ON THE EXISTENCE OF SUPPORTING BROKEN BOOK DECOMPOSITIONS FOR CONTACT FORMS IN DIMENSION 3

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ABSTRACT. We prove that in dimension 3 every nondegenerate contact form is carried by a broken book decomposition. As an application we obtain that on a closed 3-manifold, every nondegenerate Reeb vector field has either two or infinitely many periodic orbits, and two periodic orbits are possible only on the tight sphere or on a tight lens space. Moreover we get that if M is a closed oriented 3-manifold that is not a graph manifold, for example a hyperbolic manifold, then every nondegenerate Reeb vector field on M has positive topological entropy.

CONTENTS

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1. Introduction

On a closed 3-manifold M, the *Giroux correspondence* asserts that every contact structure ξ is carried by *some* open book decomposition of M: there exists a Reeb vector field for ξ transverse to the interior of the pages and tangent to the binding [Gir]. The dynamics of this specific Reeb vector field is then captured by its first-return map on a page, which is a flux-zero area-preserving diffeomorphism of a compact surface, a much simplified data. When one is interested in the dynamics of a *given* Reeb vector field this Giroux correspondence is quite unsatisfactory—though there are ways to transfer some properties of an adapted Reeb vector field to every other one through contact homology techniques [CoH, ACH]—and the question one can ask is: Is every Reeb vector field carried by some (rational) open book decomposition? Equivalently, does every Reeb vector field admit a Birkhoff section?

We give here a positive answer for the generic class of nondegenerate Reeb vector fields and the extended class of *broken book decompositions* (see Definitions 2.1 – 2.7 for details).

Theorem 1.1. *On a closed 3-manifold, there is an open C^1-neighbourhood of the set of nondegenerate Reeb vector fields such that every Reeb vector field in this neighbourhood is ∂-strongly carried by a broken book decomposition.*

A contact form and the corresponding Reeb vector field are *nondegenerate* if all the periodic orbits of the Reeb vector field are nondegenerate, namely the eigenvalues of the differentials of the return maps on small discs transverse to the periodic orbits are all different from one (even when the orbit is travelled several times). The nondegeneracy condition is generic for Reeb vector fields [CoH, Lemma 7.1].

For a vector field R on a 3-manifold, a *R-section* is a surface with boundary whose interior is embedded and transverse to R and whose boundary is immersed and composed of periodic orbits (it is sometimes called a partial section for R). A R-section is *∂-strong* when the linearised flow of R along every boundary orbit is transversal to the boundary of the surface.

A *Birkhoff section* for R is a R-section that must also intersect all orbits of R within bounded time, so that there is a well-defined first-return map in the interior of the surface. These surfaces are also known as rational global surfaces of section. A given Birkhoff section induces a rational open book decomposition of the manifold (where we do not require the usual condition of compatibility of orientations along the binding, see Section 2.2), all of whose pages are also Birkhoff sections. For this reason we refer to the boundary of a Birkhoff section as the *binding*.

Broken book decompositions are generalisations of Birkhoff sections and rational open book decompositions, reminiscent of finite energy foliations constructed by Hofer, Wysocki and Zehnder for nondegenerate Reeb vector fields on $\mathbb{S}^3$ [HWZ2]. In a broken book decomposition we allow the binding to have *broken* components, in addition to *radial* ones modelled on the classical open book case. The complement of the binding is foliated by surfaces that are relatively compact in M and whose closures in M are called the *pages*. A radial component of the binding has a tubular neighbourhood in which the pages of the broken book induce a radial foliation by annuli (as in Figure 2). The foliation in a tubular neighbourhood of a broken component has sectors that are radially foliated by annuli and sectors that are foliated by surfaces that transversally look like hyperbolas (see Figures 3 and 4). For the broken books we construct in this paper, each broken component has either two or four sectors of each type, depending on whether the component is of negative or positive hyperbolic type.

A broken book decomposition *carries*—or *supports*—a vector field R if the binding is composed of periodic orbits, while the other orbits are transverse to the interior of the pages. In particular the pages are R-sections whose boundary is contained in the binding and whose interiors give in general a non-trivial foliation of the complement of the binding, as opposed to the genuine open book case. The broken book *∂-strongly carries* the vector field R if moreover the pages are ∂-strong R-sections.

In the proof of Theorem 1.1, we construct a supporting broken book decomposition for any fixed nondegenerate Reeb vector field on a 3-manifold M from a cover of M by pseudo-holomorphic curves, given by the non-triviality of the U-map in embedded contact homology. Using a construction of Fried that glues and resolves such curves [Fri], the projected pseudo-holomorphic curves are converted into R-sections. We can then extract from this collection a finite complete system of disjoint R-sections to the Reeb vector field, meaning that their union intersects every orbit. We then complete the broken book decomposition in the complement of the chosen R-sections. The novelty in our approach is to combine pseudo-holomorphic curves with Fried's techniques.

It is worth noticing that the binding of the broken book produced by Theorem 1.1 for a contact form λ can be taken to have total action less than or equal to the *spectrum* of any class $\sigma \in ECH(M, \lambda)$ with $U(\sigma) \neq 0$, see Section 3.2 for a definition.

We believe that the notion of a (degenerate) broken book decomposition is interesting in its own right. Near the binding, the broken book foliation looks like the mapping torus of a Le Calvez transverse foliation [Lec1] and our study could also be seen as a first step towards generalising Le Calvez' theory to vector fields in three dimensions.

Birkhoff sections are often useful for understanding the dynamics of a given vector field. We hope that broken book decompositions will also help answering dynamical questions. In this direction, we give two applications of Theorem 1.1.

Weinstein conjectured in 1979 that a Reeb vector field on a closed manifold always has at least one periodic orbit [Wei]. The conjecture was proved in dimension 3 by Taubes using Seiberg-Witten Floer homology [Tau]. It is also a consequence of the U-map property we use here, and it is no surprise that our result indeed implies the existence of the binding periodic orbits. Taubes' result was then improved by Cristofaro-Gardiner and Hutchings [CrH] who proved that every Reeb vector field on a closed 3-manifold has at least two periodic orbits, following a work of Ginzburg, Hein, Hryniewicz and Macarini on $\mathbb{S}^3$ [GHHM]. It is now moreover conjectured that a Reeb vector field has either two or infinitely many periodic orbits. The existence of infinitely many periodic orbits has been established under some hypothesis (see the survey [GiG]) and Irie proved that it is generic [Iri]. Here we extend a recent result of Cristofaro-Gardiner, Hutchings and Pomerleano, originally obtained for *torsion contact structures* ξ (that is, satisfying $c_1(\xi) \in \mathrm{Tor}(H^2(M, \mathbb{Z}))$) [CHP] and prove the conjecture for an open neighbourhood of nondegenerate Reeb vector fields.

Theorem 1.2. *If M is a closed oriented 3-manifold that is not the sphere or a lens space, then there is an open C^1-neighbourhood of the set of nondegenerate Reeb vector fields on M such that every Reeb vector field in this neighbourhood has infinitely many simple periodic orbits. In the case of the sphere or a lens space, there is an open C^1-neighbourhood of the set of nondegenerate Reeb vector fields such that every Reeb vector field in this neighbourhood has either two or infinitely many periodic orbits.*

We point out that the cases where Reeb vector fields have exactly two nondegenerate periodic orbits are well-understood: they exist only on the sphere or on lens spaces, both periodic orbits are elliptic and are the core circles of a genus one Heegaard splitting of the manifold [HuT1]. In these cases, the contact structure has to be tight, since a nondegenerate Reeb vector field of an overtwisted contact structure always has a hyperbolic periodic orbit, see for example [HoK, Theorem 8.9].

Beyond the number of periodic orbits, the study of the topological entropy of Reeb vector fields started with the work of Macarini and Schlenk [MaS] and has been continued by Alves [ACH, Alv]. We recall that topological entropy measures the complexity of a flow by computing the growth of the number of “different” orbits. For flows in dimension 3, if the topological entropy is positive then the number of periodic orbits of period less than a positive number L grows exponentially with L. As another application of Theorem 1.1 we get

Theorem 1.3. *If M is a closed oriented 3-manifold that is not a graph manifold, then there is an open C^1-neighbourhood of the set of nondegenerate Reeb vector fields on M such that every Reeb vector field in this neighbourhood has positive topological entropy.*

Theorems 1.2 and 1.3 are obtained by analysing the broken binding components of the broken book decomposition. Indeed, in the nondegenerate context, a broken component of the binding has to be a hyperbolic periodic orbit. As such it has stable and unstable manifolds [Sma]. We can prove that there are heteroclinic cycles between these periodic orbits. Here a *heteroclinic cycle* is a finite sequence of orbits of the flow $\gamma_0, \dots, \gamma_n = \gamma_0$ such that every γ_i is forward and backward asymptotic to a hyperbolic periodic orbit and the forward limit orbit of γ_i is equal to the backward limit of γ_{i+1}, for $i = 0, \dots n-1$. A *homoclinic orbit* is a heteroclinic cycle with $n = 0$: an orbit that is forward and backward asymptotic to a hyperbolic periodic orbit. If there are no broken components, then we have a rational open book decomposition and Theorems 1.2 and 1.3 are deduced from an analysis of its monodromy. In particular, we obtain

Theorem 1.4. *If M is a closed oriented 3-manifold, there is an open C^1-neighbourhood of the set of strongly nondegenerate Reeb vector fields on M without homoclinic orbits such that every Reeb vector field in this neighbourhood is carried by a rational open book decomposition, or, equivalently has a Birkhoff section.*

A vector field is *strongly nondegenerate* if it is nondegenerate and the intersections of the stable and unstable manifolds of the hyperbolic periodic orbits are transverse. By a result of Katok, a strongly nondegenerate vector field with a homoclinic orbit has positive topological entropy [Kat], thus Theorem 1.4 implies that a strongly nondegenerate Reeb vector field whose topological entropy is zero is carried by a rational open book decomposition.

Our techniques, combined with Fried’s construction [Fri], also allow to establish the existence of a carrying rational open book decomposition when there is only one broken component in the binding. Supported by these constructions, we make the optimistic Conjecture 4.23 that broken book decompositions can be transformed into rational open book decompositions, and thus that nondegenerate Reeb vector fields always admit Birkhoff sections.

The results presented in this introduction are first proved for nondegenerate Reeb vector fields and extended to C^1-neighbourhoods at the end of the paper. In Section 2 we define broken book decompositions and how they carry vector fields or contact forms. The existence of broken book decompositions is established in Section 3, in particular we give a proof of Theorem 1.1 for nondegenerate Reeb vector fields. The applications of this theorem are discussed in Section 4, which is divided in seven parts. After two sections containing preliminaries, in Section 4.3 we study the invariant stable and unstable manifolds of the broken components of the binding. In particular we prove that they have to intersect. Then we give a proof of Theorem 1.2 for strongly nondegenerate Reeb vector fields in order to explain the main ideas in Section 4.4. Passing from strongly nondegenerate to nondegenerate Reeb vector fields in the proof of Theorem 1.2 is rather technical, so this proof is in Section 4.5, which also contains the proof of Theorem 1.3. In Section 4.6 we prove other applications of the existence of broken book decompositions, in particular Theorem 1.4 for nondegenerate Reeb vector fields and explain a construction that can be applied to get rid of broken components of the binding. Finally, in Section 4.7, we extend the proofs of Theorems 1.1, 1.2, 1.3 and 1.4 to an open neighbourhood of the set of nondegenerate Reeb vector fields in the C^1-topology.

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2. Reeb vector fields and broken book decompositions

This section contains the definitions underlying Theorem 1.1.

2.1. Sections and ∂-strong sections.

For a smooth non-singular vector field R on a closed 3-manifold M, we denote by $(\phi_R^t)_{t\in\mathbb{R}}$ its flow.

Definition 2.1. A *R-section* is an immersed surface S in M whose interior is embedded and transverse to R and whose boundary covers periodic orbits of R. A R-section S is a *Birkhoff section* if it intersects all the orbits of R in bounded time: there exists $T > 0$ such that for all $x \in M$ the orbit segment $\phi_R^{[0,T]}(x)$ intersects S.

Given a periodic orbit γ of R we denote by Σ_γ the unit normal bundle $(TM_\gamma/T\gamma)/\mathbb{R}_+$ to γ and by M_γ the normal blow-up of M along γ, that is the manifold $(M \setminus \gamma) \cup \Sigma_\gamma$. The vector field R being smooth, it extends to a vector field R_γ on the torus Σ_γ—and tangent to it—, and hence to a vector field on M_γ and tangent to its boundary. We abuse notation and still denote this extension R. If S is a R-section with $\gamma \in \partial S$, we denote by $\partial_\gamma S$ its extension to Σ_γ (see Figure 1).

Definition 2.2. A R-section S is *∂-strong* if, for every boundary orbit γ of S, the extension $\partial_\gamma S$ is a collection of embedded curves in Σ_γ that are transverse to the extended vector field R. If S is a Birkhoff section, S is *∂-strong* if moreover $\partial_\gamma S$ defines a transverse section for R on Σ_γ.

Remark 2.3. The ∂-strong condition is more demanding for a Birkhoff section than for a section. However, when R is a nondegenerate Reeb vector field, if we consider a ∂-strong R-section S that is also a Birkhoff section, then for every component γ of ∂S, the slope of R_γ on the boundary component Σ_γ has to be different from the one of $\partial_\gamma S$. This implies that $\partial_\gamma S$ defines a transverse section for R on Σ_γ and that S is also a ∂-strong Birkhoff section for R.

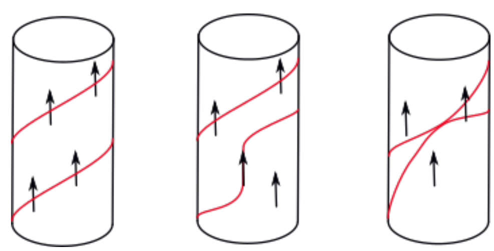

FIGURE 1. The unit normal bundle Σ_γ to a periodic orbit γ of a vector field and the extension $\partial_\gamma S$ of a R-section S to it (in red). On the left S is ∂-strong, while it is not in the center ($\partial_\gamma S$ is not transverse to R on Σ_γ) nor on the right ($\partial_\gamma S$ has a one-sided tangency point, hence it is not embedded in Σ_γ).

2.2. **Contact forms, Reeb vector fields and open books.** Recall that on a 3-manifold, a *contact form* is a non-vanishing 1-form λ such that $\lambda \wedge d\lambda$ is a volume form. Contact forms exist on every orientable 3-manifold. The *Reeb vector field* associated to a contact form λ is the unique vector field R_λ satisfying

$$i_{R_\lambda}\lambda = 1 \quad \text{and} \quad i_{R_\lambda}d\lambda = 0.$$

The vector field R_λ does not vanish and it preserves the volume $\lambda \wedge d\lambda$. It is also transverse to the so-called *contact structure* $\ker \lambda$.

Recall that a *rational open book decomposition* of a closed 3-manifold M is a pair $(K, \mathcal{F})$ where K is an oriented link called the *binding* of the open book and $\mathcal{F}$ is a foliation of $M \setminus K$ defined by a fibration of $M \setminus K$ over $\mathbb{S}^1$. Near every component k of K the foliation is as in Figure 2: in local cylindrical coordinates (r, θ, z) each leaf is the union of n surfaces $\theta = \text{cst}$, for a certain n that depends on k. A *page* of the open book is the closure of a leaf of $\mathcal{F}$ which is obtained by its union with K. In this context the pages are embedded in their interior, but only immersed along their boundary. The adjective *rational* is dropped when moreover the pages are embedded along their boundary, that is, when $n = 1$ for every boundary component. So in an *open book decomposition* each page appears exactly once along each component of the binding. In both cases we say that k is radial with respect to $\mathcal{F}$.

