Averaging sequences

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Abstract

In the spirit of Goodman-Plante average condition for the existence of a transverse invariant measure for foliations, we give an averaging condition to find tangentially smooth measures with prescribed Radon-Nikodym cocycle. Harmonic measures are examples of tangentially smooth measures for foliations and laminations. We also present sufficient hypothesis on the averaging condition under which the tangentially smooth measure is harmonic.

1 Introduction

Averaging sequences for foliations were introduced in the pioneering work of J. F. Plante [30] on the influence that the existence of transverse invariant measures exerts on the structure of a foliation. Although only the case of sub-exponential growth was dealt with in [30], Plante's approach is clearly reminiscent of the classic work of E. Følner on groups [10]. Using the same kind of ideas, S. E. Goodman and J. F. Plante exhibited an averaging condition which guarantees the existence of transverse invariant measures for foliations of compact manifolds [15].

In this paper we formulate a more general averaging condition which gives rise to a tangentially smooth measure for a compact laminated space $(M, \mathcal{F})$. This condition may be related to the η-Følner condition of [2], in the same spirit as Følner, but using a modified Riemannian metric along the leaves. The modification is done by replacing any complete Riemannian metric along the leaves with the product of the metric with some density function. Namely, given a compact laminated space and a positive cocycle defined on the equivalence relation induced by the lamination on a total transversal, we prove that an η-Følner sequence gives rise to the existence of a tangentially smooth measure whose Radon-Nikodym cocycle is the given one. Moreover, we describe sufficient hypothesis for obtaining a harmonic measure. This is the content of Theorem 4.10.

Before proving Theorem 4.10, we start by analizing the discrete case. We define an averaging condition for any equivalence relation $\mathcal{R}$ defined by a finitely generated pseudogroup acting on a compact space and any continuous cocycle

$\delta : \mathcal{R} \to \mathbb{R}^*_+$, that we call δ-averaging condition. In Theorem 3.6 we prove that the existence of a δ-averaging sequence gives a quasi-invariant measure with Radon-Nikodym cocycle δ. Under some additional conditions, in particular if δ is harmonic, the measure obtained is harmonic. In this case, our result is reminiscent of Kaimanovich's characterization of amenable equivalence relations [21].

The paper is organized as follows. In Section 2 we review some preliminaries, in particular Section 2.3 contains the proof of Goodman and Plante's theorem. The discussion of the discrete and continuous settings is splitted in two separate sections, Section 3 and Section 4, respectively, which can be read independently. In Section 5 we analyze some explicit examples. The relation between the two types of averaging sequences will be briefly discussed in the final Section 6.

2 Preliminaries

2.1 Laminations and equivalence relations

A compact space M admits a d-dimensional lamination $\mathcal{F}$ of class C^r with $1 \leq r \leq \infty$ if there exists a cover of M by open sets U_i homeomorphic to the product of an open disc P_i in $\mathbb{R}^d$ centered at the origin and a locally compact separable metrizable space T_i. Thus, if we denote the corresponding foliated chart by $\varphi_i : U_i \to P_i \times T_i$, each U_i splits into *plaques* $\varphi_i^{-1}(P_i \times \{y\})$. Each point $y \in T_i$ can also be identified with the point $\varphi_i^{-1}(0, y)$ in the *local transversal* $\varphi_i^{-1}(\{0\} \times T_i)$. In addition, the change of charts $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is given by

$$\varphi_j \circ \varphi_i^{-1}(x, y) = (\varphi_{ij}^y(x), \gamma_{ij}(y)) \tag{2.1}$$

where γ_{ij} is an homeomorphism between open subsets of T_i and T_j and φ_{ij}^y is a C^r-diffeomorphism depending continuously on y in the C^r-topology. We say that $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is a *good foliated atlas* if it satisfies the following conditions:

(i) the cover $\mathcal{U} = \{U_i\}_{i \in I}$ is locally finite, hence finite;

(ii) each open set U_i is a relatively compact subset of a foliated chart;

(iii) if $U_i \cap U_j \neq \emptyset$, there is a foliated chart containing $\overline{U_i \cap U_j}$, implying that each plaque of U_i intersects at most one plaque of U_j.

Each foliated chart U_i admits a tangentially C^r-smooth Riemannian metric $g_i = \varphi_i^* g_0$ induced from a C^r-smooth Riemannian metric g_0 on $\mathbb{R}^p$. We can glue together these local Riemannian metrics g_i to obtain a global one g using a tangentially C^r-smooth partition of unity. From Lemma 2.6 of [1], we know that any C^r lamination of a compact space equipped with a C^r foliated atlas $\mathcal{A}$ admits a C^∞ foliated atlas C^r-equivalent to $\mathcal{A}$.

A discrete equivalence relation $\mathcal{R}$ is defined by $\mathcal{F}$ on the total transversal $T = \sqcup T_i$: the equivalence classes are the traces of the leaves on T. We can see $\mathcal{R}$ as the orbit equivalence relation defined by the *holonomy pseudogroup* Γ of $\mathcal{F}$, generated by the local diffeomorphisms γ_{ij}. These homeomorphisms form a finite generating set, which we will denote $\Gamma^{(1)}$, that defines a *graphing* of $\mathcal{R}$. This means that each equivalence class $\mathcal{R}[y]$ is the set of vertices of a graph, and there is an edge joining two vertices z and w if there is $\gamma \in \Gamma^{(1)}$ such that $\gamma(z) = w$. We can define a graph metric $d_\Gamma(z,w) = \min \{ n \,/\, \exists \gamma \in \Gamma^{(n)} : g(z) = w \}$, where $\Gamma^{(n)}$ are the elements that can be expressed as words of length at most n in terms of $\Gamma^{(1)}$. A *transverse invariant measure* for $\mathcal{F}$ is a measure on T that is invariant under the action of Γ. It is quite rare for a measure of this kind to exist.

Remark 2.1. If $\mathcal{F}$ has no holonomy (*i.e.* $\Gamma_y = \{\gamma \in \Gamma \,|\, \gamma(y) = y\}$ is trivial for all $y \in T$), we can endow $\mathcal{R}$ with the topology generated by the graphs of the elements of Γ. Then $\mathcal{R}$ becomes an *étale equivalence relation*, i.e. the partial multiplication $((y,\gamma(y)),(\gamma(y),\gamma'(\gamma(y))) \in \mathcal{R} * \mathcal{R} \mapsto (y, \gamma' \circ \gamma(y)) \in \mathcal{R}$ and the inversion $(y,\gamma(y)) \in \mathcal{R} \mapsto (\gamma(y), y) \in \mathcal{R}$ are continuous, and the left and right projections $\beta : (y,z) \in \mathcal{R} \mapsto y \in T$ and $\alpha : (y,z) \in \mathcal{R} \mapsto z \in T$ are local homeomorphisms. In general, by considering the germs of the elements of Γ at the points of their domains. we can replace $\mathcal{R}$ with the *transverse holonomy groupoid* [17] that becomes similarly an *étale groupoid* [31].

2.2 Compactly generated pseudogroups

In the last section, we obtained a pseudogroup from a foliated atlas. Here we will recall the *Haefliger equivalence* for pseudogroups obtained from different atlases and its metric counterpart in the compact case that we will need later in Section 2.3. For any compact laminated space $(M, \mathcal{F})$ the holonomy pseudogroup Γ is *compactly generated* in the sense of [18], meaning that:

(i) T contains a relatively compact open set T_1 meeting all the orbits;

(ii) the reduced pseudogroup $\Gamma|_{T_1}$ (whose elements have domain and range in T_1) admits a finite generating set $\Gamma^{(1)}$ (called a *compact generation system* of Γ on T_1) so that each element $\gamma : A \to B$ of $\Gamma^{(1)}$ is the restriction of an element $\overline{\gamma}$ of Γ whose domain contains the closure of A.

