Two proofs of Taubes' theorem on strictly ergodic flows

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ABSTRACT. The purpose of this note is to give two new proofs of C. H. Taubes' result concerning the helicity of a strictly ergodic flow. A strictly ergodic flow is a uniquely ergodic volume preserving flow. The theorem states that on a 3-manifold the helicity of such flows is zero. For both proofs we use C. H. Taubes' result on the existence of periodic orbits for Reeb vector fields [**7**] and for the second one we also use the characterization of geodesible vector fields by D. Sullivan [**5**].

1. Introduction

Let X be a vector field on a closed oriented 3-manifold M, that preserves the volume form Ω. Thus the Lie derivative $L_X\Omega = 0$. Cartan's formula implies that $\omega = \iota_X\Omega$ is a closed 2-form. The vector field X is exact if ω is exact. In this case, let α be a primitive of ω. Observe that in $\mathbb{S}^3$ or in any homotopy sphere, any volume preserving vector field is exact. The helicity of the vector field X is defined as the integral

$$h(X) = \int \alpha \wedge d\alpha,$$

that does not depends on the choice of the 1-form α. The helicity is an important invariant in the study of fluid flows. When the manifold is simply connected the helicity coincides with the asymptotic linking number, as proved by Arnold [**1**].

A vector field is uniquely ergodic if its flow admits a unique invariant measure. When such a measure is a smooth volume, equivalently a volume form, the vector field is strictly ergodic.

THEOREM 1.1 (Taubes, [**6**]). *Let X be a strictly ergodic exact vector field on a closed 3-manifold M. Then $h(X) = 0$.*

Let f be the function defined by $f = \alpha(X)$ or equivalently $\alpha \wedge d\alpha = f\Omega$. Observe that $h(X)$ is the integral of f with respect to Ω. The proofs we present here give the following stronger statement:

THEOREM 1.2. *If X is a minimal exact vector field preserving the volume form Ω, then there exists an invariant measure μ such that $\int_M f\, d\mu = 0$.*

By minimal we understand a vector field whose orbits are dense in M. A strictly ergodic vector field is minimal. The minimality or strict ergodicity is used in different ways in the proofs below. In the first proof, it is used to say that X is not a Reeb vector field and thus find two invariant measures for which the integral of f is strictly positive with respect to the first measure and strictly negative with respect to the second measure. A linear combination of these invariant measures gives the measure μ in Theorem 1.2.

In the second poof, the minimality or strict ergodicity of X is used to say that X is not geodesible (in other words, X is not the Reeb vector field of a stable Hamiltonian structure). D. Sullivan's characterisation of geodesible vector fields gives the existence of the measure μ. We thank the referee for pointing out a proof of Theorem 1.1 that was explained to him by Gabriel Paternain in 2008 and differs little from the second proof in this paper. In fact, instead of using the characterisation by D. Sullivan of geodesible vector fields, one can use Theorem 5.2 of D. McDuff's article [**3**].

C. H. Taubes' proof of Theorem 1.1 relies on Seiberg-Witten equations. He uses this technique in a similar way as for proving the existence of a periodic orbit of a Reeb vector field on a 3-manifold [**7**]. The proofs presented here use the fact that a Reeb vector field has a periodic orbit and hence cannot be minimal. In the case of $\mathbb{S}^3$ or a 3-manifold with non-trivial second homotopy group, H. Hofer proved the existence of periodic orbit for Reeb vector fields using pseudoholomorphic curves [**2**].