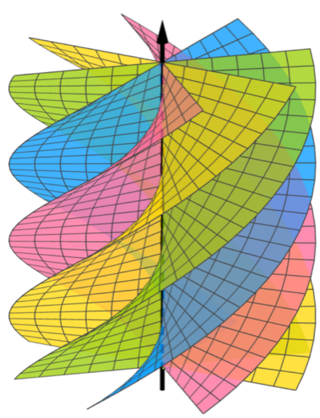

FIGURE 2. A binding component of a rational open book where every page wraps twice along the binding component. The same picture appears in a neighbourhood of a radial binding component of a broken book.

Given a contact form λ on M, a rational open book decomposition $(K, \mathcal{F})$ of M *carries* λ if its Reeb vector field R_λ is tangent to the binding K and positively transverse to the interior of the pages. Compared with the standard definition of Giroux [Gir], we do not assume here that the binding is positively tangent to the Reeb vector field with respect to the orientation induced by the pages.

In the previous context, any page S of the open book decomposition is a Birkhoff section for R_λ and the dynamics of R_λ may be studied through the first-return map $f_S : S \to S$ induced by R_λ. In particular, f_S preserves the area-form $d\lambda|_{TS}$ in the interior of S. Moreover, f_S is a *flux-zero* homeomorphism as in Definition 4.3.

2.3. **Degenerate broken book decompositions.** We generalise open book decompositions by allowing another behaviour near the binding, namely *broken* components. While similar, the definition is more general than the one of transverse foliations proposed by Hryniewicz and Salomão [HrS] and than the one of finite energy foliations of Hofer, Wysocki and Zehnder [HWZ2]. Although we mostly consider (nondegenerate) broken book decompositions, we start with a general definition.

Definition 2.4. A *degenerate broken book decomposition* of a closed 3-manifold M is a pair $(K, \mathcal{F})$ such that:

- K is a link, called the *binding*;
- $\mathcal{F}$ is a co-oriented foliation of $M \setminus K$ such that each leaf S of $\mathcal{F}$ is properly embedded in $M \setminus K$ and admits a compactification $\overline{S}$ in M which is a compact surface, called a *page*, whose boundary is contained in K.

The foliation determines the *radial* and *broken* binding components:

- a component k_r of K is radial if $\mathcal{F}$ foliates a neighbourhood of k_r by annuli all having exactly one boundary component on k_r;
- a component k_b of K is broken if in a tubular neighbourhood U of k_b such that the intersection of any leaf with U is a collection of annuli, there are two types of annuli in $\mathcal{F} \cap U$: either one boundary component contains k_b, or both boundary components are in ∂U. In the first case we speak of a *radial leaf*, and in the second of a *hyperbolic leaf*. Containing k_b in the boundary being a closed condition, the set of radial leaves is closed.

For a degenerate broken book, the pages are not assumed to be transversally smooth along the binding.

In a neighbourhood of a radial binding component, a degenerate broken book is similar to a rational open book. If K_b is empty the degenerate broken book decomposition is a rational open book decomposition.

For k_b a broken binding component the local picture is the following: around k_b alternate *radial sectors* where the foliation is made of radial leaves and *hyperbolic sectors* where the foliation looks transversally (that is, when we intersect it with a disc transverse to k_b) like hyperbolas, see Figure 3. Observe that the definition of radial and hyperbolic leaves applies only for the intersection of a leaf with a neighbourhood of k_b.

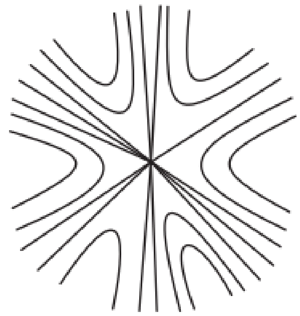

FIGURE 3. A transversal view of a degenerate broken book decomposition near a broken binding component. There are six radial sectors that alternate with six hyperbolic sectors. Note that the radial sector at 2 o'clock consists of one leaf only.

Definition 2.5. A vector field R is *carried* by a degenerate broken book decomposition $(K, \mathcal{F})$ if it is tangent to K and positively transverse to the leaves of $\mathcal{F}$. A contact form λ is *carried* by a degenerate broken book decomposition $(K, \mathcal{F})$ if its Reeb vector field R_λ is carried by $(K, \mathcal{F})$.

By definition, the pages of a degenerate broken book decomposition carrying a contact form λ are R_λ-sections. Thus, roughly speaking, a degenerate broken book decomposition carrying a Reeb vector field can be understood as a foliation by R_λ-sections, whereas a rational open book is a foliation by Birkhoff sections.

If a degenerate broken book decomposition $(K, \mathcal{F})$ carries a contact form λ, its binding K is nonempty, for otherwise the vector field R_λ would be transverse to a compact surface with empty boundary, in contradiction with Stokes' theorem.

Note that, as with rational open books in the previous section, we do not require that the orientation of the binding coming from the co-oriented pages coincides with the orientation given by R_λ, as it is often required in the classical open book case. Actually, there is not even a preferred orientation at a broken binding component since different pages coming to the binding give different orientations depending on the sector they locally stand in.

2.4. **Broken book decompositions.** Since the vector fields we consider in this article are nondegenerate, we can be a bit more restrictive concerning the broken books we consider. First the pages are assumed to be smooth. Next, recall that the periodic orbits of a nondegenerate Reeb vector field R_λ are of three types: elliptic, positive hyperbolic and negative hyperbolic, depending on whether the linearised first-return map on a small disc transverse to the periodic orbit is conjugated to an irrational rotation, has real positive eigenvalues, or has real negative eigenvalues. If a degenerate broken book $(K, \mathcal{F})$ supports a nondegenerate contact form λ, a radial component of K (that is, one in K_r) can be an elliptic or a hyperbolic periodic orbit of R_λ; while a broken component of K (one in K_b) is necessarily a hyperbolic periodic orbit of R_λ. Indeed, if there was an elliptic orbit in K_b, the vector field R_λ would fail to be transverse to $\mathcal{F}$ in the sectors where the leaves of $\mathcal{F}$ are hyperbolic-like.

Moreover, near each point of a broken component of the binding, the foliation has locally four hyperbolic sectors, separated by four radial sectors (as in Figure 4). This is due to the fact that the hyperbolic sectors cannot be transverse to the vector field if they contain no eigendirection of the broken orbit. Thus there cannot be more such sectors than eigendirections, and a given sector may not contain more than one eigendirection.

Definition 2.6. A *broken book decomposition* (or *broken book* for short) is a degenerate broken book decomposition whose pages are smooth and whose broken binding components locally have four hyperbolic sectors, separated by four radial sectors.

Since a broken component of the binding is a hyperbolic periodic orbit, it can be positive or negative. If positive, the monodromy of $\mathcal{F}$ along this orbit is the identity; and if negative the monodromy of $\mathcal{F}$ is a π-rotation, implying that the four local sectors correspond to two global sectors. A hyperbolic periodic orbit has a stable and an unstable manifold. Consider a tubular neighbourhood V of the periodic orbit and the components of the stable/unstable manifolds that contain the orbit in their intersection with V. These decompose V into four or two parts depending on the sign of the periodic orbit. We call these parts the quadrants, since their intersection with a disc transverse to the orbit defines four quadrants.

The definition of a contact form (or a vector field) carried by a degenerate broken book extends to the nondegenerate case. However it is convenient to enforce some transversality on the binding components.

Definition 2.7. A nondegenerate vector field R is ∂-*strongly carried* by a broken book decomposition $(K, \mathcal{F})$ if R is tangent to K and positively transverse to the pages of $\mathcal{F}$, and if every page of $\mathcal{F}$ is a ∂-strong R-section. A nondegenerate contact form λ is ∂-*strongly carried* by a broken book decomposition $(K, \mathcal{F})$ if its Reeb vector field R_λ is ∂-strongly carried by $(K, \mathcal{F})$.

We end this section with some general properties of broken book decompositions. When the broken binding is nonempty, we can distinguish different types of pages as follows.

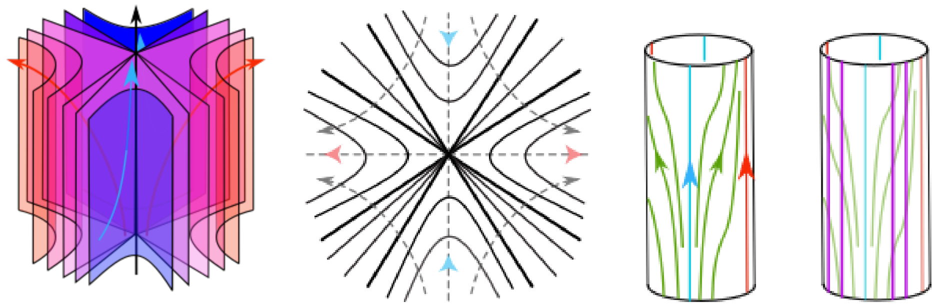

FIGURE 4. A broken book decomposition ∂-strongly carrying a nondegenerate contact form λ in a neighbourhood of a broken binding component. On the left the binding component (black) is shown, as well as some local pages of the broken book. Two unstable orbits (red) of the Reeb vector field R_λ and two stable orbits (blue) are also shown. In the center a transversal view, that is the intersection of the broken book with a disc transverse to R_λ. The rigid pages are bolded. Some orbits of R_λ are represented with green dotted lines. In particular the four local stable/unstable manifolds lie in four different hyperbolic sectors. On the right two pictures of the blow-up of the binding component with the extension of R_λ (first picture) and with the extension of the rigid pages (purple, right-most picture). The extensions of the pages are transverse to the extended vector field, hence the term ∂-strong.

Definition 2.8. A page S of a broken book $(K, \mathcal{F})$ that belongs to the interior of a 1-parameter family of pages of the form $S \times [0,1]/\sim$, where $(y,t) \sim (y,t')$ for every $y \in \partial S$ and $t, t' \in [0,1]$, is called *regular*. On the other hand, a page that is not in the interior of such a 1-parameter family is called *rigid*.

A rigid page must have at least one boundary component in the broken binding near which it locally coincides with an annulus that separates a radial sector from a hyperbolic sector. Since there are finitely many such locally separating annuli, we get that there are only finitely many rigid pages.

In the center of Figure 4 is depicted a neighbourhood of a broken binding component where the rigid pages are bolded: these are the pages that are the limit of a 1-parameter family of radial pages. On the other side, the hyperbolic pages break on them.

For $(K, \mathcal{F})$ a broken book decomposition with nonempty broken binding, in the complement of the set $\mathcal{R}$ of rigid pages the foliation $\mathcal{F}$ is a fibration over $\mathbb{R}$.

At this point, it is not clear to us whether a broken book decomposition supporting a contact form always has nonempty radial binding K_r. We can prove it is the case when every regular page is either a disk or an annulus; this result will be used in the proof of Lemma 4.12.

Lemma 2.9. *A broken book decomposition supporting a contact form λ and whose regular pages are all disks or annuli has nonempty radial binding.*

Proof. We argue by contradiction and assume that the radial binding is empty. We start from a page P_1, which is a disk or an annulus, having one boundary component on a broken binding component h_1. The circle h_1 is a hyperbolic orbit for the Reeb flow R_λ and its stable and unstable manifolds decompose a small neighbourhood of h_1 into quadrants: four quadrants if h_1 is positive hyperbolic and two quadrants if h_1 is negative hyperbolic. Denote these quadrants I, II, III, IV and I, II respectively. Each one of these quadrant contains radial leaves. Assume that near the boundary component of P_1 that we are considering, P_1 is in the quadrant I. We consider a page P_2 that is radial at h_1 and has a boundary component in a quadrant adjacent to the one containing P_1. There are two possibilities: if h_1 is positive, then we have two choices for the quadrant (II or IV) and we pick P_2 in any of them and glue P_2 to P_1 along h_1; if h_1 is negative, there is only one choice of adjacent quadrant (II) for P_2, but there are two ways to glue P_2 to P_1 and we pick any of them. If P_1 and P_2 were both disks, we would get after gluing an immersed sphere (with corner along h_1) that is positively transverse to the Reeb flow away from h_1, and obtain a contradiction using Stokes' theorem. If one of them is an annulus, say P_2, we can consider its other boundary component, going to a binding component h_2. If $h_2 = h_1$ and if P_2 arrives and leaves h_1 in two adjacent quadrants then it gives an immersed torus positively transverse to the Reeb flow (except along h_1), again a contradiction. If otherwise, we take a surface P_3 in the adjacent quadrant to the one locally containing P_2 near h_2.

Iterating this process, since there are only finitely many binding components and quadrants, we have to either end with disks on both ends and find an immersed sphere positively transverse to R_λ away from the binding components, or a cycle of surfaces $P_i, \ldots, P_{i+k}$ forming an immersed torus after gluing and still positively transverse to R_λ away from the binding components: in both cases we obtain a contradiction. $\square$

It was pointed to us by a referee that the conclusion of Lemma 2.9 holds when the pages have arbitrary genus, but at most two boundary components, with the same proof. We do not use this fact, but the statement seems interesting in its own.

3. CONSTRUCTION OF A BROKEN BOOK DECOMPOSITION FROM ECH THEORY

In this section we give a proof of Theorem 1.1 in the case of nondegenerate Reeb vector fields: for every nondegenerate contact form λ on a closed oriented 3-manifold M, we build a broken book decomposition that carries λ.

We first use embedded contact homology, in particular the U-map, to obtain for every point $z \in M$ a pseudo-holomorphic curve in the symplectization $\mathbb{R} \times M$ such that the closure of its projection to M contains z (Lemma 3.1). We then convert each of these projected curves into a ∂-strong R_λ-section (Corollary 3.3). We then prove that there is a finite collection of ∂-strong R_λ-sections, with disjoint interiors, whose union intersects every orbit and such that each connected component of the complement fibers over $\mathbb{R}$ (Lemma 3.6). This allows us to extend the finite collection of ∂-strong R_λ-sections to a foliation in the complement of a link formed by periodic orbits, thus completing a broken book decomposition.

3.1. **Pseudo-holomorphic curves.** In this section, we recall some facts concerning pseudo-holomorphic curves in symplectizations. We refer to Wendl's notes for a detailed introduction [Wen].

Fix a closed oriented 3-manifold M and a contact form λ on M whose associated Reeb vector field R_λ is nondegenerate. The symplectization is the 4-manifold $\mathbb{R} \times M$, equipped with the symplectic form $d(e^s\lambda)$, where s denotes the additional real parameter. In this context a *pseudo-holomorphic curve* is a smooth map $u : F \to \mathbb{R} \times M$ from a Riemann surface (F, j) satisfying $Tu \circ j = J \circ Tu$, where J is an almost-complex structure compatible with the symplectic structure.

We only consider pseudo-holomorphic curves that are *asymptotically cylindrical*. This means that F is a closed compact surface with finitely many punctures which are of two types, called positive and negative. At every positive (*resp.* negative) puncture $p \in F$, the curve u is *asymptotic* to a closed orbit of R_λ: this means that for some choice of local cylindrical coordinates $(s, t) \in \mathbb{R}_+ \times \mathbb{S}^1$ around p and for some periodic orbit γ of R_λ, $u(s, .)$ tends to $\{+\infty\} \times \gamma$ (*resp.* $\{-\infty\} \times \gamma$) in $\mathbb{R} \times M$ as $s \to +\infty$. In this situation, one says that γ is a *positive (resp. negative) end*, and that F *covers* γ. Composing with the projection $\pi : \mathbb{R} \times M \to M$, we see that $\pi \circ u(F)$ is a (possibly singular) surface in M bounded by some periodic orbits of R_λ. More precisely, by the nondegeneracy assumption and a theorem of Hofer, Wysocki and Zehnder [HWZ1, Thm 2.8] (see also [Sie2, Thm 2.2], [HuT2, Prop 3.2] and [Wen, Thm 3.11]), in some local cylindrical coordinates (r, θ, z) adapted to R_λ around an end γ, the projected curve $\pi \circ u(F)$ is exponentially close to a *half-helix* of the form $\theta = f(z)$. The direction $\{\theta = f(z)\}$ is given by an eigenfunction of a nonzero eigenvalue of the *asymptotic operator* associated with γ, see [Wen, Definition 3.5 and Theorem 3.11]. In particular, the asymptotic behaviour implies that the half-helix is ∂-strong along γ (see Definition 2.2).