Any probability measure ν_K on the compact set $K = \overline{T_1}$ that is preserved by the action of $\Gamma|_K$ extends to a unique Borel measure ν on T which is Γ-invariant and finite on compact sets. We refer to Lemma 3.2 of [30].

On the other hand, notice that T is covered by the domains of a family of elements of Γ with range in T_1. The union of these elements and their inverses defines the *fundamental equivalence* between the holonomy pseudogroup Γ and the reduced pseudogroup $\Gamma|_{T_1}$. This is the base concept to define the *Haefliger equivalence* of pseudogroups (see [17] and [18]):

Definition 2.2. Two pseudogroups Γ_1 and Γ_2 acting on the spaces T_1 and T_2 are *Haefliger equivalent* if they are reductions of a same pseudogroup Γ acting on the disjoint union $T = T_1 \sqcup T_1$, and both T_1 and T_2 meet all the orbits of Γ.

The choice of generators for Γ_1 and Γ_2 defines a metric graph structure on the orbits, but the Haefliger equivalence between Γ_1 and Γ_2 may not preserve their quasi-isometry type. Let us recall this concept introduced by M. Gromov [16]:

Definition 2.3. Two metric spaces (M, d) and (M', d') are *quasi-isometric* if there exists a map $f : M \to M'$ and constants $\lambda \geq 1$ and $C \geq 0$ such that

$$\frac{1}{\lambda} d(y, z) - C \leq d'(f(y), f(z)) \leq \lambda d(y, z) + C$$

for all $y, z \in M$ and $d'(y', f(M)) \leq C$ for all $y' \in M'$.

Definition 2.4 ([12],[19])**.** A Haefliger equivalence between two pseudogroups Γ_1 and Γ_2 acting on T_1 and T_2, respectively, is a *Kakutani equivalence* if Γ_1 and Γ_2 admit finite generating systems such that their orbits, endowed with the graph metric, are quasi-isometric.

According to Theorem 2.7 of [25] and Theorem 4.6 of [3], if two compactly generated pseudogroups Γ_1 and Γ_2 are Haefliger equivalent, then there are compact generating systems on T_1 and T_2, respectively, such that the pseudogroups become Kakutani equivalent. These compact generating systems are called *good* in [25] and *recurrent* in [3]. The relevance of this, is that the existence of averaging sequences depends on the quasi-isometric type of the orbits (see [3] and [24] for the details).

2.3 Existence of transverse invariant measures

In this section we will discuss a sufficient condition for the existence of a transverse invariant measure, which serves as motivation for Theorems 3.6 and 4.10. In [15], Goodman and Plante formulate the following proposition. Let us start with some definitions.

Definition 2.5. Let A be a finite subset of T and γ an element of Γ. We define the difference set

$$\Delta_\gamma A = \{x \in T \mid x \in A, \gamma(x) \notin A\} \cup \{x \in T \mid x \notin A, \gamma(x) \in A\},$$

with the convention that $\gamma(x) \notin A$ holds if $\gamma(x)$ is not defined. We denote the cardinality of A by $|A|$.

Definition 2.6. A sequence of finite subsets A_n of T is an *averaging sequence* for Γ if for all $\gamma \in \Gamma^{(1)}$ (and then for all $\gamma \in \Gamma$),

$$\lim_{n\to\infty} \frac{|\Delta_\gamma A_n|}{|A_n|} = 0.$$

Proposition 2.7 (Goodman-Plante [15])**.** *An averaging sequence* $\{A_n\}$ *gives rise to a transverse invariant measure* ν *whose support is contained in the limit set* $\lim_{n\to\infty} A_n = \{y \in T \mid \exists y_{n_k} \in A_{n_k} : y = \lim_{k\to\infty} y_{n_k}\}$.

The idea of the proof is the following. Assuming that T is compact, we may construct a Γ-invariant probability measure on T from the sequence of probability measures ν_n defined by $\nu_n(B) = |B \cap A_n|/|A_n|$ for every Borel set $B \subset T$. According to Riesz's representation theorem, each measure ν_n can be identified with a functional I_n on the space $C(T)$ of continuous real-valued functions on T. The functionals I_n are

$$I_n(f) = \frac{1}{|A_n|} \sum_{y \in A_n} f(y).$$

By passing to a subsequence, if necessary, I_n converges in the weak topology to a positive functional I which determines a unique Borel regular measure ν such that $I(f) = \int_T f d\nu$ for every $f \in C(T)$. The averaging condition implies that I and ν are Γ-invariant since for every $\gamma \in \Gamma$ and every $f \in C(T)$ with support on the range of γ, we have

$$|I(f\circ\gamma) - I(f)| \leq \|f\|_\infty \lim_{n\to\infty} \frac{|\Delta_\gamma A_n|}{|A_n|} = 0.$$

Finally, it is clear that $\nu(T) = 1$ and $supp(\nu) = \lim_{n\to\infty} A_n$.

In the non-compact case, by replacing Γ and Γ_1 with suitable reductions we can assume, without loss of generality, that the fundamental equivalence between the holonomy pseudogroup Γ and its reduction Γ_1 to a relatively compact open subset T_1 of T becomes a Kakutani equivalence for some compact generation systems on T and T_1. Then, any averaging sequence A_n for Γ defines an averaging sequence $A_n \cap K$ for $\Gamma|_K$ where $K = \overline{T}_1$ is a compact subset of T. Hence, we obtain a probability measure ν_K on K that is invariant under $\Gamma|_K$. Now, we can extend ν_K to a unique Borel measure ν on T which is Γ-invariant and finite on compact sets.

Example 2.8. Consider a graph with bounded geometry, like any orbit $\Gamma(x)$ of the holonomy pseudogroup of a compact laminated space. This graph is said to be *Følner* if it contains a sequence of finite subsets of vertices A_n such that $|\partial A_n|/|A_n| \to 0$, where ∂A_n denotes the boundary set with respect to the graph structure. Since $\Delta_\gamma A \subset \partial A \cup \gamma^{-1}(\partial A)$ for any $\gamma \in \Gamma^{(1)}$, we get that $|\Delta_\gamma A_n| \leq 2|\partial A_n|$, and we have an averaging sequence. In particular, any orbit $\Gamma(x)$ having sub-exponential growth is an example of Følner graph since

$$\liminf_{n\to\infty} \frac{|A_{n+1} - A_{n-1}|}{|A_n|} = 0,$$

where $A_n = \Gamma^{(n)}(x)$.

Using the one-to-one correspondence between foliated cycles and transverse invariant measures stablished by D. Sullivan [34], it is not difficult to show the following continuous version of Goodman-Plante's result:

Proposition 2.9 (Goodman-Plante [15]). *Let $\{V_n\}$ be an averaging sequence for $\mathcal{F}$, i.e. a sequence of compact domains V_n (of dimension d) in the leaves such that*

$$\lim_{n\to\infty} \frac{area(\partial V_n)}{vol(V_n)} = 0$$

where area denotes the $(d-1)$-volume and vol the d-volume with respect to the complete Riemannian metric along the leaves. Then $\{V_n\}$ gives rise to a transverse invariant measure ν whose support is contained in the saturated limit set $\lim_{n\to\infty} V_n = \{\, p \in M \,/\, \exists p_{n_k} \in V_{n_k} : p = \lim_{k\to\infty} p_{n_k} \,\}$.