There are two types of examples of strictly ergodic flows on 3-manifolds. First, suspensions of strictly ergodic diffeomorphisms of the 2-torus. Second quotients of the horocycle flow. The following example was explained to us by A. Verjovsky. Consider the Brieskorn manifolds

$$V_{p,q,r} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^p + z_2^q + z_3^r = 0, \ z_1^2 + z_2^2 + z_3^2 = 1\},$$

with p, q, r positive integers and two by two relative primes. Then $V_{p.q.r}$ is a homology sphere and $\mathbb{S}^3$ if p, q or r is equal to 1. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ then $V_{p,q,r}$ is a quotient of $SL(2, \mathbb{R})$ by a lattice. The horocycle flow on $SL(2, \mathbb{R})$ defines a flow in the quotient space that is a strictly ergodic flow on the 3-manifold. The horocycle flow defines a strictly ergodic flow on any compact quotient of $SL(2, \mathbb{R})$, but these quotients are not always homology spheres.

2. First proof

Consider an exact vector field X preserving the volume form Ω, then, using the same notation as in the introduction, write $\iota_X \Omega = \omega = d\alpha$ for a differential 1-form α. In this section we prove the following result:

THEOREM 2.1. *Let X be an exact volume preserving vector field. Assume that X has no periodic orbits and let f be the function defined by $\alpha \wedge d\alpha = f\Omega$. Then, there exists an invariant measure μ with $\int f \, d\mu = 0$.*

Clearly, Theorems 1.2 and 1.1 can be deduced from Theorem 2.1.

Before starting the proof, recall that a differential 1-form λ on a closed oriented 3-manifold M is a contact form if $\lambda \wedge d\lambda \neq 0$. For such a form, its Reeb vector field X is defined by the equations $\lambda(X) = 1$ and $\iota_X d\lambda = 0$. Thus X preserves λ and the volume form $\lambda \wedge d\lambda$.

PROOF. Let ϕ_t be the flow of the vector field X. The conclusion in Theorem 2.1 can be reformulated in the following way: it cannot happen that the integrals of f with respect to all the ϕ_t-invariant measures are all strictly positive or all strictly negative.

Assume the converse and let all such integrals be, for instance, strictly positive. Consider the family of 1-forms

$$\alpha_t = \frac{1}{t} \int_0^t \phi_\tau^* \alpha \, d\tau,$$

obtained by averaging α along the flow. Since the flow preserves Ω, it preserves $\omega = \iota_X \Omega$ and hence for any t the 1-form α_t is a primitive of ω with

$$\alpha_t \wedge \omega = \left(\frac{1}{t} \int_0^t f \circ \phi_\tau \, d\tau \right) \Omega.$$

Notice that among the time average functions $\bar{f}_t = \frac{1}{t} \int_0^t f \circ \phi_\tau \, d\tau$, there is a strictly positive function. Indeed, if this was not the case, there would exists a sequence of points p_{t_n} such that $\bar{f}_{t_n}(p_{t_n}) \leq 0$. But,

$$\bar{f}_{t_n}(p_{t_n}) = \frac{1}{t_n} \int_0^{t_n} f(\phi_\tau(p_{t_n})) \, d\tau = \int f \, d\mu_n,$$

where

$$\mu_n = \frac{1}{t_n} \int_0^{t_n} \delta_{\phi_\tau(p_{t_n})} \, d\tau$$

the time average of the Dirac measure $\delta_{p_{t_n}}$. Consider a Riemannian metric on M such that the norm of X is identically 1, then the length of the orbit segment from p_{t_n} to $\phi_{t_n}(p_{t_n})$ is equal to t_n. Thinking the measures μ_n as 1-currents, the mass of μ_n is 1. Extracting a weakly converging subsequence μ_{n_k} of more and more invariant measures, we find that the limit μ is a non-zero invariant measure with

$$\int f \, d\mu = \lim_{k \to \infty} \int f \, d\mu_{n_k} \leq 0,$$

giving us a contradiction to our assumption. Hence for some t we have $\alpha_t \wedge d\alpha_t = \bar{f}_t \Omega$, with

$$\bar{f}_t = \frac{1}{t} \int_0^t f \circ \phi_\tau \, d\tau > 0.$$

But then α_t is a contact form and $\alpha_t(X) = \bar{f}_t > 0$ and $\iota_X d\alpha_t = \iota_X \omega = 0$. Thus, modulo a reparametrisation by a positive function, X is the Reeb vector field of α_t. Due to Taubes' result, X possesses a periodic orbit. This contradiction concludes the proof. $\square$