The local picture of $\pi \circ u(F)$ around that given Reeb periodic orbit is similar to one page of the open book decomposition shown in Figure 2. Note that near an elliptic periodic orbit, there is a well-defined germ of the bounded-time first-return map of the Reeb flow on the corresponding cylindrical ends.

The degree of the covering $\mathbb{S}^1 \to \gamma$ induced by $t \mapsto \pi \circ \lim_{s\to+\infty} u(s,t)$ is called the *local multiplicity* at p. It turns out to be positive at positive ends and negative at negative ends. Note that, after compactifying every puncture of F with a circle, several boundary components of F may be mapped onto the same periodic orbit of R_λ. For a given periodic orbit of R_λ, the sum of the local multiplicities of all positive (*resp.* negative) boundary components that are mapped on it is called the *global positive (resp. negative) multiplicity* of this orbit.

3.2. **Pseudo-holomorphic curves from embedded contact homology.** We now gather the results from embedded contact homology used in the proof of Theorem 1.1. We refer to [Hut1] and [CHP] for a more detailed introduction to embedded contact homology. The context is unchanged: M is an oriented 3-manifold, λ is a nondegenerate contact form, and R_λ is its Reeb vector field.

The ECH-chain complex $ECC(M, \lambda)$ is generated over $\mathbb{Z}_2$ (or $\mathbb{Z}$) by finite sets of simple periodic orbits of R_λ together with multiplicities.

An ECH-holomophic curve between orbit sets Γ and Γ' is an asymptotically cylindrical pseudo-holomorphic curve such that Γ is the limit (with multiplicities) of the positive punctures and Γ' is the limit of the negative punctures. The ECH-index of an ECH-holomorphic curve is an integer, whose definition can be found in Hutchings' notes [Hut1, Def. 3.5]. The ECH-index-1 curves provide the differential for defining the complex $ECH(M, \lambda)$, while the ECH-index-2 curves provide the U-map we are interested in.

The way the ECH-index-1 and 2 curves approach their limit orbits is governed by the *partition conditions* [Hut1, Section 3.9]. These relate the global and local multiplicities of the ends. For an elliptic limit periodic orbit, these conditions are rather cumbersome, but for a positive (*resp.* negative) hyperbolic limit periodic orbit, both its positive and negative global multiplicities are only allowed to be 0 or 1 (*resp.* 0 or 2). This last condition is consistent with the way the ECH-index-1 or 2 curves involved in the definition of the differential or in the U-map break, see [Hut1, Section 5.4]: if a breaking involves a positive hyperbolic periodic orbit with multiplicity strictly larger than 1, then there is an even number of ways to glue and these contributions algebraically cancel, both for $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.

Also, thanks to the case of equality in the writhe bound when the partition conditions are satisfied [Hut1, Lemma 5.1], near a positive hyperbolic limit periodic orbit, every orbit in its stable or unstable manifold does not intersect the corresponding positive or negative cylindrical end, see Figure 5. This nice fact will not be used in the construction; it shows however that one might not expect our construction to provide supporting rational open books on the nose.

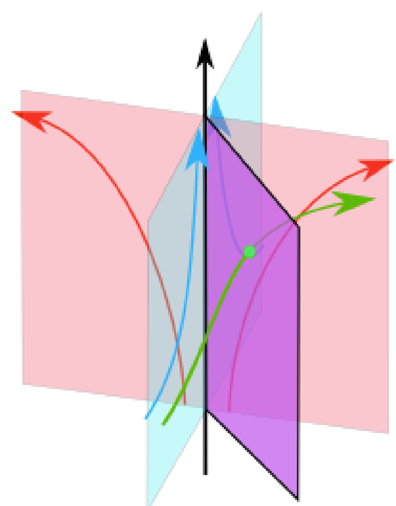

FIGURE 5. The projection in M of an ECH-index 1 or 2 curve in $\mathbb{R} \times M$, near an end which is a positive hyperbolic periodic orbit γ of R_λ. The projection of the curve (purple) does not intersect the stable and unstable manifolds of γ (blue and red), so it lies in one of the four local quadrants. It looks like an annulus bordered by γ. It intersects all the orbits of R_λ (green) in its local quadrant. As suggested by the picture, its extension to the blown-up orbit Σ_γ is transverse to the extension of R_λ (as in Figure 4 right), so it is a ∂-strong R_λ-section. The global positive multiplicity of γ is 1 and its global negative multiplicity is 0. If γ was a negative periodic orbit, the picture would look similar, except there would be a second component of the projection of the curve in the opposite quadrant.

The map $U : ECC(M, \lambda) \to ECC(M, \lambda)$ is a degree -2 map counting pseudo-holomorphic curves passing through a point $(0, z)$ of the symplectization $\mathbb{R} \times M$ of M, where z does not sit on a periodic orbit of R_λ. Now, there exists a finite orbit set $\Gamma = \Sigma_{i=1}^k \Gamma_i$ whose class $[\Gamma]$ in $ECH(M, \lambda)$ is such that $U([\Gamma]) \neq 0$. Even though the map U depends on the choice of z at the chain level, it does not at the homology one and so the fact that $U([\Gamma]) \neq 0$ does not depend on z, see [Hut1, Section 3.8]. At the chain level, $U(\Gamma)$ counts curves from Γ, so that the orbits involved all have action bounded from above by $\mathcal{A}(\Gamma)$ since U is action decreasing. The non-vanishing of the U-map is established via the naturality of the isomorphism between Heegaard Floer homology and embedded contact homology with respect to the U-map [CGH0, CGH1, CGH2, CGH3] and the non-triviality of the U-map in Heegaard Floer homology [OzS, Section 10], or via the isomorphism with Seiberg-Witten Floer homology, as explained in [CHP].

Recall that the *action* $\mathcal{A}(\gamma)$ of an orbit γ of R_λ is the integral $\int_\gamma \lambda$. The action of a collection of orbits is the sum of the actions of its elements, counted with multiplicities. By the nondegeneracy assumption, there are only finitely many periodic orbits of action less than the action $\mathcal{A}(\Gamma)$ of Γ. The *spectrum* of a class $[\Gamma] \in ECH(M, \lambda)$ is the minimal action of an orbit set representing $[\Gamma]$.

The main input from ECH-holomorphic curve theory is the following.

Lemma 3.1. *Let M be a compact 3-manifold and λ a nondegenerate contact form on M. For Γ an ECH-cycle consisting of a sum of finite sets of periodic orbits of R_λ such that $U([\Gamma]) \neq 0$, denote by $\mathcal{P}$ the finite set of periodic orbits of the Reeb vector field R_λ of action less than $\mathcal{A}(\Gamma)$. Then*

- *for every z in $M \setminus \mathcal{P}$, there exists an embedded pseudo-holomorphic curve $u : F \to \mathbb{R} \times M$ asymptotic to periodic orbits of R_λ in $\mathcal{P}$ and such that the projection to M of some point in the interior of u is z;*
- *for every z in $\mathcal{P}$, there is a similar curve such that z is either in the interior of the projection or in a boundary component of its closure. In the latter case, if the periodic orbit γ of R_λ containing z is positive (resp. negative) hyperbolic, the considered pseudo-holomorphic curve has global multiplicity 1 (resp. 2) along γ and it does not intersect locally the stable and unstable manifolds of γ.*

Proof. By definition of the U-map, for every generic $z \in M$, there is an ECH-index 2 embedded curve in $\mathbb{R} \times M$ from $\mathcal{P}$ and passing through $(0, z)$. Now, if z is fixed, it is the limit of a sequence of generic points $(z_n)_{n\in\mathbb{N}}$. Through $(0, z_n)$ passes a pseudo-holomorphic curve u_n whose positive end is in $\mathcal{P}$. By compactness for pseudo-holomorphic curves in the ECH context, there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging to a pseudo-holomorphic building, a component of which is an embedded pseudo-holomorphic curve through $(0, z)$. All the asymptotics of the limit curves are in $\mathcal{P}$, since they all have action less than $\mathcal{A}(\Gamma)$. In particular, when z is in $M \setminus \mathcal{P}$ it is contained in the interior of the projection of the curve to M.

If z is contained in one of the orbits of $\mathcal{P}$, it might be in a limit end of the curve and thus in the boundary of the closure of the projection of the curve to M.

The second item then follows from the fact that positive (*resp.* negative) hyperbolic ends of ECH-index 1 or 2 curves may only have global multiplicity 0 or 1 (*resp.* 0 or 2) and they cannot intersect the stable and unstable manifolds around an end, as discussed previously. □

Note that the compactness argument we refer to in the previous proof includes taking care of possibly unbounded genus and relative homology class, see [Hut1, Sections 3.8 and 5.3].

3.3. **From holomorphic curves to ∂-strong R_λ-sections.** Given a ∂-strong R_λ-section, the local and global multiplicities of the boundary components are defined in the same way as for pseudo-holomorphic curves: if $i : S \to M$ is a R_λ-section and c is a boundary component of S mapped onto a periodic orbit γ of R_λ, the *local multiplicity* of c is the degree of the induced cover $i : c \to \gamma$, and the *global multiplicity* of γ is the sum of the local multiplicities of the boundary components of S that are mapped on γ.

Definition 3.2. Let S be a (not necessarily connected) ∂-strong R_λ-section.

- An orbit γ of R_λ is *asymptotically linking* S if for every $T \in \mathbb{R}$ the arcs $\gamma([T, +\infty))$ and $\gamma((-\infty, T])$ intersect S.
- If γ is a periodic orbit in ∂S, consider its unit normal bundle Σ_γ. The *self-linking* of γ with S is the rotation number of the extension of R_λ to Σ_γ, with respect to the 0-slope given by $\partial_\gamma S$.

The most relevant case for us is when γ is a hyperbolic periodic orbit: in this case the stable and unstable directions around γ locally determine two or four quadrants. If the intersection of S with a tubular neighbourhood of γ remains in one of these quadrants, then the self-linking of S with γ is zero (as in Figure 5). This is the case of the ECH-index-1 or 2 curves given by Lemma 3.1.

We now promote the projected pseudo-holomorphic curves from Lemma 3.1 into R_λ-sections.

Proposition 3.3. *In the context of Lemma 3.1, for every z in M there exists a ∂-strong R_λ-section S with boundary in $\mathcal{P}$ passing through z. Moreover, if z is in $M \setminus \mathcal{P}$, it is contained in the interior of S. Every positive (resp. negative) hyperbolic orbit k in ∂S with self-linking number 0 has local multiplicity* 1 *(resp.* 2*).*

Proof. Consider the point $(0, z)$ in $\mathbb{R} \times M$. Denote by $\hat{S}_0$ the embedded pseudo-holomorphic curve from Lemma 3.1 passing through $(0, z)$. It has a finite number of points where it is tangent to the holomorphic $\langle \partial_s, R_\lambda \rangle$-plane, where s is the extra $\mathbb{R}$-coordinate. Indeed, Siefring proved that close enough to its limit end orbits in $\mathcal{P}$, the pseudo-holomorphic curve is not tangent to this plane field [Sie2, Theorem 2.2] and, by the isolated zero property for holomorphic maps, all these tangency points are isolated.

Denote by S_0 the projection of $\hat{S}_0$ in M. It contains z. The projection of the tangency points corresponds exactly to the points where S_0 is not immersed. We call these points the singular points of S_0 and we denote them by x_i, $i = 1, \dots, p$. Everywhere else S_0 is positively transverse to the Reeb vector field R_λ.

We apply the following desingularisation procedure to S_0, due to Fried for the part away from the singularities [Fri] .

Lemma 3.4. *Let R be a non-zero vector field on a 3-manifold M (here not necessarily Reeb) and S_0 be a compact surface with boundary in M, immersed away from a finite number of interior points, whose boundary is tangent to R and whose interior is positively transverse to R away from its singularities. Assume that, for every boundary orbit $\gamma \subset \partial S_0$, the surface S_0 extends to the unit normal bundle Σ_γ into a collection of immersed curves transverse to the extension of R and in general position. Then there exist $T > 0$ and a ∂-strong R-section S that satisfies $\partial S \subset \partial S_0$ and that intersects all arcs of orbit of the form $\phi_R^{(-T,T)}(x)$ for $x \in S_0$.*

Proof. We modify S_0, first away from its singular points, and then around the singular points. First we put S_0 in general position by a generic perturbation, keeping it transverse to R away from the singular points: we make its self-intersections transverse and its triple self-intersection points isolated.

Since the singular points are disjoint from the boundary components, we surround each singular point x_i, $i = 1, \dots, p$, by a small ball B_i of the form of a flow box $D_i^2 \times [-1, 1]$ that does not intersect ∂S_0, where the $[-1, 1]$-direction is tangent to R, so that the singular point x_i is at the center and the boundary discs $D_i^2 \times \{\pm 1\}$ are disjoint from S_0. Then S_0 only intersects ∂B_i along its vertical boundary $(\partial D_i^2) \times [-1, 1]$.

On $M \setminus (\cup_{i=1}^p B_i)$ the surface $S_0 \setminus (\cup_{i=1}^p B_i)$ is immersed. We first treat the part $S_0 \setminus (\cup_{i=1}^p B_i)$ of S_0. We blow-up the manifold M along the boundary circles ∂S_0 to obtain a compact manifold $M_{\partial S_0}$ bounded by 2-tori. By assumption, S_0 extends to an immersed compact surface still denoted by S_0 on $M_{\partial S_0}$. This extension is transverse to the extension of R along $\partial M_{\partial S_0}$ and only has transverse self-intersections even in $\partial M_{\partial S_0}$.

The surface S_0 has a transversal given by R so that we can resolve its self-intersections coherently to get an embedded surface S_1 in $M_{\partial S_0} \setminus (\cup_{i=1}^p B_i)$, positively transverse to R, as in Figure 6. Each boundary torus of $M_{\partial S_0}$ has well-defined meridians: they correspond to the unit normal bundle of points in ∂S_0. We can then isotope the resolved boundary in $\partial M_{\partial S_0}$, by an isotopy transversal to R, so that it becomes either transverse to the foliation by meridians or equal to a collection of meridian circles (see Figure 7). This isotopy can be extended to an isotopy of $S_0 \setminus (\cup_{i=1}^p B_i)$ transverse to R and supported in a small neighbourhood of $\partial M_{\partial S_0}$.

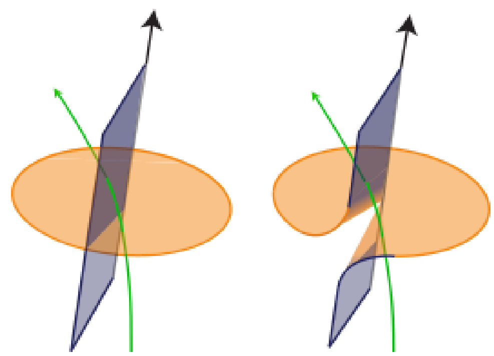

FIGURE 6. How to resolve a line of intersections transversally to the flow.

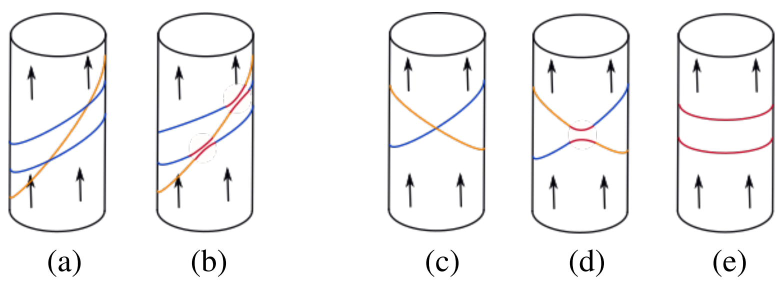

FIGURE 7. The blow-up Σ_γ of a boundary orbit γ in ∂S_0. (a, c) The boundary $\partial_\gamma S_0$ is a collection of curves transverse to R on Σ_γ. (b, d) One resolves the double points of $\partial_\gamma S_0$ transversally to R. (b) If the obtained curves are transverse to the meridians, one stops here. (e) If the obtained curves are isotopic to meridians, one makes an isotopy of S_1 and $\partial_\gamma S_1$ that turn them into meridians.