Recall that a *foliated d-form* $\alpha \in \Omega^d(\mathcal{F})$ is a family of differentiable d-forms over the plaques of $\mathcal{A}$ depending continuously on the transverse parameter and which agree on the intersection of each pair of foliated charts. A *foliated r-cycle* is a continuous linear functional $\xi : \Omega^d(\mathcal{F}) \to \mathbb{R}$ strictly positive on strictly positive forms and null on exact forms with respect to the leafwise exterior derivative $d_{\mathcal{F}}$. Any averaging sequence V_n defines the sequence of foliated currents

$$\xi_n(\alpha) = \frac{1}{\mathrm{vol}(V_n)} \int_{V_n} \alpha$$

where α is a foliated d-form. By passing to a subsequence, if necessary, we have a limit current $\xi = \lim_{n\to\infty} \xi_n$. Since the boundaries of the domains V_n vanish asymptotically, Stokes' theorem implies that ξ is a foliated d-cycle [34].

3 Averaging sequences in the discrete setting

The main objective of this section is to prove the existence of a harmonic measure for an étale equivalence relation $\mathcal{R}$ that contains a modified averaging sequence. Initially, we will assume that $\mathcal{R}$ is given by a free action of a pseudogroup Γ on a compact space T, but some generalizations will be discussed later. In Section 3.1, we will define a weighted measure on the equivalence classes, that will allow us to recall the notion of *modified averaging sequence* introduced by V. A. Kaimanovich in [21] and [23]. Given a continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$, the *Radon-Nikodym problem* is to determine the set of probability measures ν on T which are quasi-invariant and admit δ as their Radon-Nikodym derivative [32]. Theorem 3.6 gives a positive answer to this problem in the presence of a modified averaging sequence.

3.1 Quasi-invariants measures

Let ν be a quasi-invariant measure on T. As usual, we will assume that ν is a regular Borel measure that is finite on compact sets. Integrating the counting measures on the fibers of the left projection $\beta(y,z) = y$ with respect to ν gives the *left counting measure* $d\widetilde{\nu}(y,z) = d\nu(y)$. Indeed, for each Borel set $A \subset \mathcal{R}$, we define $\widetilde{\nu}(A) = \int |A^y| d\mu(y)$ where $|A^y|$ is the cardinal of the set $A^y = \{z \in T/(y,z) \in A\} \subset \mathcal{R}[y]$. The same is valid for the right projection $\alpha(y,z) = z$ and we get the *right counting measure* $d\widetilde{\nu}^{-1}(y,z) = d\widetilde{\nu}(z,y) = d\nu(z)$. Then $\widetilde{\nu}$ and $\widetilde{\nu}^{-1}$ are equivalent measures if and only if ν is quasi-invariant, in which case the Radon-Nikodym derivative is given by $\delta(y,z) = d\widetilde{\nu}/d\widetilde{\nu}^{-1}(y,z)$. We refer to [28], [21], [31] and [32].

Definition 3.1. A *cocycle with values in* $\mathbb{R}_+^*$ is a map $\delta : \mathcal{R} \to \mathbb{R}_+^*$ satisfying $\delta(x,y)\delta(y,z) = \delta(x,z)$ for all $(x,y),(y,z) \in \mathcal{R}$.

The map δ is known as the *Radon-Nikodym cocycle* of $(\mathcal{R}, T, \nu)$.

Definition 3.2. Given a cocycle $\delta : \mathcal{R} \to \mathbb{R}_+^*$, the measure $|\cdot|_y$ on $\mathcal{R}[y]$ is given by $|z|_y = \delta(z,y)$ for all $z \in \mathcal{R}[y]$. Then, for a finite subset $A \subset \mathcal{R}[y]$,

$$|A|_y = \sum_{z\in A} \delta(z,y).$$

3.2 Discrete averaging sequences

We are interested in giving a sufficient condition to solve the Radon-Nikodym problem in the discrete setting. We will state this condition using the notion of modified averaging sequence (see [21] and [23]):

Definition 3.3. Let $\delta : \mathcal{R} \to \mathbb{R}_+^*$ be a cocycle of $\mathcal{R}$. Let $\{A_n\}$ be a sequence of finite subsets of T such that $A_n \subset \mathcal{R}[y_n]$ for each $n \in \mathbb{N}$. We will say that $\{A_n\}$ is a δ*-averaging sequence for* Γ if

$$\lim_{n\to\infty} \frac{|\Delta_\gamma A_n|_{y_n}}{|A_n|_{y_n}} = 0$$

for all $\gamma \in \Gamma^{(1)}$. An equivalence class $\mathcal{R}[y]$ is δ*-Følner* if $\mathcal{R}[y]$ contains an δ-averaging sequence $\{A_n\}$ such that $|\partial A_n|_y/|A_n|_y \to 0$ as $n \to +\infty$.

By choosing a finite generating set for Γ, we can realize each equivalence class $\mathcal{R}[y]$ as the set of vertices of a graph. We will write $z \sim w$ for each pair of neighboring vertices z and w joined by an edge in $\mathcal{R}[y]$, and $deg(z)$ the number of neighbors of $z \in \mathcal{R}[y]$. We will use $\mathcal{D}$ to denote the set of discontinuities of the degree function deg. Let ν be a quasi-invariant measure on T, and denote by $D : L^\infty(T,\nu) \to L^\infty(T,\nu)$ the Markov operator defined by

$$Df(y) = \frac{1}{deg(y)} \sum_{z\sim y} f(z).$$

We denote by D^* the dual operator acting on the space of positive Borel measures on T, and by $\Delta : L^\infty(T,\nu) \to L^\infty(T,\nu)$ the Laplace operator defined by $\Delta f(y) = Df(y) - f(y)$.

Definition 3.4. A quasi-invariante measure ν on T is *harmonic* or *stationary* (for the simple random walk on $\mathcal{R}$) if for every bounded measurable function $f : T \to \mathbb{R}$, we have $\int \Delta f \, d\nu = 0$.

Proposition 3.5 ([29])**.** *For a quasi-invariant measure ν on T the following are equivalent:*

(i) ν is harmonic;

(ii) $D^*\nu = \nu$;

(iii) the Radon-Nikodym cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$ is harmonic, i.e. for ν-almost every $y \in T$ and every $z \in \mathcal{R}[y]$, we have

$$\delta(z, y) = \frac{1}{deg(z)} \sum_{w \sim z} \delta(w, y).$$

Theorem 3.6. *Let $\mathcal{R}$ be the orbit equivalence relation defined by a finitely generated pseudogroup Γ acting freely on a compact space T. Let $\delta : \mathcal{R} \to \mathbb{R}^*_+$ be a continuous cocycle. Then:*

i) any δ-averaging sequence $\{A_n\}$ gives rise to a positive Borel measure ν on T whose support is contained in the limit set of $\{A_n\}$, which is quasi-invariant and has δ as Radon-Nikodym cocycle;

ii) moreover, if δ is harmonic and $\nu(\mathcal{D}) = 0$, then ν is a harmonic measure.