3. Second proof

A non-singular vector field is geodesible if there exists a Riemannian metric on the ambient manifold making all the orbits geodesics. As remarked in [**5**], a vector field is geodesible if and only if there exists a differential 1-form λ such that $\lambda(X) = 1$ and $\iota_X d\lambda = 0$. Observe that geodesible vector fields form a larger family than Reeb vector fields, since the form λ might be a closed 1-form. Geodesible volume preserving vector fields are also known in the literature as Reeb vector fields of stable Hamiltonian structures.

THEOREM 3.1 (Sullivan [5]). *A vector field is geodesible if and only if any foliation cycle cannot be approximated by tangent 2-chains.*

A foliation cycle is a flat current that is obtained from positive linear combinations of the Dirac currents defined by X. We recall that foliation cycles are in one to one correspondence with the invariant measures of the flow of X, as proved by D. Sullivan [4]. The term tangent 2-chain means a flat current of integration on sets that are tangent to the vector field X, this implies that the 2-chain evaluated on any 2-form whose kernel contains X must be equal to zero. We can now give the second proof to Theorem 1.1.

PROOF. We use the previous notation, that is Ω is the invariant volume form and $\iota_X \Omega = \omega = d\alpha$.

If the flow of X is strictly ergodic then it is a minimal flow: every orbit is dense. Assume that X is geodesible, then there exists a differential 1-form λ such that $\lambda(X) = 1$ and $\iota_X d\lambda = 0$. In particular, λ and $\lambda \wedge d\lambda$ are invariant forms. Since the latter is a differential 3-form we can write $\lambda \wedge d\lambda = g\Omega$ for some function g. Since both Ω and $\lambda \wedge d\lambda$ are invariant forms, the function g is invariant and by the minimality of the flow it is constant. If $g \neq 0$, the vector field X is the Reeb vector field of the contact form λ, which contradicts minimality by the existence of a periodic orbit [7]. If $g = 0$, observe that $d\lambda = 0$ since X is not in the kernel of λ. Then λ is a closed 1-form positive on X. Tischler's theorem [8] implies that M fibers over the circle and each fiber is a global section for the flow. This means, that the fibers are closed 2-manifold S transverse to the vector field and the flow is the suspension of a diffeomorphism of S. This is impossible, an exact vector field cannot admit a global section as a consequence of Stokes' theorem:

$$0 = \int_{\partial S} \alpha = \int_S d\alpha = \int_S \omega > 0,$$

where the last inequality follows since ω is an area form on S.

Thus X is not geodesible. Since X is strictly ergodic and invariant measures are in one to one correspondence with foliation cycles [4], there is only one foliation cycle Z that corresponds to the volume form Ω. Then

$$Z(\alpha) = \int_M \alpha \wedge d\alpha = \int_M f\Omega = h(X),$$

as remarked in page 232 of [4]. Then by Theorem 3.1 there exists a sequence of tangent 2-chains B_n such that ∂B_n tends to Z. Since X is in the kernel of $d\alpha$ and is tangent to B_n, we have that $B_n(d\alpha) = 0$. Thus,

$$Z(\alpha) = \lim_{n \to \infty} \partial B_n(\alpha) = \lim_{n \to \infty} B_n(d\alpha) = 0.$$

$\square$

REMARK 3.2. If X is minimal but not strictly ergodic, the proof above implies that X is not geoedesible. Then there is a the foliation cycle approximated by the boundaries of tangent 2-chains, that is not necessarily the foliation cycle associated to the invariant volume form. Thus the proof above provides the existence of an invariant measure μ for which $\int_M f \, d\mu = 0$, proving Theorem 1.2.

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