Once this is done, the isotoped surface gives an immersed surface S_1 in $M \setminus (\cup_{i=1}^p B_i)$ after blow-down. In case the intersection of the isotoped surface with a boundary component Σ_γ of $\partial M_{\partial S_0}$ is along a meridian circle, the blow-down surface does not have γ as a boundary component any more and crosses γ transversally. The other components are ∂-strong.

The resolution of the self-intersections, described above, along a line of double-points ending in a boundary component is pictured in Figure 6: before and after the resolution. The resolution of triple points of intersection, coming generically from the transverse intersections of two branches of double points, are not an issue, since we can locally resolve one branch after another in any order and extend this resolution away, see Figure 8. Similarly, there might be lines of double points ending at the boundary of the balls B_i surrounding singular points. We resolve them outside, up to the boundary of the balls.

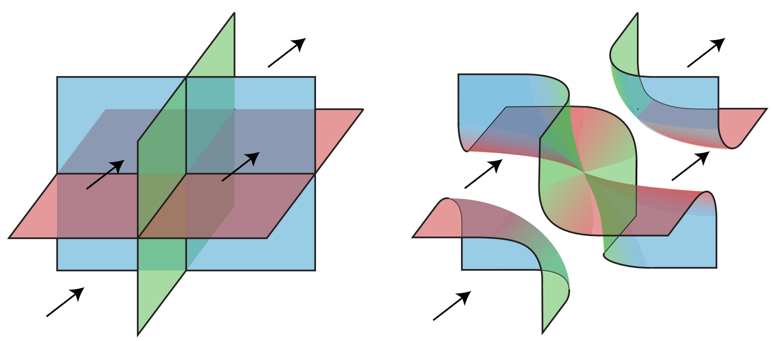

FIGURE 8. How to resolve a triple point of self-intersections. One other way to picture what happens is to first resolve the intersection of the union of two surfaces and then add and resolve the intersections with a third one.

We are left with the part inside the balls B_i. The new surface S_1 now intersects every sphere ∂B_i along an embedded collection of circles contained in the vertical part $(\partial D^2) \times [-1, 1]$ and transverse to R, i.e. the $[-1, 1]$, direction. We can extend S_1 inside the balls B_i by an embedded collection of disks transverse to R. We get a surface S which is a ∂-strong R-section.

Finally, note that all transformations can be made locally, except the isotopy of the resolved boundary in $\partial M_{\partial S_0}$. For the latter, we can move the surface along the flow for a time at most the period of the boundary orbit. So taking for T the maximal period of an orbit in ∂S_0 yields the last part of the statement. □

In order to finish the proof of Proposition 3.3, we apply Lemma 3.4 to S_0 and R_λ to get a surface S and remark that it is easy to perform these surgery operations and keep the constraint of passing through the point z. Since the embedded parts of S_0 are ∂-strong, as the projection of an ECH curve, it satisfies the hypothesis of Lemma 3.4. Then S can be taken to be ∂-strong .

Finally, along every positive or negative hyperbolic periodic orbit of ∂S having self-linking number 0 with S (see Definition 3.2), the surface S is made of longitudinal annuli that do not intersect the stable and unstable directions. Since these annuli are embedded, their coordinates in a (longitude-meridian) basis are primitive, hence they can only have local degree 1 if the orbit is positive hyperbolic or 2 if the orbit is negative hyperbolic. □

Observe that the surface S constructed above might have several connected components, but each connected component is a ∂-strong R_λ-section with boundary, because a closed surface cannot be transverse to a Reeb vector field.

Recall that $\mathcal{P}$ denotes the set of those periodic orbits of R_λ whose action is smaller than $\mathcal{A}(\Gamma)$. We now analyse what happens around the orbits of $\mathcal{P}$.

Lemma 3.5. *In the context of Lemma 3.1, let k be a periodic orbit of R_λ that belongs to $\mathcal{P}$. Then one of the following holds:*

(1) *there exists a ∂-strong R_λ-section transverse to k;*
(2) *there exists a ∂-strong R_λ-section containing k in its boundary and whose self-linking with k is non-zero;*
(3) *there exists a ∂-strong R_λ-section containing k in its boundary and whose self-linking with k is zero. In this case k is hyperbolic and each one of the quadrants transversally delimited by the stable and unstable manifolds of k is intersected by at least one ∂-strong R_λ-section having k as a boundary component.*

Proof. Let z be a point of k. Consider a small disc D containing z and transverse to k, and take a sequence z_n of generic points in $D \setminus \{z\}$ converging to z. In $\mathbb{R} \times M$, take a sequence u_n of ECH-index 2 embedded pseudo-holomorphic curves through $(0, z_n)$. In the limit pseudo-holomorphic building, there is a pseudo-holomorphic curve containing $(0, z)$, whose projection S_0 to M contains z. Then at least one of the following holds:

(a) S_0 is transverse to k;
(b) S_0 contains k in its boundary and its self-linking with k is non-zero (as we will see later, this is the case when k is an elliptic boundary orbit, but the partition conditions explained in Section 3.2 imply that it cannot happen if k is a positive hyperbolic boundary orbit);
(c) S_0 contains k in its boundary and its self-linking with k is zero.

Observe that these are not disjoint cases: if S_0 is not embedded, a periodic orbit can be in the boundary of S_0 and intersect transversely S_0 in its interior.

Applying the resolution procedure of Lemma 3.4 to a surface S_0 satisfying at least one of the first two cases, we obtain a ∂-strong R_λ-section S. Then either S is transverse to k or ∂S contains k. In the latter case the resolution procedure can change the self-linking number along k. If k satisfies (a) and (c) as boundary orbit of S_0, then k will have non-zero self-linking number with respect to S. If k satisfies (b) alone, or (a) and (b), then the self-linking number of k with respect to S will still be non-zero.

Now assume that $k \in \partial S_0$ satisfies (c) only. The limit curves satisfy the asymptotic conditions of [HWZ1, Theorem 2.8], [Sie1] [Sie2, Theorem 2.2], [HuT2, Proposition 3.2], [Wen, Theorem 3.11]: the order 1 expansion of the planar coordinate $z(s,t)$ (transverse to k in M) says that along the boundary a limit curve is tangent (at the first order) to an annular half-helix. Then k cannot be elliptic: in that case the self-linking of S with k would be irrational, which is absurd. Hence k is hyperbolic and its tubular neighbourhood has two or four quadrants delimited by its stable and unstable manifolds. From now on we assume that the points z_n all belong to the same quadrant while converging to z.

Now the transversality to the Reeb vector field implies that the half-helix has to stay in one quadrant. Indeed if it changes quadrant, in order to stay transverse to R_λ, it can only change in one direction. So, in order to come back to the original quadrant after following k for one longitude, it has to wrap around k: in this case the self-linking number would be positive, a contradiction.

There are two cases depending on whether k is at an intermediate level of the limit building with non-trivial positive and negative ends, or there is only a non-trivial positive and negative end (possibly followed by connectors).

In the first case, for the limit building, the positive and negative ends at k both have to follow a half-helix contained in a quadrant of k. If none of these quadrants is the one containing the sequence z_n, then, close to the breaking curve, the curve has to enter and exit the quadrant containing z_n. We then look at its intersection with the stable and unstable manifolds delimiting the quadrant. This is a collection of circles that have to all be positively transverse to the Reeb vector field foliating the stable/unstable manifolds. In particular, none of the circles is contractible in the stable/unstable manifolds. The non-contractible circles have a fixed co-orientation given by the Reeb vector field: when the curve enters a quadrant along a stable/unstable manifold, it cannot exit along the same stable/unstable manifold nor along the other unstable/stable manifold. So this situation cannot happen: the limit curve has to be in the quadrant containing z_n near one end.

In the second case, the limit building has only one non-trivial end on k (though there can be connectors) defining a half-helix (which is more a half-annulus) . The sequence of pseudo-holomorphic curves u_n is converging to this half-helix and k must be an end of the u_n's. Now we have by the same argument as in the first case that z_n has to be either on the same quadrant than the half-helix of u_n or in the same quadrant than the one of the limit curve, otherwise we have a subsurface near the end of u_n for n large enough that is crossing a quadrant.

In both cases, we see that the limit building has a component with an end approaching k by an annulus in the corresponding quadrant. □

We can now assemble all the ∂-strong R_λ-sections together, by possibly changing some of them.

Lemma 3.6. *In the context of Lemma 3.1, there exists a finite number of ∂-strong R_λ-sections with disjoint interiors, intersecting all orbits of R_λ.*

If an orbit of R_λ is not asymptotically linking this collection of sections, it converges to one of their boundary components, which is a hyperbolic periodic orbit k with local multiplicity 1 or 2. In this case, each one of the quadrants transversally delimited by the stable and unstable manifolds of k is intersected by at least one ∂-strong R_λ-section having the orbit as a boundary component.

Proof. Start with a finite cover by flow-boxes of the complement of a small enough open neighbourhood of $\mathcal{P}$. Through every point in a flow-box, Corollary 3.3 provides an embedded ∂-strong R_λ-section with boundary in the orbits of $\mathcal{P}$. Since the closure of every flow-box is compact, there is a finite collection of surfaces intersecting every portion of orbit in the flow-box. Using the desingularisation process described in Lemma 3.4, we can resolve the intersections between the R_λ-sections to obtain a finite collection of disjoint ∂-strong R_λ-sections $S_1, \dots, S_n$ that intersect all the orbits in the complement of an open neighbourhood of $\mathcal{P}$ and whose boundary is contained in $\mathcal{P}$.

We now analyse what happens in the neighbourhood of $\mathcal{P}$. Let k be an orbit in $\mathcal{P}$. If at least one of the surfaces $S_1, \dots, S_n$ is transverse to k, then by disjointness all are transverse to k (or do not even intersect k), in which case there is nothing to do.

In the case where none of $S_1, \dots, S_n$ is transverse to k, we apply Lemma 3.5 to k. In the case of Lemma 3.5 (1), we add the ∂-strong R_λ-section S given by Lemma 3.5 to the collection $S_1, \dots, S_n$ and we resolve once more the intersections of $S \cup S_1 \cup \dots \cup S_n$ using the common transverse direction R_λ. We obtain a new collection of ∂-strong R_λ-sections with disjoint interiors intersecting k. The case of Lemma 3.5 (2) is similar and we add and resolve again the section S with $S_1 \cup \dots \cup S_n$; k is in the boundary of the new section which intersects every orbit nearby k.

The remaining case, Lemma 3.5 (3), arises when k is hyperbolic and the ∂-strong R_λ-sections given by Lemma 3.5 have self-linking equal to zero. Then the stable and unstable manifolds of the periodic orbit k transversally delimit four quadrants in a neighbourhood of a point in the orbit, and Lemma 3.5

yields four surfaces (two if k has negative eigenvalues) that cover all the quadrants around k. Again, we add these new ∂-strong R_λ-sections to our previous collection of ∂-strong R_λ-sections and then resolve the intersections of this new family using the common transverse direction R_λ. □

Finally we have a ∂-strong R_λ-section S (possibly disconnected), so that every orbit of R_λ is either a boundary component or intersects S transversely. The union K of the boundary orbits of S is a subset of $\mathcal{P}$. If every orbit of K is asymptotically linking S, we get a rational open book. However, we can have boundary components where some orbits of R_λ accumulate without intersecting the corresponding surface. These boundary components are necessarily hyperbolic periodic orbits and they all have local multiplicity one or two. We have now all the elements to prove Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.6 gives an ∂-strong R_λ-section S intersecting every orbit of the Reeb flow R_λ, and we want to turn it into a broken book decomposition. Said differently, the ∂-strong R_λ-section S forms a trivial lamination of $M \setminus K$, and we have to extend S into a foliation of $M \setminus K$.

For convenience, we first double all the components of S who have at least one boundary component on a hyperbolic periodic orbit and are not asymptotically linking the orbits in their stable/unstable manifolds. The two copies are separated in their interior by pushing along the flow of R_λ. We keep the notation S for this new ∂-strong R_λ-section. We then cut M along S and delete standard Morse type neighbourhoods of the hyperbolic orbits with self-linking number equal to zero of ∂S as in Figure 9. We denote by M_0 the resulting manifold.

We claim that M_0 is a sutured manifold, foliated by compact R_λ-intervals. Indeed, observe that when R_λ is asymptotically linking S near an orbit k of ∂S, the flow of R_λ near k has a well-defined first-return map on S. These orbits are then decomposed by S into compact segments. When we are near a positive hyperbolic periodic orbit k_b where the flow is not asymptotically linking with S, then S intersects a Morse type tubular neighbourhood of k_b in at least 8 annuli, at least two in each quadrant (because of the doubling operation). Between two annuli in the same quadrant, the orbits of R_λ are locally going from one annulus to the other, thus an orbit is decomposed into compact intervals. If two consecutive annuli belong to different adjacent quadrants, then they are co-oriented by R_λ and can be pushed in the direction of the invariant manifold of k_b separating them and glued to form an annulus transverse to R_λ. Again every local orbit of R_λ ends or starts in finite time on some (possibly glued) annulus. Thus M_0 is an I-bundle with oriented fibers, hence a product, and the conclusion follows.

To finish the proof, we just have to glue back the Morse-type neighbourhood of the hyperbolic orbits with self-linking number equal to 0 of ∂S

to M_0, that we foliate with the local model of a broken book, as on the bottom of Figure 9. The construction near a negative hyperbolic periodic orbits is analogous. □

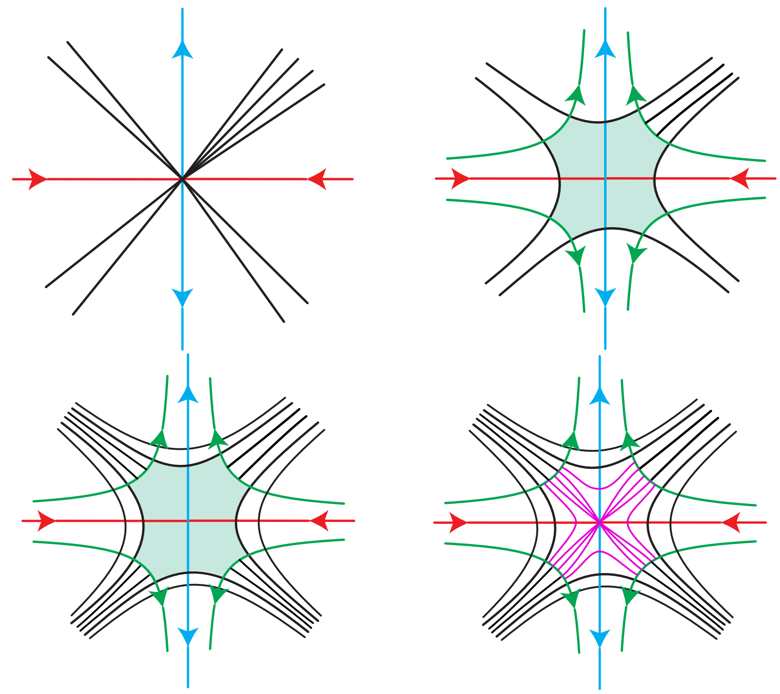

FIGURE 9. A transversal view of a broken component k_b of the binding. On the top left, the ∂-strong R_λ-section S around k_b after the doubling procedure is shown (black). It is made of an even number of annuli—the point being to have at least 2—in each quadrant. For building the foliation of $M \setminus K$, one reconnects the first and last part of S in each quadrant in the neighbourhood of k_b, removes a smaller neighbourhood (green), and cuts the resulting manifold along the (modified) S. The result is a trivial I-bundle (top right). From the foliation of the trivial I-bundle (bottom left), one adds the neighbourhood of k_b back, foliated with the local model of a broken binding orbit (bottom right).