Proof. We start by constructing a sequence of probability measures ν_n given by $\nu_n(B) = |B \cap A_n|_{y_n} / |A_n|_{y_n}$ for every Borel subset B of T. By passing to a subsequence, the sequence ν_n converges in the weak topology to a positive Borel measure ν on T. First, we will prove that ν is a quasi-invariant measure having a Radon-Nikodym cocycle equal to δ. For every local transformation $\gamma \in \Gamma$ an every function $f \in C(T)$ with support on the range of γ, we have

$$\int f(z) \, d(\gamma_* \nu)(z) = \int f(\gamma(y)) \, d\nu(y) = \lim_{n \to \infty} \frac{1}{|A_n|_{y_n}} \sum_{y \in A_n} f(\gamma(y)) \delta(y, y_n)$$

and

$$\begin{aligned} \int f(y)\delta(z, y) \, d\nu(y) &= \lim_{n \to \infty} \frac{1}{|A_n|_{y_n}} \sum_{y \in A_n} f(y)\delta(\gamma(y), y)\delta(y, y_n) \\ &= \lim_{n \to \infty} \frac{1}{|A_n|_{y_n}} \sum_{y \in A_n} f(y)\delta(\gamma(y), y_n) \end{aligned}$$

where $z = \gamma(y)$. Therefore

$$
\begin{aligned}
0 &\leq \left| \int f(z)\, d(\gamma_* \nu)(z) - \int f(y)\delta(z,y)\, d\nu(y) \right| \\
&\leq \lim_{n\to\infty} \frac{1}{|A_n|_{y_n}} \left| \sum_{y\in A_n} f(\gamma(y))\delta(y,y_n) - f(y)\delta(\gamma(y),y_n) \right| \\
&\leq \lim_{n\to\infty} \|f\|_\infty \frac{|\Delta_\gamma A_n|_{y_n}}{|A_n|_{y_n}} \;=\; 0
\end{aligned}
$$

and thus

$$
\int f(z)\, d(\gamma_*\nu)(z) = \int f(y)\delta(z,y)\, d\nu(y),
$$

proving the claim.

We will now prove that if δ is harmonic and $\nu(\mathcal{D}) = 0$, then ν is a harmonic measure. Observe that if $\nu(\mathcal{D}) = 0$, then Δf is continuous ν-almost everywhere and therefore

$$
\int \Delta f\, d\nu = \lim_{n\to\infty} \int \Delta f\, d\nu_n
$$

for all $f \in C(T)$. If δ is harmonic, we have

$$
\begin{aligned}
\int \Delta f(y)\, d\nu_n(y) &= \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} \left(\frac{1}{deg(y)} \sum_{z\sim y} f(z) - f(y) \right) \delta(y,y_n) \\
&= \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} \frac{1}{deg(y)} \sum_{z\sim y} f(z)\delta(y,y_n) - f(y) \left(\frac{1}{deg(y)} \sum_{z\sim y} \delta(z,y_n) \right) \\
&= \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} \frac{1}{deg(y)} \sum_{z\sim y} f(z)\delta(y,y_n) - f(y)\delta(z,y_n)
\end{aligned}
$$

and then

$$
\begin{aligned}
0 &\leq \left| \int \Delta f(y)\, d\nu(y) \right| \\
&\leq \lim_{n\to\infty} \frac{1}{|A_n|_{y_n}} \left| \sum_{y\in A_n} \sum_{z\sim y} f(z)\delta(y,y_n) - f(y)\delta(z,y_n) \right| \\
&\leq \lim_{n\to\infty} \|f\|_\infty \sum_{\gamma\in\Gamma^{(1)}} \frac{|\Delta_\gamma A_n|_{y_n}}{|A_n|_{y_n}} \;\leq\; \lim_{n\to\infty} 2\, \|f\|_\infty\, |\Gamma^{(1)}|\, \frac{|\partial A_n|_{y_n}}{|A_n|_{y_n}} \;=\; 0,
\end{aligned}
$$

that is ν is a harmonic measure. □

A similar result can be found in [33]. In general, the second part of Theorem 3.6 remains valid when the Laplace operator Δ preserves continuous functions. This is always true when $\mathcal{D} = \emptyset$, as in the following case:

Corollary 3.7. *Let $\mathcal{R}$ be the orbit equivalence relation defined by a group of finite type Γ acting freely on a compact space T. Let $\delta : \mathcal{R} \to \mathbb{R}_+^*$ be a continuous harmonic cocycle. Any δ-averaging sequence $\{A_n\}$ gives rise to a harmonic measure ν on T supported by the limit set of $\{A_n\}$.* □

Arguing as for usual averaging sequences, we can extend Theorem 3.6 to any compactly generated pseudogroup Γ acting freely on a locally compact Polish space T. Moreover, in the 0-dimensional case, the degree function is again continuous. This applies in particular to solenoids [5] and laminations defined by repetitive graphs (introduced in [13] and studied in [1], [7] and [26]):

Corollary 3.8. *Let $\mathcal{R}$ be the orbit equivalence relation defined by compactly generated pseudogroup Γ acting freely on a locally compact separable 0-dimensional space T. Let $\delta : \mathcal{R} \to \mathbb{R}_+^*$ be a continuous harmonic cocycle. Any δ-averaging sequence $\{A_n\}$ gives rise to a harmonic measure ν on T supported by the limit set of $\{A_n\}$.* □

In order to extend Theorem 3.6 to non-free actions, we can adopt two different strategies. Let us first recall that the notion of equivalence relation is enough to describe the transverse structure of a lamination in the Borel context. More precisely, any Borel or topological lamination $\mathcal{F}$ induces a Borel equivalence relation $\mathcal{R}$ on a total transversal T (compare to Remark 2.1) defined by the action of the holonomy pseudogroup. We refer to the Ph.D. thesis of M. Bermúdez [6] for the definition of a Borel lamination. If $\mathcal{R}$ is a discrete Borel equivalence relation defined by the action of a Borel pseudogroup Γ acting on a compact space T and if $\delta : \mathcal{R} \to \mathbb{R}_+^*$ is a Borel cocycle, then the proof of Theorem 3.6 remains valid. In the topological context, Theorem 3.6 is not exactly equivalent to the situation above because the transverse holonomy groupoid and the equivalence relation are only Borel isomorphic on the residual set of leaves without holonomy. Another strategy consists in replacing étale equivalence relations with étale groupoids, and proving that averaging sequences for stationary cocycles define stationary measures on groupoids. Details will be reported elsewhere.

4 Averaging sequences in the continuous setting

We are interested in stating Theorem 3.6 in the continuous setting, namely for a compact laminated space $(M, \mathcal{F})$. Instead of working with quasi-invariant measures, we are going to use tangentially smooth measures. These form a larger class than harmonic measures. As previously mentioned, transverse invariant measures for foliations are rather rare, but harmonic measures always exist. Harmonic measures were introduced by L. Garnett in [11]. In Sections 4.1 and 4.2 we will study these measures and recall some notation. In Section 4.3 we will construct a differential foliated 1-form from a given cocycle. Finally, in Section 4.4 we will use this foliated form to prove the continuous analogue of Theorem 3.6.

4.1 Tangentially smooth measures

Consider now a regular Borel measure μ on M. Using a C^r foliated atlas $\mathcal{A}$, we can give a local decomposition $\mu = \int \lambda_i^y d\nu_i(y)$ on each foliated chart U_i, where λ_i^y is a measure on the plaque $\varphi_i^{-1}(P_i \times \{y\})$ and ν_i a measure on T_i. Here, in order to define the foliated Laplace operator $\Delta_{\mathcal{F}}$, we can always assume that $r \geq 3$ up to C^1-equivalence of foliated atlases, and we fix a tangentially C^r-smooth Riemannian metric g along the leaves of $\mathcal{F}$.

Definition 4.1 ([2])**.** A measure μ on M is *tangentially smooth* if for every $i \in I$ and ν_i-almost every $y \in T_i$, the measures λ_i^y are absolutely continuous with respect to the Riemannian volume *dvol* restricted to the plaque passing through y, and the density functions $h_i(x,y) = d\lambda_i^y/dvol(x,y)$ are smooth functions of class C^{r-1} on the plaques.

