Consequence:

$G = \langle S \rangle$ hyp group. $\exists L$ s.t. if

$s_1 \cdots s_n = 1$ with $s_i \in S$, then some

subword of length $\leq L$ is not geodesic.

(i.e, for some $j - i \leq L$ have $|s_i \cdots s_j| < j - i$)

pf: Take L as above. If every subword $\leq L$ is geod

$s_1, \dots, s_n$ is a L-local geod in Cayley graph,

hence $(2,0)$ quasi-geod & $d(s_n, id) \geq \frac{n}{2} > 0$ $\boxtimes$

Dehn's Algorithm for word problem

☆ linear time!

to decide if $w = s_1 \cdots s_n = 1$,

Check for a non geodesic subword of Length L

- If find one, shorten it to get w'. Repeat
- If none exists, $w \neq 1$

$\Longrightarrow$

- Every hyp group is finitely presented;
  $G = \langle S \mid R \rangle$ where $R = \left\{ \begin{array}{l} \text{trivial words of} \\ \text{length } \leq 2L \end{array} \right\}$
- for $w = s_1 \cdots s_n = 1$
  $\text{area}(w) = \min \left\{ k \mid s_1 \cdots s_n = \begin{array}{l} \text{product of } k \text{ conjugates} \\ \text{of relators} \end{array} \right\}$

  in free group

  combinatorial area in Cayley 2-complex

Algorithm shows $\text{area}(w) \leq n$

☆ hyp groups have linear isoperimetric inequality!

(Gromov): Any space with subquadratic isoperimetric function is hyperbolic.

G finitely generated by S $\rightsquigarrow$ word metric $|\cdot|_S$.

$\sigma_n = \#\{ g \in G \mid |g|_S = n \}$ size of sphere of radius n

(Cannon) If G is hyperbolic, then power series

$$\sum_{n=0}^{\infty} \sigma_n z^n$$ defines a rational function $\frac{p(z)}{q(z)}$

(numbers encoded by finite amount of data)

IV – Extensions

Now that we know hyperbolic spaces are great, natural question is:

Q: How can we build new hyperbolic spaces / groups, (say out of hyperbolic pieces)?

Bestvina – Feighn Combination Theorem '92

Suppose X is a finite graph of spaces st
- univ. covers $\tilde{X}_{e_i}$, $\tilde{X}_v$ of edge + vertex spaces are δ-hyp
- gluings $\tilde{X}_e \hookrightarrow \tilde{X}_v$ are qi-embeddings
- the "flaring condition" is satisfied.

Then the univ cover $\tilde{X}$ is Gromov hyperbolic

- Applies to graphs of hyperbolic groups! (provided have flaring + qi inclusions)
- Proof uses flaring condition to show subquadratic isoperimetric function, hence hyperbolic.

If K + Q are groups, an extension of K by Q is any group G fitting into a short exact sequence

$$\boxed{1 \to K \to G \to Q \to 1}$$

Basic Question: If K + Q are hyperbolic, when will G be hyperbolic as well?

K & Q hyp is only interesting case:

- Mosher '96: K & G hyp $\Rightarrow$ Q hyp
- presentations give examples with G (e.g free) & Q hyperbolic

<u>Ex</u> Stupid extension

$$1 \to K \to K \times Q \to Q \to 1$$

is never hyperbolic provided $|K|, |Q| = \infty$

☆ Depends on <u>how</u> K & Q are combined.

SES induces <u>monodromy ρ</u>: $Q \to Out(K)$ "outer action"

Exercise

$\gamma \mapsto$ conjugate by $\tilde{\gamma}$.

for G to be hyperbolic:

- ρ must be pretty complicated. (w/ $ker(\rho)$ finite)

  (if $\exists$ $\gamma \in Q$, $h \in K$ inf order st $\rho(\gamma)$ fixes conj class h, then $\mathbb{Z} \oplus \mathbb{Z} \leq G$, so not hyp!)

  $\rightsquigarrow$ restrictions on $Out(K)$ (e.g must be infinite)

- JSJ theory of hyp grps (Rips Sela):

  If K torsion free, must have

  K = free product of surface groups & free groups.

This leads us to study of $Out(F_n)$, $Out(\pi_1(surf)) = MCG$

& hyp extns of <u>free groups</u> & <u>surface groups</u>

Do hyperbolic extensions exist?

First Example: Σ closed surface, $f: \Sigma \to \Sigma$

$\rightsquigarrow$ mapping torus $M_f = \Sigma \times [0,1] / (x,1) \sim (f(x),0)$

fibration
$$\begin{array}{ccc} \Sigma & \to & M_f \\ & & \downarrow \\ & & S^1 \end{array}$$

(long exact seq of fibration)

$\rightsquigarrow$ $1 \to \pi_1(\Sigma) \to \pi_1(M_f) \to \mathbb{Z} \to 1$

A beautiful Story!

(Thurston '70s) f pseudo-Anosov $\Rightarrow$ M_f admits a hyp Riem. metric

$\Uparrow$ $\Downarrow$

$\pi_1(M_f)$ hyperbolic

Second Example

$\varphi \in Out(F_n)$ $\rightsquigarrow$ semidirect product

$1 \to F_n \to F_n \rtimes_\varphi \mathbb{Z} \to \mathbb{Z} \to 1$

$\| $

$\langle F_n, t \mid t^{-1} w t = \varphi(w) \ \forall w \in F_n \rangle$

(Brinkmann '00) hyperbolic $\Leftrightarrow$ φ is atoroidal

(no power fixes a nontriv. conjugacy class)

(Mosher '97) examples hyp extns

$1 \to \pi_1(\Sigma) \to G \to F_2 \to 1$

Uses:

- Bestvina-Feighn combination thm
- train tracks to get flaring.

(Farb–Mosher, Hamenstädt, Kent–Leininger)

$\|$ theory of hyperbolic extensions of surf grps $\pi_1(\Sigma)$

— convex cocompact subgroups of $MCG(\Sigma)$.