Remark 3.7. If R_λ has a Birkhoff section S, then every embedded projected pseudo-holomorphic curve P gives rise to a new Birkhoff section for R_λ by the constructions above. Indeed, the resolution procedure described before applies to $S \cup P$. The new binding corresponds to ∂S together with the asymptotic orbits of the holomorphic curve in ∂P. Regardless whether these asymptotic orbits intersect S in its interior or coincide with an original orbit in ∂S, the resolution procedure provides that the self-linking numbers all become strictly positive after resolving the intersections of P and S (and all orbits in ∂S keep a strictly positive self-linking number). In particular, the abundance of embedded projected pseudo-holomorphic curves given by the non-triviality of the U-map in embedded contact homology, or the differential, typically furnishes many Birkhoff sections for the same Reeb vector field.

4. Applications

The purpose of this section is to use broken book decompositions to study the dynamics of nondegenerate Reeb vector fields. The proofs are always divided into two cases: when the broken book decomposition is a rational open book decomposition ($K_b = \emptyset$), in which case one studies the diffeomorphism given by the first-return map on a page; or when the broken binding is nonempty ($K_b \neq \emptyset$), in which case the stable and unstable manifolds of the broken binding components provide enough information to prove Theorems 1.2 and 1.3.

In Section 4.3 we prove that these invariant manifolds always intersect. The case in which the intersections are transverse is simpler and the proof of Theorem 1.2 in this situation is explained in Section 4.4. The proof without the transversality assumption requires considering several cases and destroying one-sided intersections. This proof is given in Section 4.5, the results there provide also a proof for Theorem 1.3. In Section 4.6 we discuss how to change a broken book decomposition with exactly one broken component into a rational open book decomposition supporting the same Reeb vector field, proving Theorem 1.4. Finally in Section 4.7, we discuss the extension of Theorems 1.2, 1.3 and 1.4 to an open neighbourhood of the set of nondegenerate vector fields or strongly nondegenerate vector fields, accordingly.

4.1. **Flows on 3-manifolds: entropy and horseshoes.** In this section we consider a general non-singular vector field on a closed 3-manifold and recall certain aspects of the dynamics that we use in the proofs of Section 4.

Let $\phi^t : M \to M$ be a flow on a compact 3-manifold. The flow needs to be C^2 for most of the results below. The topological entropy $h_{top}(\phi^t)$ is a non-negative number that measures the complexity of the flow. We review a definition, originally introduced by Bowen [Bow]. We first endow M with a metric d. Given $T > 0$ we define

$$d_T(x, y) = \max_{t \in [0,T]} d(\phi^t(x), \phi^t(y)),$$

for any pair of points $x, y \in M$. A subset $S \subset M$ is (T,ϵ)-separated if for all $x \neq y$ in S we have that $d_T(x,y) > \epsilon$. Let $N(T,\epsilon)$ be the maximal cardinality of a (T,ϵ)-separated subset of M. The topological entropy is defined as

$$h_{top}(\phi^t) = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log(N(T,\epsilon)).$$

A Smale horseshoe is a compact invariant set of the flow, in which the dynamics is semi-conjugate to the suspension of a shift of finite type, that is the prototypical example of chaotic dynamics, see for example Section 1.9 of the book [KaH]. We ask further that the map realising the semi-conjugation is continuous with respect to the metric d in M and the natural metric in the shift-space. There is a strong relation between positive entropy and horseshoes (see for example [LiS] for the case of flows or the original work of Katok [Kat] for diffeomorphisms):

Theorem 4.1. *Let ϕ^t be a C^2-flow on a closed 3-manifold generated by a non-vanishing vector field. Then $h_{top}(\phi^t) > 0$ holds if and only if there exists a Smale horseshoe subsystem of the flow.*

As a consequence, if the topological entropy is positive the number of hyperbolic periodic orbits grows exponentially with respect to the period (see Theorem 1.1 in [LiS]). A Smale horseshoe persists under small perturbations, hence having positive entropy is an open condition for 3-dimensional flows.

A Smale horseshoe can be obtained from a transverse homoclinic intersection, see for example Theorem 6.5.5 in [KaH]. Recall that a *homoclinic* orbit lies in the intersection of the stable and unstable manifolds of the same hyperbolic periodic orbit, while a *heteroclinic* orbit is an orbit that lies in the intersection of a stable manifold of a hyperbolic periodic orbit and an unstable manifold of another hyperbolic periodic orbit. From the above discussion one can conclude that having a transverse homoclinic intersection implies that the topological entropy of the flow is positive and that there are infinitely many periodic orbits.

We use the term *homo/heteroclinic orbit* for an orbit that is either homoclinic or heteroclinic.

Definition 4.2. Consider a flow ϕ^t and two hyperbolic periodic orbits (not necessarily distinct) with a homo/heteroclinic orbit connecting them.

(1) The orbits have a *homo/heteroclinic connection* if the corresponding stable and unstable manifolds coincide.
(2) The orbits have a *crossing homo/heteroclinic intersection* if the stable and unstable manifolds cross topologically, where crossing is in the topological sense of Burns and Weiss [BuW]. It means that there exists an embedded disk D in M transverse to the flow such that the

intersection of D with the stable and unstable manifolds contains two arcs γ_{st} and γ_{un} with boundaries in ∂D and whose endpoints alternate in ∂D (in particular they have to intersect and "cross").

(3) The orbits have a *one-sided homo/heteroclinic intersection*, or orbit, if the stable and unstable manifolds intersect and do not cross.

Note that these definitions include the case where the stable and unstable manifolds intersect along an interval transverse to the flow and either cross or stay on one side at the boundary components.

The transversality condition to obtain a Smale horseshoe can be weakened. A heteroclinic cycle is a sequence of hyperbolic periodic orbits $k_0, k_1, \ldots, k_{n-1}, k_n = k_0$ such that the unstable manifold of k_i intersects the stable manifold of k_{i+1} for every $0 \leq i \leq n-1$. In dimension 3, a heteroclinic cycle such that the intersection between the unstable manifold of k_0 and the stable manifold of k_1 is a crossing intersection and all other intersections are one-sided suffices to have positive topological entropy, see [BuW, Theorem 2.2].

4.2. **Reeb vector fields supported by rational open books.** In this section, we consider the case in which a nondegenerate Reeb vector field R_λ is supported by a rational open book decomposition. We review known conclusions on the dynamics. In particular we prove, at the end of the section, that such a vector field has two or infinitely many periodic orbits, see Corollary 4.8.

Let (M, λ) be a contact closed 3-manifold such that the Reeb vector field R_λ is supported by a rational open book decomposition. Recall that this consists of a finite collection K of periodic orbits called binding orbits, and a foliation of $M \setminus K$ by Birkhoff sections that radially foliate a neighbourhood of K. These Birkhoff sections are called the pages.

Taking such a page S of the open book, there is a well-defined homeomorphism $h : S \to S$ given by the first-return map along the flow of R_λ. Moreover, the differential 2-form $d\lambda$ defines an exact area form on S that is invariant under h. The map h satisfies a stronger condition than being area-preserving. Consider the induced map

$$h_* - I : H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z}).$$

Definition 4.3. A homeomorphism $h : S \to S$ has *flux-zero* if for every curve γ whose homology class is in $\ker(h_* - I)$, we have that γ and $h(\gamma)$ cobound a $d\lambda$-area zero 2-chain.

Observe that this definition is valid for any exact area form. If h is given by the first-return map of a Reeb vector field, it is clear that it is a flux-zero homeomorphism: take a curve γ whose homology class is in $\ker(h_* - I)$, then γ and $h(\gamma)$ bound a 2-chain in S and also bound a surface tangent to the Reeb vector field in the ambient manifold M. The integral of $d\lambda$ on the union of the 2-chain and the surface has to be equal to zero by Stokes' theorem and is equal to the integral of $d\lambda$ on the 2-chain.

The relation between flux-zero homeomorphisms and Reeb vector fields was studied in the paper [CHL]. Given a surface S, possibly with boundary, and a homeomorphism $h : S \to S$, we can consider the mapping torus $\Sigma(S,h) = S \times [0,1]/(x,0) \sim (h(x),1)$ that is a 3-manifold with boundary if $\partial S \neq \emptyset$. The vector field ∂_t, where t is the $[0,1]$-coordinate, has as first-return map on $S \times \{0\}$ the map h. Observe that all the orbits of this flow are non-contractible, by construction.

The relation is given by Giroux's lemma [CHL, Lemma 2.3] when h is the identity near ∂S:

Lemma 4.4. *Let $(S, d\lambda)$ be a compact oriented surface endowed with an exact area form and $h : S \to S$ a flux-zero homeomorphism that is the identity near ∂S. Then there exists a contact form α on $\Sigma(S,h)$ whose Reeb vector field R_α corresponds to ∂_t and has h as first-return map on $S \times \{0\}$.*

We now recall the Nielsen-Thurston classification [Thu].

Theorem 4.5. *If S is a compact surface, possibly with boundary, and $h : S \to S$ a homeomorphism, then there is a map $h_0 : S \to S$ isotopic to h (not relative to the boundary) such that at least one of the following holds:*

- *h_0 is periodic, meaning that some power of h_0 is the identity map;*
- *h_0 preserves some finite union of disjoint simple closed curves on S, in this case h_0 is called reducible;*
- *h_0 is pseudo-Anosov.*

The three types are not mutually exclusive, but a pseudo-Anosov homeomorphism is neither periodic nor reducible.

When h_0 is reducible, one can cut S along a finite collection of simple closed curves to obtain a homeomorphism of a smaller surface (possibly disconnected) and apply Theorem 4.5 to each of the pieces. Iterating this process, one obtains a map isotopic to h that breaks up into periodic pieces and pseudo-Anosov pieces (see for example Section 13.3 in [FaM]). By abuse of notation, we call this final map h_0.

In terms of the dynamics of the original map h, if h_0 has a pseudo-Anosov piece, then the map h has positive topological entropy and infinitely many periodic orbits. We now use this classification to prove the following statement.

While writing this paper, we learned about a more direct and general proof (for homeomorphisms) by Le Calvez [Lec2].

Theorem 4.6. *Let S be a compact surface with boundary different from the disk or the annulus and $\omega = d\beta$ an exact area form on S. If $h : S \to S$ is an area-preserving diffeomorphism with zero-flux then h has infinitely many different periodic points.*

Proof. By Giroux's Lemma 4.4, the zero-flux condition implies that h can be realised as the first-return map of the flow of a Reeb vector field on $\Sigma(S, h)$. Consider the Nielsen-Thurston decomposition of h. If it has a pseudo-Anosov component, then the conclusion of the theorem classically holds by Nielsen-Thurston theory, even without any conservative hypothesis. Otherwise, all the pieces of h in the decomposition are periodic and there is a power h^n of h that is isotopic to the identity on each piece. This means that h^n is isotopic to a composition of Dehn twists on disjoint curves. Note that it is sufficient to show that h^n has infinitely many periodic points.

If h^n is isotopic to the identity, the conclusion is given by a theorem of Franks and Handel [FrH], extended by Cristofaro-Gardiner, Hutchings and Pomerleano to fit exactly our case with boundary [CHP, Proposition 5.1].

If h^n is not isotopic to the identity, consider the Nielsen-Thurston representative h_0 of h^n given by a product of Dehn twists along disjoint annuli and at least one annulus A is not boundary parallel. We now realise h_0 as the first-return map of a Reeb vector field R_0 on S in $\Sigma(S, h^n)$. This flow has no contractible periodic orbits since it is transverse to the fibration over $\mathbb{S}^1$ and thus can be used to compute cylindrical contact homology. In the mapping torus of A, we have $\mathbb{S}^1$-families of periodic Reeb orbits realising infinitely many slopes in the suspended thickened torus. A generic perturbation transforms an $\mathbb{S}^1$-family into an elliptic and a positive hyperbolic periodic orbit. These all give generators for the cylindrical contact homology, since the other orbits (corresponding to periodic points of h_0) belong to different Nielsen classes. The invariance of cylindrical contact homology, see [BaH, HuN], suffices to conclude that the Reeb orbits given by the mapping torus of h^n are at least the number given by the rank of the cylindrical contact homology computed with R_0, i.e. an infinite number. $\square$

We obtain two corollaries of Theorem 4.6 for Reeb vector fields.

Corollary 4.7. *Let R_λ be a Reeb vector field on a closed 3-manifold M that is ∂-strongly supported by a rational open book decomposition. Let S be a page and h the first-return map to S. Assume that the Nielsen-Thurston decomposition of h has a pseudo-Anosov component. Then R_λ has positive entropy and infinitely many periodic orbits.*

Corollary 4.8. *Let R_λ be a Reeb vector field on a closed 3-manifold M that is ∂-strongly supported by a rational open book decomposition. Then R_λ has two or infinitely many periodic orbits.*

Proof. The ∂-strong hypothesis implies the existence of the first return map of the Reeb vector field on the interior of a page S, which extends continuously to its boundary ∂S.

If S is a disk or an annulus the conclusion follows by Proposition 5.1 of [CHP]: there are either 2 or infinitely many periodic orbits.

In all other cases, observe that the first-return map on the page S might not extend smoothly to ∂S. In that case, we filter the cylindrical contact homology complex by the intersection number of orbits with the page, that is equal to the period of the corresponding periodic points of h. We can then modify the map h near ∂S to a zero-flux area-preserving diffeomorphism h_k so that (1) the modified monodromy h_k extends smoothly to ∂S, (2) the orbits of period less than or equal to k in the Nielsen classes not parallel to the boundary are not affected and (3) h_k is isotopic to h and so has the same Nielsen-Thurston representative h_0. The arguments developed in the previous proof then apply to show the existence of periodic points of h_k with period bounded above by k. These are also periodic points of h with period bounded by k. Hence, h has infinitely many periodic orbits. □

4.3. **Heteroclinic and homoclinic intersections.** In this section we study the hyperbolic orbits in the broken components of the binding of a supporting broken book, combined with the Reeb property of the flow. After completing this work, we realised that some arguments in this section were also used in [HWZ2].

For $k \in K_b$, a *branch of the unstable manifold of* k or *unstable branch*, denoted by $V^u(k)$, is a connected component of the unstable manifold of k cut along k. It coincides with the unstable manifold minus k when k is negative hyperbolic, but it consists of half of it when k is positive hyperbolic. In the same way we use the notation $V^s(k)$ for a branch of the stable manifold of k.

Lemma 4.9. *Let R_λ be a Reeb vector field for a contact form λ carried by a broken book decomposition $(K, \mathcal{F})$. Then, for every $k_0 \in K_b$, every unstable branch $V^u(k_0)$ (*resp.* stable branch $V^s(k_0)$) contains a homo/heteroclinic orbit asymptotic at $+\infty$, resp. $-\infty$, to a broken component of K_b.*

Proof. An unstable branch $V^u(k_0)$ is made of an $\mathbb{S}^1$-family of orbits of R_λ, asymptotic to k_0 at $-\infty$. Each of these $\mathbb{S}^1$-family of orbits is a cylinder in M with a boundary component in k_0, that is injectively immersed in M since its portion near k_0 for time $t < -T$, with T large enough, is embedded.

We now argue by contradiction. Consider the finite collection of all the rigid pages $\mathcal{R} = \{S_0, \dots, S_k\}$ of the broken book decomposition (see Definition 2.8). If no orbit in $V^u(k_0)$ limits to a broken component of K_b at $+\infty$, then $V^u(k_0)$ has a return map on $\mathcal{R}$ which is well-defined. Then all the connected components of $V^u(k_0) \cap \mathcal{R}$ are compact and there are infinitely many of them, hence there is a rigid page, say S_0, such that $V^u(k_0) \cap S_0$ has infinitely many connected components. Since $V^u(k_0)$ is an injectively immersed cylinder, the intersection with S_0 forms an infinite embedded collection C_0 of closed curves in S_0.

Observe that $d\lambda$ is an area form on S_0. We claim that only a finite number of the curves in C_0 can be contractible in S_0. Two contractible components of C_0 bound disks D and D' in S_0, these disks have the same $d\lambda$-area. Indeed D and D' can be completed by an annular piece $A \subset V^u(k_0)$ tangent to R_λ to form a sphere. Applying Stokes' theorem we obtain

$$0 = \int_{D \cup A \cup D'} d\lambda = \int_D d\lambda + \int_A d\lambda - \int_{D'} d\lambda = \int_D d\lambda - \int_{D'} d\lambda, \tag{1}$$

because $d\lambda$ vanishes along A. Note that Equation (1) also implies that D is disjoint from D', since ∂D is disjoint from $\partial D'$. Since the total area of S_0 is bounded, there are only finitely many contractible curves in C_0, as we wanted to prove.