Observe that the local decomposition of μ is not necessarily unique. Let $\mu|_{U_i} = \int \lambda_i^y d\nu_i(y) = \int \bar{\lambda}_i^y d\bar{\nu}_i(y)$ be two decompositions. Then we obtain

$$\int_{T_i} \int_{P_i \times \{y\}} h_i(x,y)\, dvol(x,y)\, d\nu_i(y) = \int_{T_i} \int_{P_i \times \{y\}} \bar{h}_i(x,y)\, dvol(x,y)\, d\bar{\nu}_i(y),$$

and we can consider the Radon-Nikodym derivative $\delta_i(y) = d\nu_i/d\bar{\nu}_i(y)$ such that $\bar{h}_i(x,y) = \delta_i(y) h_i(x,y)$. This situation arises naturally in the intersection of two foliated charts U_i and U_j. Indeed, if $U_i \cap U_j \neq \emptyset$, we have that $\mu|_{U_i \cap U_j} = \int \lambda_i^y d\nu_i(y) = \int \lambda_j^y d\nu_j(y)$. Thus, as before, we deduce that

$$\delta_{ij}(y) = d\nu_i/d((\gamma_{ji})_* \nu_j)(y) = \frac{h_j(\varphi_{ij}^y(x), \gamma_{ij}(y))}{h_i(x,y)}. \tag{4.1}$$

Then the functions h_i verify that $\log h_j - \log h_i = \log \delta_{ij}$ on $U_i \cap U_j$. Since δ_{ij} is a function on T_i, we have that $d_{\mathcal{F}} \log h_i = d_{\mathcal{F}} \log h_j$. Then $\eta = d_{\mathcal{F}} \log h_i$ is a well-defined foliated 1-form of class C^{r-2} along the leaves, which makes possible to estimate the transverse measure distortion under the holonomy.

Definition 4.2. The foliated 1-form η is the *modular form* of μ.

4.2 Harmonic measures

We start by recalling the definition given by L. Garnett in [11]:

Definition 4.3. We will say that μ is *harmonic* if $\int \Delta_{\mathcal{F}} f d\mu = 0$ for every continuous tangentially C^{r-1}-smooth function $f : M \to \mathbb{R}$.

According to Theorem 1 of [11], any harmonic measure is an example of tangentially smooth measure since the densities h_i are positive harmonic functions of class C^{r-1} on the plaques. In particular, any transverse invariant measure combined with the Riemannian volume on the leaves gives a harmonic measure which is called *completely invariant*. A harmonic measure μ is completely invariant if and only if $\eta = 0$ (we refer to corollary 5.5 of A. Candel's paper [8]).

In the general harmonic case, the following proposition states some properties of the modular form. This proposition is a refined version of Lemma 4.19 on page 116 of the Ph.D. thesis of B. Deroin [9].

Proposition 4.4 ([9])**.** *If μ is a harmonic measure, then η is a bounded foliated 1-form which admits a uniformly tangentially Lipschitz primitive $\log h$ on the residual set of leaves without holonomy.*

Proof. Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ be a good C^r foliated atlas of $(M, \mathcal{F})$, and h_i the local density functions of μ. Let us first observe that since the functions h_i coincide on the intersections of the plaques modulo multiplication by a constant, they define a primitive of the induced 1-form on the holonomy covering of each leaf L. If $\mathcal{F}$ has no essential holonomy, the functions $\log h_i$ can be glued together to obtain a measurable global primitive $\log h$ of η. In general, the modular form η admits a continuous primitive $\log h$ on the residual set of leaves without holonomy. Now, let us assume that $\mathcal{A}$ is a refinement of a good atlas $\mathcal{A}' = \{(U'_i, \phi'_i)\}_{i \in I}$, and h'_i are the corresponding local densities. Thus, every plaque of U_i is relatively compact in a plaque of U'_i. In fact, using a vertical reparametrization, we can suppose that $\phi_i^{-1}(P_i \times \{y\}) \subset (\phi'_i)^{-1}(P'_i \times \{y\})$ for every $y \in T_i$. There exists a relatively compact open set $V \subset P'_i$ such that $\phi_i^{-1}(P_i \times \{y\}) \subset (\phi'_i)^{-1}(V \times \{y\})$ for every $y \in T_i$. Since h_i is harmonic, the Harnack inequality implies the existence of a constant $C_i > 0$ such that

$$\frac{1}{C_i} \leq \frac{h_i(x, y)}{h_i(x_0, y)} \leq C_i, \tag{4.2}$$

for all $x, x_0 \in P_i$ and for all $y \in T_i$. Since the atlases $\mathcal{A}$ and $\mathcal{A}'$ are finite, the primitive $\log h$ is uniformly Lipschitz in the tangential coordinate x. □

4.3 Modular form associated to a cocycle

We will now describe how to construct a modular form $\eta \in \Omega^1(\mathcal{F})$ from a Borel or continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$. For simplicity, $\mathcal{R}$ is endowed here with the natural Borel or topological structure induced by the structure of Borel or topological groupoid on the transverse holonomy groupoid G formed by the germs $< \gamma >_y$ of the elements γ of Γ at the points y of their domains, see [28]. The natural projection $(\beta, \alpha) :< \gamma >_y \in G \mapsto (y, \gamma(y)) \in \mathcal{R}$ becomes an isomorphism of Borel or topological groupoids in restriction to the residual set of leaves without holonomy. Equivalently, we can consider a Borel or continuous cocycle $\delta : G \to \mathbb{R}^*_+$ projectable on $\mathcal{R}$.

We start by considering tangentially C^r-smooth Borel or continuous functions $c_{ki} : U_i \cap U_k \to \mathbb{R}$ given by

$$c_{ki}(\varphi_k^{-1}(x, y)) = log\, \delta_{ki}(y)$$

where $\delta_{ki}(y) = \delta(y, \gamma_{ki}(y))$ for all $(x, y) \in P_k \times T_k$. By choosing a tangentially C^r-smooth partition of unity $\{\rho_i\}_{i=1}^m$ subordinated to the foliated atlas $\mathcal{A}$, we can glue the functions c_{ki} obtaining tangentially C^r-smooth Borel or continuous functions $c_i : U_i \to \mathbb{R}$ given by

$$c_i = \sum_{k=1}^{m} \rho_k c_{ki}.$$

The cocycle condition implies that $c_{ij} = c_{kj} - c_{ki}$, so that

$$c_j - c_i = \sum_{k=1}^{m} \rho_k c_{kj} - \sum_{k=1}^{m} \rho_k c_{ki} = \left(\sum_{k=1}^{m} \rho_k \right) c_{ij} = c_{ij}.$$

Hence, for each $i = 1, \ldots, m$ we can define a tangentially C^{r-1}-smooth Borel or continuous foliated 1-form

$$\eta_i = \sum_{k=1}^{m} (d_{\mathcal{F}} \rho_k)\, c_{ki}$$

on U_i. Each local 1-form η_i is exact

$$\eta_i = \sum_{k=1}^{m} (d_{\mathcal{F}} \rho_k)\, c_{ki} = d_{\mathcal{F}} c_i = d_{\mathcal{F}} \log h_i$$

where $h_i = e^{c_i} : U_i \to \mathbb{R}^*_+$ is a Borel or continuous function of class C^r along the leaves.

