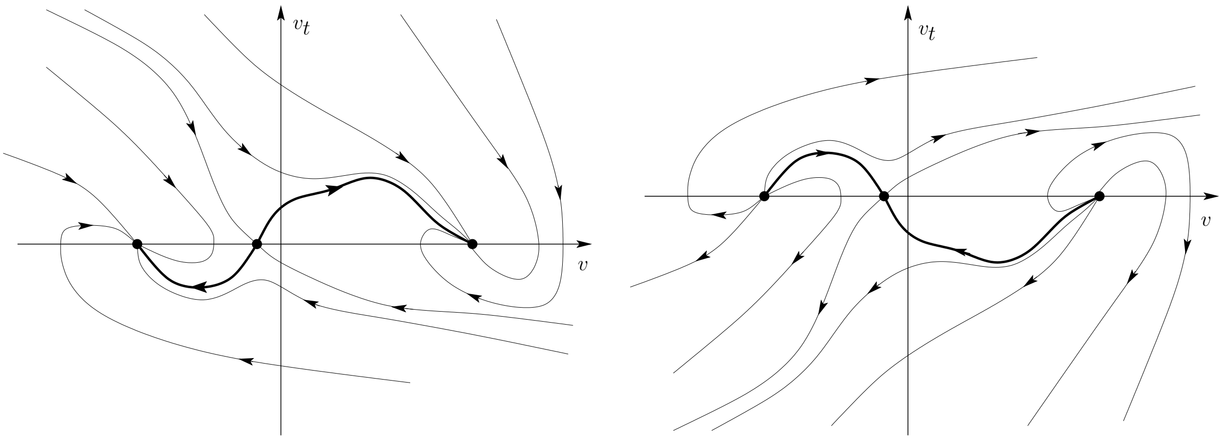


Figure 1: *Right: the flow of the dynamical system $S(t)$ generated by the feedback control $u = -\gamma(x)v_t$. The flow is represented in the phase plane $(v, v_t)$. The compact global attractor $\mathcal{A}$ (in bold) consists in three equilibrium points and two heteroclinic orbits connecting them. Left: the flow generated by the feedback control $u = \gamma(x)v_t$. It is deduced from the flow of $S(t)$ by reversing time and orientation of the second coordinate.*

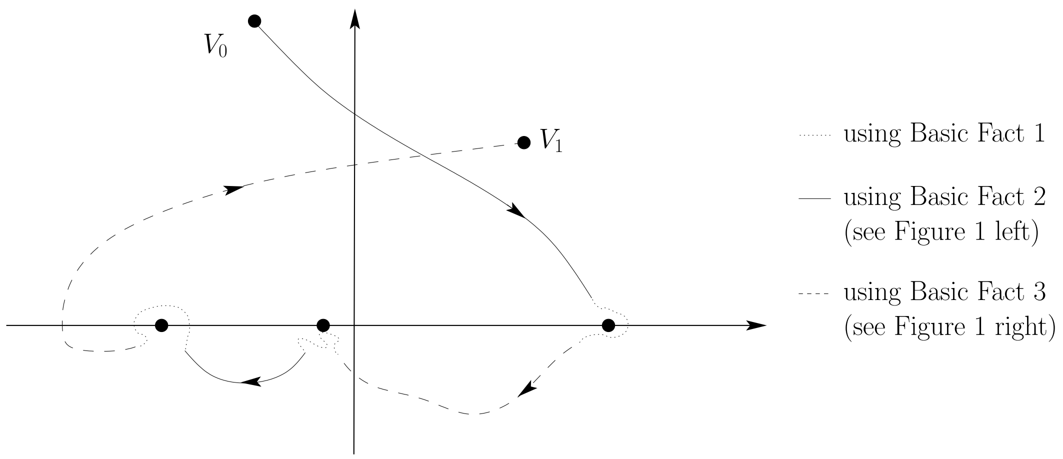


Figure 2: *An example of global control using successively the Basic Facts 1, 2 and 3 introduced at the beginning of the proof of Theorem 1. The resulting trajectory consists in switching from one of the flows of Figure 1 to the other, using in between the local controls near the equilibrium points.*

the proofs of several control results for the nonlinear wave equations (see [33] for example). Of course, our proof of Theorem 1 is not really constructive, because the computation of heteroclinic orbits is not explicit. But it is reasonable to expect some cases where approximations of the heteroclinic orbits are numerically known. In this case, our proof of Theorem 1 gives an explicit way to compute an approximated control to connect two equilibrium points. By the way, the question whether Theorem 1 can be proved by using a fixed point method as the one of [33], or not, is an open interesting question. In any case, we think that our method of proof is new and represents an interesting alternative to the proof by Leray-Schauder theorem, which is actually available only for weak nonlinearities.

As noticed after Theorem 6, it is reasonable to think the attractor $\mathcal{A}$ of (3.4) as a finite number of isolated equilibrium points connected by heteroclinic orbits, as in Figure 1. In this case, one can go from any equilibrium point to another one by following a finite number of heteroclinic orbits, forward or backward in time. In this point of view, the steps 5 and 6 of the proof of Theorem 1 may seem unnatural. However, the above generic framework is not true in general: $\mathcal{A}$ may contain a continuum of equilibrium points, which precludes the connection between two parts of $\mathcal{A}$ via heteroclinic orbits. That is why, in the proof of Theorem 1, we used the topological connectedness of $\mathcal{A}$ and not the connectedness in the sense of path of heteroclinic orbits.

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