Thus infinitely many components of C_0 are not contractible in S_0, so at least two have to cobound an annulus A' in S_0. The annulus A' is transverse to R_λ and its boundary components cobound by construction an annulus $A'' \subset V^u(k_0)$. We now apply Stokes' theorem to the torus $A \cup A''$ and get

$$0 = \int_{A' \cup A''} d\lambda = \int_{A'} d\lambda > 0,$$

a contradiction.

Hence each unstable (*resp.* stable) branch of k_0 contains an orbit that is forward (*resp.* backward) asymptotic to a component of K_b. □

Remark that in Equation (1) the $d\lambda$-area of the disc D is equal to the action of the periodic orbit k_0.

We point out a consequence of the previous proof.

Corollary 4.10. *Let $(K, \mathcal{F})$ be a broken book decomposition with $K_b \neq \emptyset$ and R_λ a Reeb vector field supported by $(K, \mathcal{F})$. Then for every $k \in K_b$ and for every rigid page S, there is a natural number $n(k, S)$, depending on the action of k, the $d\lambda$-area and the topology of the page S, that bounds the number of closed curves in the intersection of the stable/unstable branches of k with S.*

Similarly, every embedded cylinder tangent to the flow and with one boundary component of action $\mathcal{A} > 0$ intersects the union of the rigid pages $\mathcal{R}$ in less than $n(\mathcal{A}, \mathcal{R})$ of circles.

Recall that a heteroclinic orbit from k_0 to k_1 is an orbit contained in the unstable manifold of k_0 and in the stable manifold of k_1, that a heteroclinic chain based at k_0 is a sequence of broken components $(k_0, k_1, \dots, k_{n-1}, k_n)$ so that there is a heteroclinic orbit γ_i from k_i to k_{i+1} for every $0 \leq i \leq n-1$, and that a homo/heteroclinic cycle based at k_0 is either a heteroclinic cycle based at k_0 or the cycle (k_0, k_0) if there is a homoclinic orbit from k_0 to k_0.

Lemma 4.11. *There exists $k_0 \in K_b$ such that*

(a) *either k_0 is positive hyperbolic and it has two homo/heteroclinic cycles $(k_0, k_1, \dots, , k_{n-1}, k_n = k_0)$ and $(k_0, k'_1, \dots, , k'_{l-1}, k'_l = k_0)$ based at k_0 so that the orbits γ_0 and γ'_0 are contained in different unstable branches of k_0,*

(b) *or k_0 is negative hyperbolic and it has one homo/heteroclinic cycle based at k_0.*

Proof. The argument is of graph-theoretical nature. Consider the directed graph G_b whose vertices are the broken components of the binding and such that there is one edge from k_0 to k_1 if there is a heteroclinic orbit between them. In particular G_b may have loops (if there are homoclinic orbits) and double edges (if both unstable branches of k_0 intersect one or both stable branches of k_1).

Recall that a strongly connected component of a directed graph is a subgraph where any two vertices may be connected by an oriented path and which is maximal with respect to this property. The quotient of the graph by the strongly connected components being acyclic and finite, there is a strongly connected component in G_b with no outgoing edge. We consider such a strongly connected component which is a sink, and denote it by G'_b. Let k_0 be an orbit in G'_b.

If k_0 is a negative hyperbolic orbit, by Lemma 4.9, k_0 has at least one outgoing edge. Denote by $k_1 \in G'_b$ the end of this edge. If $k_1 = k_0$ we are done. Otherwise, by strong connectivity there is a path in G'_b from k_1 to k_0. Concatenating the edge (k_0, k_1) with this path yields a heteroclinic cycle.

If k_0 is positive hyperbolic, by Lemma 4.9, k_0 has at least two outgoing edges, corresponding to the two unstable branches of k_0. Denote by k_1 and k'_1 the respective ends of these two edges. Then by strong connectivity there are two paths in G'_b from k_1 to k_0 and from k'_1 to k_0 respectively. As before, concatenating the edges (k_0, k_1) and (k_0, k'_1) with these paths yields the two desired homo/heteroclinic cycles. Observe that k_1 and k'_1 might be equal, but γ_1 and γ'_1 are different. □

4.4. **Theorem 1.2 for strongly nondegenerate Reeb vector fields.** We now prove Theorem 1.2 when the Reeb vector field is *strongly nondegenerate*, that is when the invariant manifolds of the broken binding orbits intersect transversally. Then in Section 4.5 we remove this transversality hypothesis and in Section 4.7 we extend the proof to an open neighbourhood.

Observe that the strong nondegeneracy is a weaker hypothesis than being *Kupka-Smale*, since a Kupka-Smale vector field has in addition all its periodic orbits hyperbolic. The strongly nondegenerate condition is generic for vector fields due to [Kup, Sma]. Another proof is due to Peixoto [Pei]. The latter extends to the Reeb context, so that strongly nondegenerate condition is also generic for Reeb vector fields. It can also be derived from the results of Robinson [Rob] which addresses the cases of symplectic diffeomorphisms and Hamiltonian vector fields. All the needed perturbations are local and make no difference between area-preserving and flux-zero properties. Giroux's Lemma 4.4 allows to convert local transverse area-preserving perturbations into Reeb perturbations.

Theorem 1.2 follows from the fact that if the broken book has broken components in its binding, then Lemma 4.11 implies that there are homo/heteroclinic cycles. The strongly nondegenerate hypothesis then gives a transverse homoclinic intersection, implying the existence of a Smale horseshoe. Theorem 4.1 then implies that the flow has positive topological entropy and infinitely many periodic orbit.

If there are no broken components in the binding, the broken book is a rational open book. Then, Corollary 4.8 together with Remark 2.3 completes the proof of Theorem 1.2 for strongly nondegenerate Reeb vector fields.

4.5. **Theorems 1.2 and 1.3 for nondegenerate Reeb vector fields.** We now prove Theorem 1.2 in the case where we drop the strong nondegeneracy hypothesis, and obtain the result for nondegenerate Reeb vector fields. Consider two hyperbolic periodic orbits (not necessarily different) with a homo/heteroclinic orbit connecting them. If the corresponding intersection of stable/unstable manifolds is not transverse then it is a connection, a crossing intersection or a one-sided intersection as in Definition 4.2.

After the proof in Section 4.2, in particular Corollary 4.8 and Remark 2.3, in order to prove Theorem 1.2 for nondegenerate Reeb vector fields we need to consider only those supported by broken book decompositions with $K_b \neq \emptyset$. By Lemma 4.11 there is a heteroclinic cycle and as discussed in Section 4.1, if at least one of the intersections in this cycle is a crossing intersection, then the vector field has positive topological entropy and infinitely many periodic orbits. Hence we consider the cases in which all the intersections are either connections or one-sided. We treat differently homo/heteroclinic connections and one-sided intersections because in the Reeb context, a homo/heteroclinic connection cannot be eliminated by a local perturbation of the Reeb vector field: one cannot displace a transverse circle from itself with a flux-zero map close to the identity.

We first prove Theorem 1.2 when all the homoclinic and heteroclinic intersections between the orbits of K_b are connections (Lemma 4.12). We then consider the case of an unstable manifold that has only one-sided intersections with the stable manifolds of the orbits in K_b. The idea is to destroy the intersections, one by one, by perturbing the Reeb vector field and obtain a new Reeb vector field supported by the same broken book decomposition. For any given number $L > 0$, the new Reeb vector field has a periodic orbit $k \in K_b$ whose unstable manifold intersects L times the set of rigid pages along circles. This contradicts Corollary 4.10. This is done in Proposition 4.16. Before that, we order the intersections along a stable or unstable manifold.

We start with the case when there are only complete connections.

Lemma 4.12. *Let $(K, \mathcal{F})$ be a broken book decomposition with $K_b \neq \emptyset$ supporting a nondegenerate Reeb vector field R_λ. Assume that all the homoclinic or heteroclinic intersections between orbits in K_b are connections. Then R_λ has infinitely many different simple periodic orbits.*

Proof. The idea is to cut M along the stable/unstable manifolds of the broken components of the binding to obtain a manifold with torus boundary.

Let M' be the metric completion of M minus the stable/unstable manifolds of the broken components of the binding. Then M' is a 3-manifold with boundary and corners, possibly disconnected.

The boundary of M' is made of copies of the stable and unstable manifolds of orbits in K_b and has corners along the copies of orbits of K_b. The Reeb vector field is tangent to the boundary, while the foliation $\mathcal{F}$ is now transverse to the boundary. The foliation $\mathcal{F}$ has two types of singularities, both along circles: it is singular along the radial components in K_r and along the corners of M' that corresponded to quadrants of a hyperbolic orbit in which $\mathcal{F}$ had a radial sector with nonempty interior. This configuration implies that (each connected component of) $M' \setminus K_r$ fibers over $\mathbb{S}^1$ and that the Reeb vector field has a first-return map defined on the interior of each one of these fibers.

Let N be a connected component of M'. If the fibers are neither disks, nor annuli, we obtain the existence of infinitely many periodic orbits using Theorem 4.6. Thus assume that in each connected component of M' the fibers are either disks or annuli. Then the regular pages of $(K, \mathcal{F})$ are disks or annuli and, by Lemma 2.9, K_r is nonempty.

A component N containing an element of K_r cannot have disk pages. So there is a component N of M' where all pages are annuli, having one boundary component on $k_r \in K_r$ contained in the interior of N and the other one on the boundary of N. Note that this implies that N is a solid torus. The boundary of N is decomposed into the annuli given by the homoclinic or heteroclinic connections, each annulus being bounded by two (not necessarily different) components of K_b.

The orbits of R_λ define a 1-foliation on each annulus. Observe that no annulus is foliated by Reeb components of R_λ, since R_λ is geodesible [Sul]: here in the case of a Reeb component, $d\lambda$ would be zero on the annulus, while the integral of λ would be nonzero on the boundary, in contradiction with Stokes' theorem. Hence, the periodic orbits in the 2-torus ∂N turn in the same direction, are attracting on one side and repelling on the other side since we are alternately passing from a stable manifold to an unstable manifold.

We now claim that we can change the fibration of the interior $N \setminus k_r$ by another fibration by annuli, close to the previous one (in terms of their tangent plane fields), so that it remains transverse to R_λ in the interior and becomes transverse to R_λ along the boundary. Indeed, outside of a neighbourhood of ∂N, the Reeb vector field is away from the tangent plane field of the fibration by a fixed factor, in particular near k_r. Near ∂N, the Reeb vector field gets close to the tangent plane field of the fibration and is tangent to it along $K_b \cap \partial N$, but with a fixed direction since there are no Reeb components for the flow restricted to the boundary. Thus we can tilt the fibration in the other direction to make it everywhere transverse (see Figure 10). This operation changes the slope by which the fibration approaches k_r and the boundary ∂N.

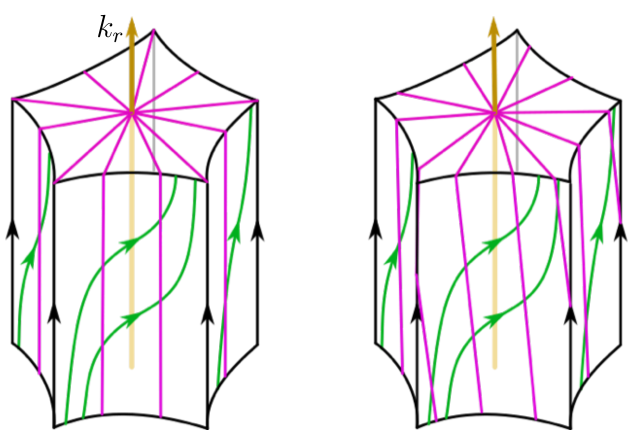

FIGURE 10. A tube $N$with a radial binding orbit k_r. The boundary ∂N contains several (here 5) broken binding orbits (black). The annuli between these broken binding orbits are connections. On the left, the pages of the broken book foliate N by annuli (purple) which are transverse to R_λ (green), except at the broken binding orbits. On the right, one tilts the foliation by annuli by a small amount, so that they are now transverse to all orbits, including the binding orbits.

Since the fibers are now everywhere transverse to R_λ, there is a well-defined first-return map that extends to the boundary to give an area-preserving homeomorphism of a closed annulus. Observe that this annulus is not necessarily embedded along $k_r \cup K_b$, but the map is well-defined. The boundary of any such annular page intersects at least one component of K_b, so that the first-return map to this annulus has at least one periodic point in the boundary. A theorem of Franks implies that there are infinitely many periodic points [Fra, Theorem 3.5]. □

We now prove that an unstable manifold of an orbit in K_b that does not coincide with the stable manifold of some orbit of K_b must have a crossing intersection with some unstable manifold of an orbit in K_b. In order to prove such a statement we will destroy, one by one, one-sided intersections without creating new ones, by perturbing the Reeb vector field. To do so, we first have to order the intersections.

Consider an orbit segment γ that is not an entire periodic orbit. Thus γ has no self-intersection and we say that γ is a *simple orbit segment*.

Definition 4.13. The *length* of a simple orbit segment γ is the number of components of $\gamma \setminus \mathcal{R}$, where $\mathcal{R}$ denotes the set of rigid pages. The length of a heteroclinic chain is the sum of the length of its components.

Observe that the length of an orbit of the binding K is zero. The length of a full heteroclinic or homoclinic orbit is bounded, hence in $V^u(k)$ and $V^s(k)$ for $k \in K_b$, the heteroclinic and homoclinic orbits are partially ordered.

We also want to consider convergence of sequences of orbits. For that we consider a small neighbourhood $N(K_b)$ of K_b, made of the disjoint union of neighbourhoods $N(k)$ of each $k \in K_b$. These neighbourhoods are taken to be standard Morse type neighbourhoods, as in Figure 9. Hence any orbit that enters and exits $N(k)$ has to intersect at least one rigid page inside $N(k)$. If γ is a simple orbit segment, set $\hat{\gamma} = \gamma \setminus N(K_b)$ that is a collection of simple orbit segments.

The first part of the following lemma is a tautology following from the definition of the length, and the second part follows by compactness.

Lemma 4.14. *Every simple orbit segment γ of length equal to $L > 0$ intersects the set of rigid pages $\mathcal{R}$ exactly $L - 1$ times.*

For every $L > 0$, there exists $N > 0$ such that if γ is a simple orbit segment of length bounded by L, then the action of $\hat{\gamma}$ is bounded from above by N.

We now consider convergence of sequences of orbits and orbit segments using the Hausdorff topology (which could probably be replaced by a stronger topology).

Lemma 4.15 (Compactness lemma)**.** *Given $L > 0$, the set of homo/heteroclinic orbits and simple orbit segments of length less than L admits a natural compactification by chains of heteroclinic orbits and simple orbit segments whose length is at most L.*

Proof. Let (γ_n) be a sequence of homo/heteroclinic orbits of length bounded by L. Then every orbit γ_n passes through less than L components of $N(K_b)$ and we can extract a subsequence such that the orbits in the subsequence have the same pattern of crossings with $N(K_b)$. Then all orbit segments $\hat{\gamma_n}$ have bounded action by Lemma 4.14 and are, up to extracting a subsequence, converging to a collection of orbit segments. Inside a component $N(k_1)$ of $N(K_b)$, the sequence $(\gamma_n \cap N(k_1))$ has either bounded or unbounded action.

In the first case, there is a subsequence that converges to a simple orbit segment in $N(k_1) \setminus k_1$.

In the second case, there is a subsequence that converges to the concatenation of an orbit segment in $V^s(k_1)$, followed by k_1 travelled for an arbitrarily long time, and then by an orbit segment in $V^u(k_1)$. Thus a subsequence of (γ_n) converges to a heteroclinic chain. It is then immediate that the chain has length less than L and Lemma 4.15 follows.