Proposition 4.5. *There is a well defined Borel or continuous closed foliated 1-form $\eta \in \Omega^1(\mathcal{F})$ such that $\eta|_{U_i} = \eta_i$.*

Proof. For each pair $i, j \in \{1, \ldots, m\}$, we have that:

$$\eta_j - \eta_i = \sum_{k=1}^{m} (d_{\mathcal{F}} \rho_k)\, c_{kj} - \sum_{k=1}^{m} (d_{\mathcal{F}} \rho_k)\, c_{ki} = \left(\sum_{k=1}^{m} d_{\mathcal{F}} \rho_k \right) c_{ij} = 0$$

on $U_i \cap U_j$. Then the 1-form η is well defined, Borel or continuous, and closed. □

Definition 4.6. The foliated 1-form η is the modular form of δ.

Remarks 4.7. (i) The modular form η depends on the choice of the partition of unity, but its cohomology class does not depend.

(ii) As for harmonic measures, the modular form η of a Borel or continuous cocycle δ admits a Borel or continuous primitive $\log h$ on the residual set of leaves without holonomy. Thus, assuming that $\mathcal{F}$ has no holonomy (or passing to the holonomy covers of the leaves), we may find a global Borel or continuous primitive on M (respectively, a Borel or continuous primitive on the holonomy groupoid $Hol(\mathcal{F})$), see [2].

4.4 Continuous averaging sequences

In the present setting, we can reformulate the *Radon-Nikodym problem* as the problem of determining tangentially smooth measures μ on M which admit η as their modular form. The aim of this section is to establish Theorem 3.6 for laminations. First, we need a continuous analogue of Definition 3.3. Consider a d-dimensional lamination $\mathcal{F}$ of class C^r on a compact space M, endowed with a tangentially C^r-smooth Riemannian metric g, and a continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$. The modular form η admits a continuous tangentially C^r-smooth primitive $\log h$ on the residual set of leaves without holonomy. On each leaf without holonomy L_y passing through $y \in T$, we can multiply g by the normalized density function $h/h(y)$ in order to obtain a *modified metric* $(h/h(y))g$.

Definition 4.8. Let $\{V_n\}$ be a sequence of compact domains with boundary contained in a sequence of leaves without holonomy L_{y_n}. We will say that $\{V_n\}$ is a *η-averaging sequence for $\mathcal{F}$* if

$$\lim_{n\to\infty} \frac{\mathrm{area}_\eta(\partial V_n)}{\mathrm{vol}_\eta(V_n)} = 0$$

where area_η denotes the $(d-1)$-volume and vol_η the d-volume with respect to the modified metric along L_{y_n}. A leaf L_y is *η-Følner* if it contains an η-averaging sequence $\{V_n\}$ such that $\mathrm{area}_\eta(\partial V_n)/\mathrm{vol}_\eta(V_n) \to 0$ as $n \to +\infty$.

Remarks 4.9. (i) The isoperimetric ratio $\mathrm{area}_\eta(\partial V_n)/\mathrm{vol}_\eta(V_n)$ does not depend on the choice of y nor h in the second definition. This justifies the notation, which is slightly different from the one used in [2].

(ii) When μ is a completely invariant harmonic measure, the normalized density function is equal to 1 and thus the modified volume and the Riemannian volume coincide. Hence, we recover the common definition of averaging sequence.

(iii) For harmonic measures, Harnack's inequalities (4.2) imply that the modified volume of the plaques and the modified area of their boundaries remain uniformly bounded.

Theorem 4.10. *Let $(M, \mathcal{F})$ be a C^r lamination of a compact space M, $1 \leq r \leq \infty$, and let $\mathcal{R}$ be the equivalence relation induced by $\mathcal{F}$ on a total transversal T. Consider a continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$, and let η be the modular form of δ. Assume that $\mathcal{F}$ admits a foliated atlas such that the modified volume of the plaques is bounded. Then:*

i) any η-averaging sequence $\{V_n\}$ for $\mathcal{F}$ gives rise to a tangentially smooth measure μ whose support is contained in the limit set of $\{V_n\}$ and whose modular form is equal to η;

ii) moreover, if η has a primitive $\log h$ such that h is a harmonic function, then μ is a harmonic measure.

Proof. As in the discrete case, we will start by constructing a sequence of foliated d-currents

$$\xi_n(\alpha) = \frac{1}{\mathrm{vol}_\eta(V_n)} \int_{V_n} \frac{h}{h(y_n)} \alpha,$$

where α is a foliated d-form. By passing to a subsequence, the sequence ξ_n converges to a foliated d-current ξ. Let μ be the measure on M associated with the current ξ. For every function $f \in C(T)$, we have $\int f \, d\mu = \xi(f\omega)$ where $\omega = dvol$ is the volume form along the leaves.

Now, we will prove that μ is a tangentially smooth measure with modular form η. Consider a good C^r foliated atlas $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ obtained by refinement from a given good atlas, and whose plaques have bounded modified volume. As we mentioned before, up to C^1-equivalence, we can assume now that $r \geq 3$. Since the modified volume of the plaques of $\mathcal{A}$ and the modified area of their boundaries remain bounded, the traces $A_n = V_n \cap T$ of the domains V_n on the total transversal T form a δ-averaging sequence, as in Definition 3.3. In fact, since V_n is covered by the plaques P_y of $\mathcal{A}$ centered at the points y of A_n, we have that:

$$\mathrm{vol}_\eta(V_n) = \int_{V_n} \omega_\eta \leq \sum_{y \in A_n} \int_{P_y} \omega_\eta = \sum_{y \in A_n} \left(\int_{P_y} \frac{h(x,y)}{h(0,y)} dvol(x,y) \right) \delta(y, y_n)$$

where ω_η is the modified volume form along the leaves and $h(x,y)$ denotes the density function restricted to a foliated chart U_y containing the plaque P_y. Then there is a constant $C > 0$ such that $\mathrm{vol}_\eta(V_n) \leq C|A_n|_{y_n}$ Actually, we can choose $C > 0$ such that $\frac{1}{C} \leq \mathrm{vol}_\eta(V_n)/|A_n|_{y_n} \leq C$. Thus, by passing to a subsequence, we may assume that the ratio $\mathrm{vol}_\eta(V_n)/|A_n|_{y_n}$ converges to a constant $c > 0$. Now, as stated in the proof of Theorem 3.6, we may also assume that the sequence of measures $\nu_n(B) = |B \cap A_n|_{y_n}/|A_n|_{y_n}$ converge to a quasi-invariant measure ν on T whose Radon-Nikodym derivative is equal to δ. Combined with the modified Riemannian volume along the leaves, this transverse measure gives us a tangentially smooth measure μ' on M. Thus, for every function $f \in C(M)$ with support in U_i, we have

$$\int f \, d\mu' = \int_{T_i} \int_{P_i \times \{y\}} f(x,y) \, \frac{h_i(x,y)}{h_i(0,y)} \, dvol(x,y) \, d\nu(y).$$

Then

$$\begin{aligned} \int f \, d\mu' &= \lim_{n \to +\infty} \frac{1}{|A_n|_{y_n}} \sum_{y \in V_n \cap T_i} \left(\int_{P_i \times \{y\}} f(x,y) \, \frac{h_i(x,y)}{h_i(0,y)} \, dvol(x,y) \right) \delta(y, y_n) \\ &= \lim_{n \to +\infty} \frac{1}{|A_n|_{y_n}} \sum_{y \in V_n \cap T_i} \int_{P_i \times \{y\}} f \, \omega_\eta. \end{aligned} \tag{4.3}$$

On the other hand, by definition, we have

$$\begin{aligned}\int f\,d\mu &= \xi(f\omega) = \lim_{n\to+\infty}\frac{1}{\mathrm{vol}_\eta(V_n)}\int_{V_n} f\omega_\eta \\ &= \lim_{n\to+\infty}\frac{1}{\mathrm{vol}_\eta(V_n)}\sum_{y\in V_n\cap T_i}\int_{P_i\times\{y\}} f\,\omega_\eta \qquad (4.4)\end{aligned}$$

Comparing identities (4.3) and (4.4), we deduce that $\mu = \frac{1}{c}\,\mu'$ is a tangentially smooth measure with modular form η.

To conclude, we will prove that μ is harmonic when h is harmonic. We will start by denoting $h_n = h/h(y_n)$ the normalized density functions on the leaves L_{y_n}. Since the Laplace operator $\Delta_{\mathcal{F}}$ preserves continuous functions, we have that

$$\int \Delta_{\mathcal{F}} f\,d\mu = \lim_{n\to\infty}\frac{1}{\mathrm{vol}_h(V_n)}\int_{V_n}(\Delta_{\mathcal{F}} f)\,h_n\,\omega,$$

for all $f \in C(T)$. Green's formula implies that

$$\int_{V_n}(\Delta_{\mathcal{F}} f)\,h_n\,\omega = \int_{V_n}((\Delta_{\mathcal{F}} f)\,h_n - f\,(\Delta_{\mathcal{F}} h_n)\,\omega = \int_{\partial V_n} h_n\,\iota_{grad(f)}\omega - f\,\iota_{grad(h_n)}\omega.$$

Since h_n is harmonic, we have

$$\int_{\partial V_n}\iota_{grad(h_n)}\omega = \int_{V_n}\mathrm{div}(grad(h_n))\,\omega = \int_{V_n}(\Delta_{\mathcal{F}} h_n)\,\omega = 0$$

and then

$$0 \le \left|\int_{\partial V_n} f\iota_{grad(h_n)}\omega\right| \le \|f\|_\infty \int_{\partial V_n}\iota_{grad(h_n)}\omega = 0$$

for all $n \in \mathbb{N}$. On the other hand, since f is bounded, there exists a constant $k > 0$ depending only on f such that we have

$$0 \le \left|\frac{1}{\mathrm{vol}_h(V_n)}\int_{\partial V_n} h_n\,\iota_{grad(f)}\omega\right| \le \lim_{n\to\infty} k\frac{\mathrm{area}_\eta(\partial V_n)}{\mathrm{vol}_\eta(V_n)} = 0$$

and therefore

$$\int \Delta_{\mathcal{F}} f\,d\mu = \lim_{n\to\infty}\frac{1}{\mathrm{vol}_h(V_n)}\int_{V_n}(\Delta_{\mathcal{F}} f)\,h_n\,\omega = 0,$$

that is μ is a harmonic measure. □

Remarks 4.11. (i) If $\delta : \mathcal{R} \to \mathbb{R}^*_+$ is a Borel cocycle with modular form η, Theorem 4.10 remains also valid. So any η-averaging sequence for $\mathcal{F}$ gives rise to a tangentially smooth measure μ that is harmonic when η admits a primitive $\log h$ such that h is a harmonic function.

(ii) According to Remark 4.7.(ii), the notion of η-Følner may be applied to the holonomy covers of the leaves of $\mathcal{F}$. Thus, it suffices to replace $\mathcal{F}$ with the lifted lamination in the holonomy groupoid $Hol(\mathcal{F})$, in order to globalize the previous result. As in the discrete setting, details will be precised elsewhere.

5 Examples

5.1 Discrete averaging sequences for amenable non Følner actions.

There are amenable actions of non amenable discrete groups whose orbits contain averaging sequences [23]. For example, let $\partial\Gamma$ be the space of ends of the free group Γ with two generators α and β whose elements are infinite words $x = \gamma_1\gamma_2\dots$ with letters γ_n in $\Phi = \{\alpha^{\pm 1}, \beta^{\pm 1}\}$. If ν denotes the equidistributed probability measure on $\partial\Gamma$ (such that all cylinders consisting of infinite words with fixed first n letters have the same measure), then Γ acts essentially freely on $\partial\Gamma$ by sending each generator γ and each infinite word $x = \gamma_1\gamma_2\dots$ to $\gamma.x = \gamma\gamma_1\gamma_2\dots$. Since this action is amenable, according to Theorem 2 of [21] (see also Proposition 4.1 of [2]), we know that ν-almost every orbit is δ-Følner (where δ is the Radon-Nikodym derivative of ν). We will recall here an explicit construction by V. A. Kaimanovich in [23].

For each $x \in \partial\Gamma$, let $b_x : \Gamma \to \mathbb{R}$ be the *Busemann function* defined by

$$b_x(\gamma) = \lim_{n\to+\infty} \big(d_\Gamma(\gamma, x_{[n]}) - d_\Gamma(1, x_{[n]})\big)$$

where d_Γ is the Cayley graph metric, $x_{[n]}$ is the word consisting of first n letters of x and 1 is the identity element. The level sets $H_k(x) = \{\, \gamma \in \Gamma \,/\, b_x(\gamma) = k \,\}$ are the *horospheres* centered at x. The Radon-Nikodym derivative of ν is given by

$$\delta(\gamma^{-1}.x, x) = \frac{d\gamma.\nu}{d\nu}(x) = 3^{-b_x(\gamma)}$$

where $\gamma.\nu$ is the translation of ν by γ. Since $|\cdot|_x = \delta(\cdot, x)$ is a harmonic measure on $\Gamma.x$, ν is also a harmonic measure. In fact, as stated in Theorem 17.4 of [22], ν is the unique harmonic probability measure on $\partial\Gamma$.

Let A^x_n be the set of all points $\gamma^{-1}.x$ in $\Gamma.x$ such that $0 \le b_x(\gamma) = d_\Gamma(1,\gamma) \le n$. Since $|A^x_n \cap H_k(x)|_x = \sum_{b_x(\gamma)=d_\Gamma(1,\gamma)=k} \delta(\gamma^{-1}.x, x) = 3^k \frac{1}{3^k} = 1$ for all $0 \le k \le n$, we have that $|A_n|_x = n+1$. But $\partial A^x_n = \{1\} \cup (A^x_n \cap H_n(x))$ and so $|\partial A^x_n|_x = 2$. The δ-averaging sequence $\{A^x_n\}$ defines a harmonic measure (which is equal to ν up to multiplication by a constant).

5.2 Averaging sequences for hyperbolic surfaces.

The geodesic and horocycle flows are classical examples of flows on the unitary tangent bundle of a compact hyperbolic surface. They are given by the right actions of the diagonal subgroup

$$D = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \Big| \; t \in \mathbb{R} \right\}$$

and the unipotent subgroup

$$H^+ = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \Big| \; s \in \mathbb{R} \right\}$$

of $G = PSL(2,\mathbb{R})$ on the quotient $\Gamma\backslash G$ by the left action of a uniform lattice Γ. If $\mathbb{H}$ denotes the hyperbolic plane, we can identify $\Gamma\backslash G$ with the unitary tangent bundle of the compact hyperbolic surface $\Gamma\backslash\mathbb{H}$. The right action of the normalizer A of H^+ in $PSL(2,\mathcal{R})$ defines a foliation $\mathcal{F}$ by Riemann surfaces on $\Gamma\backslash G$. Since A is an amenable group, $\mathcal{F}$ is an amenable non Følner foliation. Moreover, there is an A-invariant measure μ on $\Gamma\backslash G$. In [11], L. Garnett proved that μ is a harmonic measure by describing its density function on a foliated chart.