The proof for sequences of orbit segments is analogous. □

We can now prove the main result for one-sided intersections.

Proposition 4.16. *Let $(K, \mathcal{F})$ be a broken book decomposition supporting a nondegenerate Reeb vector field R_λ. If an unstable branch $V^u(k)$ of some orbit $k \in K_b$ does not coincide with a stable manifold of an orbit in K_b, then it contains a crossing intersection.*

Proof. The proof of this result is not straightforward and will involve proving Lemma 4.17. The proof of Lemma 4.17 uses Lemmas 4.18, 4.19 and 4.20.

We know by Lemma 4.9 that $V^u(k)$ must intersect stable manifolds of periodic orbits in K_b. We argue by contradiction and assume that $V^u(k)$ has no crossing intersection. Then $V^u(k)$ must contain only one-sided intersections.

Recall that $V^u(k)$ is a cylinder bounded on one side by k and which is foliated by orbits of R_λ. We travel along $V^u(k)$, starting from k.

Consider the set $\mathcal{R}$ of rigid pages of the broken book decomposition. Then $M \setminus \mathcal{R}$ is formed of product-type components. Since R_λ is transverse to $\mathcal{R}$, after leaving k, the unstable branch $V^u(k)$ enters successively some of these components along core circles $C_1, C_2, \dots$ of the cylinder $V^u(k)$, see Figure 11. Note that these entering circles are indeed ordered by the flow in $V^u(k)$. Also it exits each component along the circle that is an entering circle of the next component. This happens until the cylinder $V^u(k)$

enters a component P, along a circle C_P, that contains in its boundary an orbit $k' \in K_b$ such that there is an orbit of a point in C_P asymptotic to k' that stays in P. This means in particular that $V^u(k)$ and $V^s(k')$ intersect.

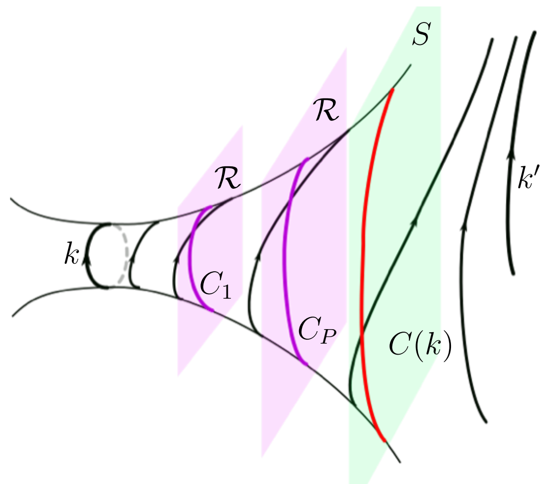

FIGURE 11. The unstable branch $V^u(k)$ of some broken orbit, in case it coincides with no stable manifold (Proposition 4.16). The orbits of R_λ escape from k. When travelling away from k one meets some rigid pages (purple) along circles $C_1, C_2, \ldots$. Since there are connections in $V^u(k)$, there is an orbit in the stable manifold $V^s(k')$ for some broken orbit k'. The circle C_P is the last circle in $V^u(k) \cap \mathcal{R}$. The region at the right of C_P is denoted by P. The surface S is a regular page (green) of the broken book intersecting $V^u(k)$ in P, along a circle $C(k)$ (red).

Before arriving near k', the intersections of $V^u(k)$ with the regular pages of $(K, \mathcal{F})$ also occur along circles. We pick a regular page S of $(K, \mathcal{F})$ in P. Then the 1-manifold $V^u(k) \cap S$ contains an embedded circle $C(k)$—the one after the last entrance circle C_P in P—and $V^s(k') \cap S$ contains another embedded circle $C(k')$—the first intersection of $V^s(k')$ with S when flowing backward from k'—so that the circles $C(k)$ and $C(k')$ intersect along a nonempty compact set Δ, containing only one-sided intersections. Indeed every point of Δ is located on an heteroclinic intersection from k to k', all of those being one-sided by assumption. The one-sided intersections can be on one side of $C(k)$ or on the other, thus we further decompose Δ as the disjoint union of two compact sets Δ_+ and Δ_-, depending on the side of tangency, see Figure 12. They could *a priori* be both nonempty.

The circle $C(k')$ is contained in a branch of $V^s(k')$. The side Δ_+ further determines a quadrant Q_+ of k' in which $C(k)$ lies near Δ_+.

The idea of the rest of the proof is to destroy, inductively, the one-sided intersections of $V^u(k)$, starting from those passing through Δ and to find a new Reeb vector field supported by the same broken book decomposition, but such that $V^u(k)$ does not contain any heteroclinic orbit up to a certain length. The important consequence of Corollary 4.10 is that the length up to which we need to eliminate the intersections is *a priori* bounded uniformly with respect to the features of the rigid pages (genus and area) and the length of the binding orbits. Let L be a length that provides a contradiction to Corollary 4.10.

Lemma 4.17. *If the component of $V^u(k')$ is not a complete connection, i.e. it does not coincide with the stable manifold of an orbit in K_b, we can slightly modify R_λ in order to eliminate Δ_+ and Δ_-, without creating extra intersections of $V^u(k)$ of length less than or equal to L.*

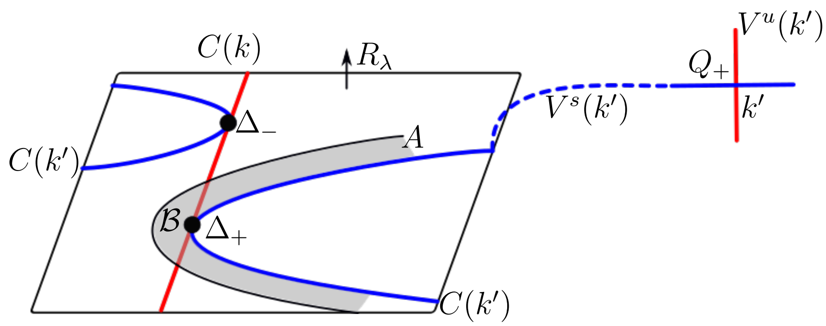

FIGURE 12. The surface S contains circles $C(k) = V^u(k) \cap S$ (red) and $C(k') = V^s(k') \cap S$ (blue), which are tangent to one another. Depending on the relative position at these contact points, they belong to Δ_+ or Δ_-. Δ_+ determines a quadrant Q_+ at k', as well as an unstable branch of $V^u(k')$. An annulus A (gray) in $Q_+ \cap S$ is also shown.

Proof of Lemma 4.17. We now explain how to eliminate the intersections from Δ_+. Recall that they determine near k' a quadrant Q_+ delimited by the branch of $V^s(k')$ containing Δ_+ and an unstable branch of $V^u(k')$.

Let us first observe the following important property.

Lemma 4.18. *Every intersection of the branch $V^u(k')$ and any $V^s(k'')$, with $k'' \in K_b$, is non-crossing or complete, and on the other side of Q_+ (i.e. locally $V^s(k'')$ does not meet Q_+).*

Proof. We consider an annulus A' transverse to R_λ and which is a thickening of an essential curve in $V^u(k')$ near k', see Figure 13. If there was either a crossing intersection or a non-crossing intersection on the side of Q^+ between $V^u(k')$ and some $V^s(k'')$, with $k'' \in K_b$, then $A' \cap Q_+ \cap V^s(k'')$ would contain an arc δ anchored in $V^u(k')$. The half-infinite strip swept by the backward flow of δ would intersect S into a set containing a curve spiralling to $C(k')$ on the side of Q_+, and thus there would be crossing intersections between $V^u(k)$ and $V^s(k'')$, contradicting the hypothesis on $V^u(k)$. □

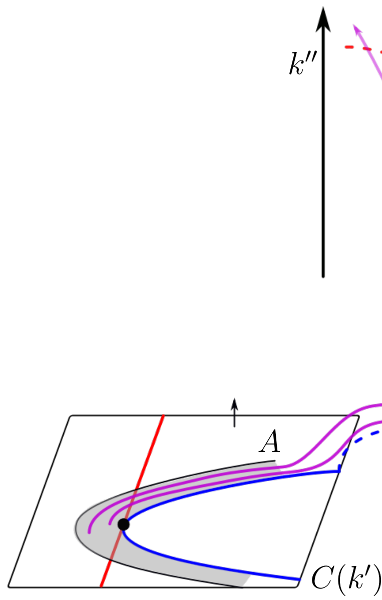

FIGURE 13. Proof of Lemma 4.18: a crossing intersection between $V^u(k')$ and some $V^s(k'')$ on the side of Q_+. An annulus A' transverse to R_λ and to $V^u(k')$ is shown in gray. The intersection $A' \cap Q_+ \cap V^s(k'')$ constrains an arc δ (purple) anchored in $V^u(k')$ (purple dot). By flowing backward, we obtain an arc still anchored in $V^u(k')$ and visiting the annulus A.

More generally, if A' is an embedded annulus transverse to R_λ which is a thickening of an essential closed curve a in $V^s(k')$ near some $k' \in K_b$, we say that some $V^s(k'')$, $k'' \in K_b$, *spirals to* $V^s(k')$ if $V^s(k'') \cap A'$ contains a sequence of arcs a_n that is uniformly converging to the universal cover $\pi : \hat{a} \to a$ in C^1_{loc}-norm. This definition does not depend on the choice of A'.

Let $A = C(k') \times [0, \eta] \subset S$ be a small embedded annulus with $C(k') = C(k') \times \{0\}$, situated on the side of Q_+ in S, see Figure 12.

Lemma 4.19. *If A is small enough, then its interior $C(k') \times (0, \eta)$ does not intersect any homo/heteroclinic orbit of length $< L$ in the stable manifold of an orbit $k'' \in K_b$.*

Proof. Assume for a contradiction that for any A there is an intersection with an orbit in a stable manifold whose length is at most L. Then we find a sequence (x_n) of points in A approaching $C(k')$, so that their orbits (γ_n) are of length at most L and are asymptotic at $+\infty$ to elements in K_b. Using the Compactness Lemma 4.15, up to extracting a subsequence, we can assume that (x_n) converges to a point $x \in C(k')$ and (γ_n) converges to the union of:

(1) a sequence of orbits $\delta_0, \dots, \delta_l$, where δ_0 is a half-infinite orbit from x to $k' = k_0$, and, for $1 \leq i \leq l$, δ_i is a heteroclinic orbit from $k_{i-1} \in K_b$ to $k_i \in K_b$;
(2) the collection of periodic orbits $k_i \in K_b$ for $i = 0, 1, \dots, l$, travelled for an arbitrarily long time.

Iterating the argument of the proof of Lemma 4.18, we see that, following the sequence of connections by the backward flow from k_l at each connection, for $i \leq l-1$, a portion of $V^s(k_l)$ spirals to the component $V^s(k_i)$ containing δ_i from the side containing γ_n (for n large enough). Note here that for the propagation of the spiralling property, there are two slightly different cases that can occur: $V^s(k_l)$ spirals to the component $V^s(k_{i+1})$ and either $V^s(k_l)$ intersects $V^u(k_i)$ in which case it spirals to $V^s(k_i)$ by the proof of Lemma 4.18, or it does not intersect $V^u(k_i)$ (because it spirals on the side of $V^s(k_{i+1})$ that is not intersecting $V^u(k_i)$ near δ_{i+1}), but still since $V^s(k_{i+1})$ spirals to $V^s(k_i)$, so does $V^s(k_l)$ which spirals to $V^s(k_{i+1})$.

Thus finally a portion of $V^s(k_l)$ spirals to $V^s(k')$ from the side of A and there are crossing intersections of $V^u(k)$ with $V^s(k_l)$. A contradiction to the hypothesis on $V^u(k)$. □

Thus, if A is small enough, there is a collection of arcs in $C(k') \times \{\eta\}$ and a collection of arcs in $C(k)$ containing Δ_+ that cobound a family $\mathcal{B}$ of bigons in A whose interior does not intersect any homo/heteroclinic orbit of length $< L$, see Figure 12. We want to push $C(k)$ in $C(k') \times (0, \eta]$ near Δ_+. However this cannot be done by a modification supported in A and there is a risk of messing up the dynamics if there is some recurrence in length less than L. We need the extra control:

Lemma 4.20. *There exists a neighbourhood U of Δ_+ in S such that there is no orbit of length at most L from $U \cap \operatorname{int}(A)$ to the components of $U \setminus \mathcal{B}$.*

Proof. Arguing by contradiction, we can find as in the proof of Lemma 4.19 a sequence (γ_n) of orbits of length at most L, starting from $U \cap \operatorname{int}(A)$ and ending outside of $\mathcal{B}$, that converges to a sequence of orbits $\delta_0, \dots, \delta_l$ such that δ_0 is an half-infinite orbit starting from a point x of Δ_+ and going to $k' = k_0$; δ_i for $1 \leq i \leq l-1$ is a heteroclinic orbit from $k_{i-1} \in K_b$ to $k_i \in K_b$; and δ_l is a half-infinite orbit from k to a point y in Δ_+. With the same reasoning than in Lemma 4.18 and Lemma 4.19, we see that the backward flow of $C(k') \cap U$ has to come back spiralling along $C(k')$ on the side of A, thus creating crossing intersections with $V^u(k)$, a contradiction. □

The proof of Lemma 4.17 follows from the combination of Lemmas 4.19 and 4.20. First we choose the neighbourhood U from Lemma 4.20 that does not intersect Δ_-.

We consider a flux-zero area-preserving diffeomorphism ψ supported in U that pushes $C(k) \cap U$ inside $(C(k') \times (0, \eta]) \cap U$. The modification of R_λ will be done in a positive-flow-box product thickening of U by modifying the direction of R_λ so that the circle $C(k)$ entering the thickening of U will be mapped to its image by ψ when exiting, using Giroux' realisation Lemma 4.4. This locally eliminates Δ_+. Now, thanks to Lemmas 4.19 and 4.20, we see that we do not create extra connections of length $\leq L$ for k. Indeed the modified part of $C(k)$ has no connections of length $\leq L$ that are not passing again in U by Lemma 4.19; moreover they are not passing again through $U \setminus \mathcal{B}$ by Lemma 4.20. If they enter again through $\mathcal{B}$ then they exit in $\mathrm{int}(A) \cap U$ and the same reasoning applies again.

We can apply the same argument to eliminate Δ_-. ☐

Back to the proof of Proposition 4.16, we are left with the case in which the unstable component $V^u(k')$ is a complete connection to an orbit k''. We repeat the argument of Lemma 4.17: either the corresponding unstable manifold of k'' is a complete connection, or we can eliminate Δ_+. The sequence of successive complete connections we are following is dictated at each step by the side which contains Δ_+ that have a natural co-orientation. In particular, we cannot use twice the same connection, since otherwise the corresponding cycle of complete connections would have, by tracing it back, to go through $V^s(k')$, which is not a complete connection, a contradiction. Since there are finitely many complete connections, this process stops and we can always eliminate $\Delta_\pm$. Arguing by induction, we eliminate successively all intersections from k of length less or equal to L, without creating new ones, and obtain a contradiction to Corollary 4.10 applied to the modified Reeb vector field for the unstable manifold $V^u(k)$. This terminates the proof of Proposition 4.16. ☐

We can now prove Theorems 1.2 and 1.3 for nondegenerate Reeb flows. In view of Lemma 4.12 and Proposition 4.16, to prove Theorem 1.2 we need to consider the case when there is at least one crossing intersection between the components of K_b.

Lemma 4.21. *Let $(K, \mathcal{F})$ be a broken book decomposition supporting a nondegenerate Reeb vector field R_λ. Assume that there is at least one homo/heteroclinic intersection between broken components of the binding K_b, with a crossing of stable and unstable manifolds. Then R_λ has positive topological entropy and infinitely many periodic orbits.*

Proof. If we have a homoclinic intersection with a crossing, as in the hypothesis, we are done. So assume that we have a heteroclinic intersection. We consider the set $\mathcal{C}$ of complete connections between components of K_b. As before cut M along $\mathcal{C}$ to get a manifold M' with boundary and corners. We let K'_b be the collection of periodic orbits of K_b, when viewed in M'. The set K'_b may contain several copies of the same orbit of K_b.

By hypothesis there is at least one heteroclinic orbit in M' between elements of K'_b along which there is a crossing intersection. Hence it is not in $\partial M'$. Note that for every component T of $\partial M'$, the total number of stable and unstable branches of an orbit $k_b \in T \subset \partial M'$ that are not themselves contained in T is even, since there are as many stable than unstable manifolds in T (every heteroclinic or homoclinic connection in T involves a stable and an unstable manifold).

Consider a connected component N of M' that contains a crossing intersection, and let $k \in K'_b$ be the orbit whose unstable manifold $V^u(k)$ is involved in this intersection. Following the heteroclinic intersections from $V^u(k)$, as in Lemma 4.11, we get a sequence of heteroclinic intersections. We claim that this sequence stays inside N. Indeed, if it arrives to a periodic orbit in $K'_b \cap \partial N$ along a stable manifold, then the two branches of the unstable manifold of this periodic orbit are in N. We can thus construct a sequence such that all the stable and unstable manifolds involved are in N. To see this, one can collapse every component of ∂N to a generalised hyperbolic orbit with $2n$ stable/unstable manifolds: we identify all the periodic orbits in a connected component of ∂N and get rid of the stable and unstable branches in this boundary component. This is doable since there are no Reeb components for the Reeb flow on the boundary. We now have a closed manifold together with a Reeb vector field without connections. We also have a degenerate broken book decomposition ∂-strongly carrying the flow. The degeneracy is under control with again local models near degenerate binding components transversally given by hyperbolic phase portraits with $2n$ separatrices (as in Figure 3 for $n = 3$). In this slightly generalised context, Lemma 4.9 and Proposition 4.16 still apply straightforward and imply that there is a cycle with only crossing intersections.

Near this cycle, we obtain a crossing homoclinic intersection in N, with a horseshoe in N that is also contained in M. Positivity of topological entropy and the existence of infinitely many different periodic orbits comes from [BuW, Theorem 2.2]. □

We now prove Theorem 1.3, for nondegenerate Reeb vector fields, stating that on a 3-manifold that is not graphed, every nondegenerate Reeb vector field has positive topological entropy.

Proof of Theorem 1.3 for nondegenerate Reeb flows. A nondegenerate Reeb vector field is carried by some broken book decomposition. If there is no broken component in the binding, then the broken book is in fact a rational open book. If M is not a graph manifold, then the monodromy of this rational open book must contain a pseudo-Anosov component in its Nielsen-Thurston decomposition. The first-return map of the Reeb vector field on a page provided by the ∂-strong property is homotopic to the Nielsen-Thurston monodromy, so its topological entropy is bounded from below by the latter one, that is positive [FLP, Exposé 10, IV].

If the binding of the broken book has broken components then all elements of K_b that do not belong to complete connections contain, by Proposition 4.16, a crossing intersection. This proves the positivity of the entropy in this case.

If all stable and unstable manifolds of elements of K_b are complete connections, then as in Lemma 4.12, they decompose M into partial open books and if M is not graphed, then one of them must have some pseudo-Anosov monodromy piece in its Nielsen-Thurston decomposition and we obtain positive topological entropy. ☐

We end this section with a remark that partially answers Question 1.8 of [CHP]: *Is it true that if M is not $\mathbb{S}^3$ nor a lens space and if R_λ is a nondegenerate Reeb vector field on M, then it has a simple positive hyperbolic periodic orbit?*

We have obtained so far that if R_λ is strongly nondegenerate, then it either has positive topological entropy or admits a Birkhoff section on which the first return map of the flow h has a Nielsen-Thurston decomposition into periodic pieces. If R_λ has positive topological entropy, Katok's theorem implies the existence of a transverse horseshoe and hence of a simple positive hyperbolic periodic orbit. This is for example the case when the ambient manifold is not graphed, where we thus can answer positively Cristofaro-Gardiner, Hutchings and Pomerleano's question for strongly nondegenerate Reeb flows. If in the Nielsen-Thurston decomposition the map h has at least two periodic pieces, the proof of Theorem 4.6 adapted to the case of periodic Nielsen-Thurston pieces gives again the existence of simple positive hyperbolic periodic orbits, which are odd degree generators of cylindrical contact homology coming from the positive hyperbolic generators in the Morse-Bott families. The remaining cases to study is when h is isotopic to a periodic map. In this situation, one can use Lefschetz-Hopf indices computations to conclude when the Birkhoff section has strictly positive genus.

4.6. **Eliminating broken components.** In this section we explore the possibility of obtaining a rational open book decomposition from a broken book decomposition. The results here can be applied to constructions by de Paulo and Salomão [dPS], to obtain an open book decomposition. These results also give a proof of Theorem 1.4.

Theorem 4.22. *Let R_λ be a strongly nondegenerate Reeb vector field for a contact form λ carried by a broken book decomposition $(K, \mathcal{F})$. Assume that K contains at most one broken component. Then R_λ has a Birkhoff section.*

Proof. Denote by k_0 the broken component in the binding K and assume first that it is a positive hyperbolic periodic orbit. By the proof of Lemma 4.11, each of the two branches of the unstable manifold of k_0 intersect at least one stable branch of k_0, and each of the two stable branches intersect at least one unstable branch. Therefore, up to a symmetry, there are two orbits γ_a and γ_b such that γ_a belongs to both the west unstable branch and the south stable branch of k_0, and γ_b belongs to both the east unstable manifold and the north stable manifold of k_0 (see Figure 14).

Consider a small local transverse section D to R_λ at a point of k_0 and the induced first-return map f.

By taking a small transverse rectangle r_W intersecting γ_a and considering its images by f, the standard horseshoe construction [KaH, Section 6.5] implies that there is an iterate f^{k_W} such that r_W and $f^{k_W}(r_W)$ intersect and the intersection contains a fixed point of f^{r_W} that we denote by p_a (the pentagon on Figure 14). Similarly there is a transverse rectangle r_E intersecting γ_b and a suitable iterate f^{k_E} such that $f^{k_E}(r_E)$ intersects r_E and the intersection contains a fixed point of f^{k_E}, denoted by p_b. Denote by k_a and k_b the periodic orbits of R_λ corresponding to p_a and p_b respectively. More generally [KaH, Sections 1.9 and 6.5], for every word w in the alphabet $\{a, b\}$, one can find a periodic point p_w of f that follows k_a, and k_b in the order given by w. In particular there is periodic orbit k_{ab} that intersects D in two points p_{ab} and p_{ba} so that it remains close to γ_a for one period of k_a and close to γ_b for the next period of k_b.

Now consider an arc connecting p_a to p_{ab}. When pushed by the flow, it describes a certain rectangle R_1 and comes back to an arc connecting p_a to p_{ba}. Likewise an arc connecting p_b to p_{ba} describes a rectangle R_2 whose opposite side is an arc connecting p_b to p_{ab} (see Figure 15). Together these four arcs form a parallelogram P in D which contains $D \cap k_0$ in its interior. The union of P and the two rectangles R_1 and R_2, forms an immersed topological pair of pants, which can be smoothed into a surface S_0 transverse to R_λ. The main properties of S_0 is that it is bounded by k_a, k_b and k_{ab}, and it is transverse to k_0.

Now consider a page S of the foliation $\mathcal{F}$ having k_0 in its boundary, with zero asymptotic linking number, take the union $S_0 \cup S$, and use the

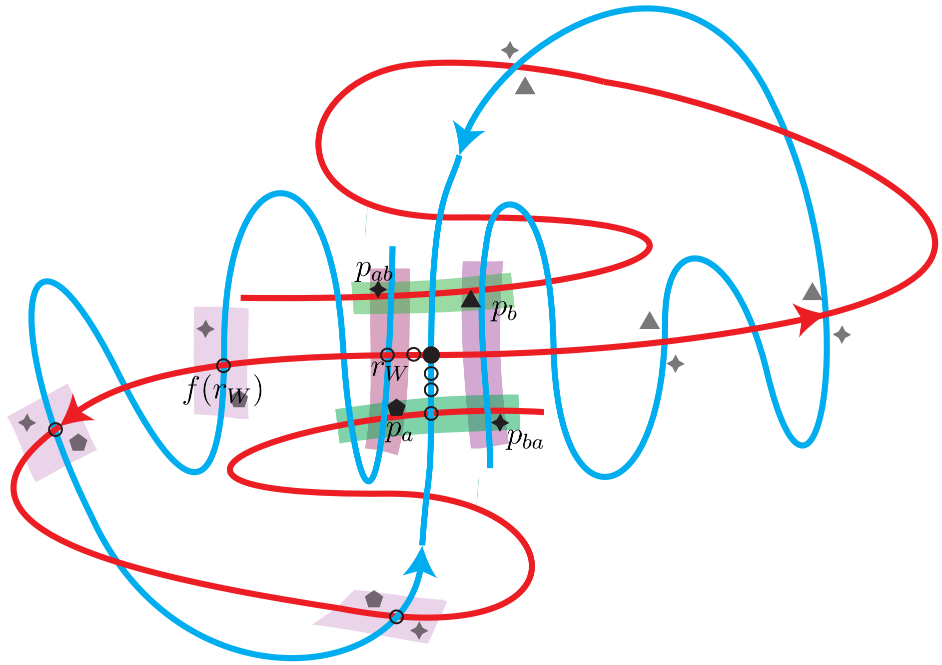

FIGURE 14. A transverse view of the orbit k_0 (full black point in the center) and its stable/unstable manifolds (in blue/red respectively). The empty circle denote the orbit γ_a which sits at the intersection of the W-unstable branch and the S-stable branch of k_0. Two small transverse rectangles r_W and r_E in the W- and E-parts are shown in purple, together with their images by suitable iterates f^{k_W} and f^{k_E} of the first-return map f, in green. In the intersection $r_W \cap f^{k_W}(r_W)$ lies a fixed point p_a of f^{k_W}, represented by a black pentagon, and its f-orbit is shown with gray pentagons. Similarly a fixed point p_b of f^{k_E} is represented by a black triangle, and its f-orbit by gray triangles. Finally a fixed point p_{ab} of $f^{k_W+k_E}$ is represented by a black 4-star, as well as its image p_{ba} by f^{k_W}.

flow direction R_λ to desingularise the arcs and circles of intersection as in Lemma 3.4 (see also Figure 6). The obtained surface intersects any small enough tubular neighbourhood of k_0 along one meridian plus one or two longitudes. Therefore k_0 is not anymore in the broken part of the binding, but is part of the boundary of the new surface. Also the surface F_0 intersects k_a, k_b, and k_{ab}, so that these periodic orbits link positively the union $F_0 \cup S$.

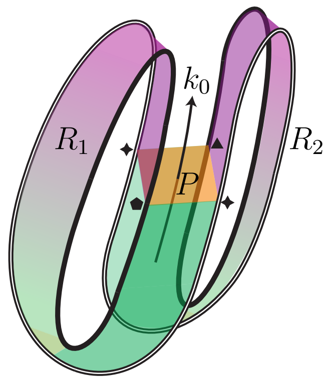

FIGURE 15. The union of the rectangles R_1 and R_2 (whose color change from purple to green along the flow) with the parallelogram P (orange) which lies in D yields a topological pair of pants. It can be smoothed into a surface transverse to the Reeb flow, bounded by the periodic orbits k_a, k_b and k_{ab}, and whose interior intersects k_0.

The resulting surface is then a genuine (rational) Birkhoff section for the Reeb vector field. We can thus obtain an open book decomposition from it adapted to the Reeb vector field. Observe that the orbits k_0, k_a, k_b, and k_{ab} are boundary components of radial type with respect to the new foliation.

The case where k_0 is a negative hyperbolic periodic orbit is treated in the same manner. The difference is that now one needs to consider the second iterate of the return map to a local transversal to the periodic orbit in order to have Figure 14. □

In Lemma 4.11 we proved the existence of homo/heteroclinic cycles between the components of K_b. We could try to use the local Fried's construction of pair of pants from the proof of the previous Theorem 4.22 to inductively decrease the number of broken binding components of a supporting open book to finally get a Birkhoff section. Unfortunately, Lemma 4.11 does not seem sufficient to make sure that S_0 intersects k_0, since the two heteroclinic cycles might involve twice the same stable or unstable branches (see Figures 16 and 14).

Conjecture 4.23. *Every Reeb vector field has a Birkhoff section.*

It follows from the considerations in Section 4.4 that if a strongly nondegenerate Reeb vector field has no homoclinic orbit, then any of its supporting broken book decomposition is in fact a rational open book, providing a proof of Theorem 1.4. Moreover, if a strongly nondegenerate Reeb vector field has at most one periodic orbit having a heteroclinic cycle then, by Lemma 4.9, it

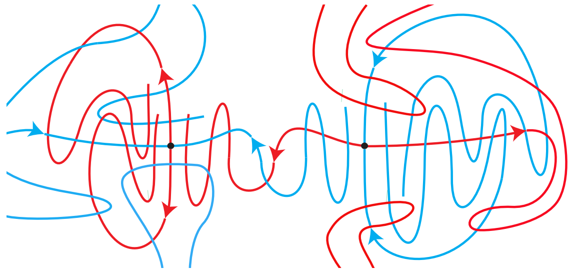

FIGURE 16. Two hyperbolic orbits and their stable/unstable manifolds at which one cannot directly apply Fried's construction.

has a supporting broken book with at most one broken binding component, and by Theorem 4.22, a Birkhoff section.

4.7. **Openness of the various properties.** In this last section, we discuss the nondegeneracy hypothesis and observe that our Theorems 1.1, 1.2, 1.3 and 1.4 hold for an open set of contact forms containing nondegenerate ones or strongly nondegenerate ones, accordingly.

Indeed, the arguments only use the fact that the periodic orbits in the binding are nondegenerate and do not care about nondegeneracy of "long" periodic orbits. The binding orbits are a subset of the set of orbits $\mathcal{P}$ for a nondegenerate contact form λ that shows up in Lemma 3.1. The set $\mathcal{P}$ contains all periodic orbits of R_λ whose actions are less than $\mathcal{A}(\Gamma)$, where Γ is a representative of a homology class whose image by the U-map does not vanish. As we will see below, the action of an orbit set realising a class which is not in the kernel of U can be *a priori* bounded in a neighbourhood of λ, a manifestation of the continuity of the spectral invariants.

Precisely, there exists a C^2-neighbourhood $N(\lambda)$ of λ such that for every nondegenerate contact form λ' in $N(\lambda)$, there is a representative Γ' for λ' of a nonzero class in $ECH(M, \lambda')$ whose image under the U-map is nonzero and whose total action is *a priori* bounded by some $L > 0$, depending only on $N(\lambda)$. This is a consequence of the existence of the cobordism maps in ECH (defined through Seiberg-Witten homology) given in [Hut2, Theorem 2.3]. We now shrink $N(\lambda)$ so that moreover for every form λ' in $N(\lambda)$, the periodic Reeb orbits of λ' of action less than L, forming a set $\mathcal{P}'$, are all nondegenerate – note this is an open condition. Now, if λ' is a contact form in $N(\lambda)$, possibly degenerate, first its periodic Reeb orbits of action less than L are nondegenerate and second it can be approximated by a sequence of nondegenerate contact forms λ'_n in $N(\lambda)$ whose periodic orbits of actions less than L coincide with those of λ'. Since λ'_n is nondegenerate, the conclusion of Lemma 3.1 holds for λ'_n and $\mathcal{P}'$: through every point z

in M, one can find a projected holomorphic curve through z and with asymptotics in $\mathcal{P}'$. By compactness for pseudo-holomorphic curves in the ECH context [Hut1, Sections 3.8 and 5.3], this property also holds for λ' and $\mathcal{P}'$. This is all we need to find a supporting broken book for λ', with binding in the nondegenerate set $\mathcal{P}'$. The rest of the arguments to prove Theorems 1.1, 1.2, 1.3 and 1.4 then carry over to λ'.

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