We can identify G/A with the boundary $\partial\mathbb{H}$ by sending each coset of A in G to the center of the horocycle defined by the corresponding coset of H^+ in G. For each point $z \in \mathbb{H}$, there is a unique probability measure ν_z on $\partial\mathbb{H}$ which is invariant by the action of all isometries of $\mathbb{H}$ fixing z. This measure is the image of the normalized Lebesgue measure on the circle of the tangent plane at z under the exponential map, and is called the *visual measure* at z. According to Proposition 2 of [11], the normalized density function is given by $d\nu_z/d\nu_{z_0}(x)$ where $z, z_0 \in \mathbb{H}$ and $x \in \partial\mathbb{H}$. In particular, for $x = \infty$, we have that

$$\frac{d\nu_z}{d\nu_{z_0}}(\infty) = \frac{y}{y_0}$$

where $z = x + iy$ and $z_0 = x_0 + iy_0$. In the leaf passing through $x = \infty$, the sequence
$V_n^\infty = \{\, z \in \mathbb{H} \,|\, -1 \leq x \leq 1 \,,\, e^{-n} \leq y \leq 1 \,\}$ becomes a η-averaging sequence (where η is the modular form of μ). Indeed, on the one hand, we have that

$$\text{area}_\eta(V_n^\infty) = \int_{V_n^\infty} \frac{d\nu_z}{d\nu_i}(\infty)\, dvol(z) = \int_{V_n^\infty} y\, \frac{dx \wedge dy}{y^2} = \int_1^1 dx \int_{e^{-n}}^1 \frac{dy}{y} = 2n.$$

On the other hand, the modified length of a smooth curve $\sigma(t) = x(t) + iy(t)$ (with $0 \leq t \leq l$) is given by $\text{length}_\eta(\sigma) = \int_0^l \sqrt{x'(t)^2 + y'(t)^2} dt$, and so we have that

$$\text{length}_\eta(\partial V_n^\infty) = 2(2 + (1 - e^n)) \leq 6.$$

As before, this η-averaging sequence defines a harmonic measure (which is equal to μ up to multiplication by a constant). In fact, all leaves are η-Følner since for each point $x \in \partial\mathbb{H}$ obtained as the image of ∞ under $g \in G$, the sets $V_n^x = g(V_n^\infty)$ form a η-averaging sequence in the leaf passing through x.

5.3 Averaging sequences for torus bundles over the circle.

To conclude, we will present other examples of foliations on homogeneous spaces studied by É. Ghys and V. Sergiescu in [14]. Each matrix $A \in SL(2,\mathbb{Z})$ with $|tr(A)| > 2$ defines a natural representation $\varphi : \mathbb{Z} \to Aut(\mathbb{Z}^2)$ which extends to a representation $\Phi : \mathbb{R} \to Aut(\mathbb{R}^2)$ given by $\Phi(t) = A^t$. If $\lambda > 1$ and $\lambda^{-1} < 1$ are the eigenvalues of A, then Φ is conjugated to the representation Φ_0 given by

$$\Phi_0(t) = \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{pmatrix}.$$

Let T_A^3 be the homogeneous space obtained as the quotient of the Lie group $G = \mathbb{R}^2 \rtimes_\Phi \mathbb{R}$ with group law $(x,y,t).(x',y',t') = ((x,y)+A^t(x',y'),t+t')$ by the uniform lattice $\Gamma = \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$ with a similar law. Observe that G is isomorphic to the solvable group $Sol^3 = \mathbb{R}^2 \rtimes_{\Phi_0} \mathbb{R}$ with group law $(x,y,t).(x',y',t') = (x+\lambda^t x', y+\lambda^{-t}y', t+t')$ (where x and y are the first and second coordinate with respect to the eigenbasis) and T_A^3 is diffeomorphic to the quotient of Sol^3 by a uniform lattice Γ_0. The right action of the image A of the monomorphism

$$(a,b) \in \mathbb{R} \rtimes \mathbb{R}_+^* \mapsto \left(a, 0, \frac{\log b}{\log \lambda}\right) \in Sol^3$$

defines a foliation $\mathcal{F}$ on T_A^3. The Lebesgue measure on T_A^3 defined by the volume form $\Omega = dx \wedge dy \wedge dt$ is a tangentially smooth measure. Since the Riemannian volume along the right orbits is given by

$$\frac{da \wedge db}{b^2} = (\log \lambda)\, \lambda^{-t} dx \wedge dt$$

the density function is equal to $\frac{\lambda^t}{\log \lambda}$. In the orbit of the identity element, the sequence $V_n = \{\,(a,b) \in A \,/\, -1 \leq a \leq 1\,,\, e^{-n \log \lambda} \leq b \leq 1\,\}$ becomes a η-averaging sequence (where η is the modular form of μ). Indeed, on one hand, we have that

$$\text{area}_\eta(V_n) = \int_{V_n} \frac{1}{\log \lambda} \lambda^t (\log \lambda) \lambda^{-t} dx \wedge dt = \int_1^1 dx \int_{-n}^0 dt = 2n.$$

On the other hand, the modified length of a smooth curve $\sigma(t) = (a(t), b(t)$ (with $0 \leq t \leq L$) is given by $\text{length}_\eta(\sigma) = \int_0^L \sqrt{a'(t)^2 + b'(t)^2} dt$, and so we have that

$$\text{length}_\eta(\partial V_n) = 2(2 + (1 - e^{n \log \lambda})) \leq 6.$$

By replacing the orbit corresponding to $y = 0$ with another orbit, it is easy to see that all leaves are η-Følner. As in the previous example, all η-averaging sequences define (up to multiplication by a constant) the same harmonic measure: the Lebesgue measure.

6 Final comments

6.1 Discrete and continuous averaging sequences

Comparing the discrete and continuous settings, a natural question arises: what is the relation between δ-averaging and η-averaging sequences? Let us first notice that repeating the same argument as in the classical case (see Theorem 4.1 of [24]), the boundedness condition derived from Harnack's inequalities in Remark 4.9.(iii) implies that *the leaf L_y is η-Følner if and only if the equivalence class $\mathcal{R}[y]$ is δ-Følner*. But then, what is the relation between the harmonic measures defined by δ-averaging and η-averaging sequences? In this case, the answer is more subtle, and we have to use an important result of R. Lyons and D. Sullivan [27], completed later by V. A. Kaimanovich [20] and independently by W. Ballman and F. Ledrappier [4], about the discretization of harmonic functions on Riemannian manifolds. First, according to Theorem 6 of [27], if μ is a harmonic measure, then the transverse measure ν (well defined up to equivalence) is π-harmonic where π is a transition kernel defining a random walk on $\mathcal{R}$ different from the simple random walk considered in Definition 3.4. Reciprocally, assuming that T admits a relatively compact neighborhood which meets almost every leaf in a recurrent set, the Main Theorem of [4] implies that μ is harmonic if ν is π-harmonic.

6.2 Amenability

It is not casual that all examples in Section 5 are amenable: according to a resul by V. A. Kaimanovich [21], amenable foliations admit always averaging sequences. In fact, if $\mathcal{F}$ is an amenable foliation with respect to a tangentially smooth measure μ, then $\mathcal{F}$ is η- Følner, i.e. μ-almost every leaf is η-Følner, see Proposition 4.3 of [2]. This paper can be viewed as a sequel of [2] where we proved that minimal η-Følner foliations are μ-amenable (assuming that the modified volume of the plaques is bounded). To complete the series, we have to prove that any foliation is amenable with respect to a tangentially smooth measure μ constructed from an averaging sequence using Theorem 4.10.

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