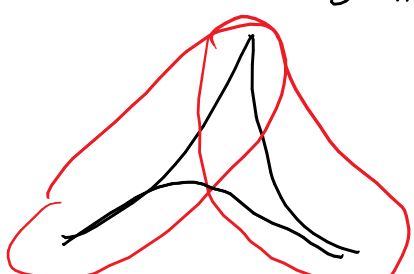
Lecture III — Hyperbolicity

A a <u>geodesic</u> in a metric space (X,d) is an isometric embedding $\gamma : I \to X$ (i.e $d(\gamma(s),\gamma(t)) = |s-t|$) for $I \subset \mathbb{R}$ an interval.

* we obscure map vs image.

Call X <u>geodesic</u> if any 2 pts can be joined by a geod.

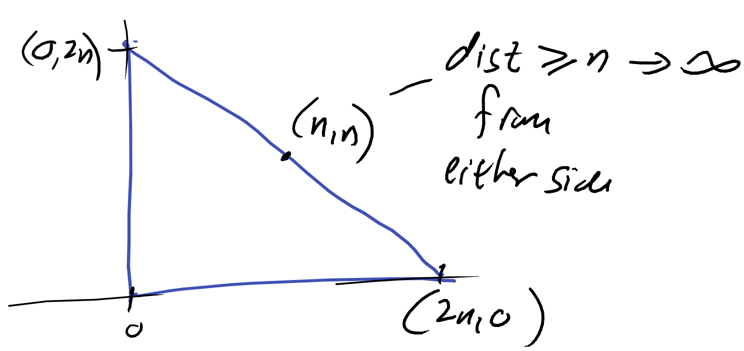
<u>Def</u> (Gromov) For $\delta > 0$, a geodesic metric space is <u>δ-hyperbolic</u> if every geodesic triangle is δ-slim.
i.e: each side is contained in the δ-nbhd of union of other 2.



<u>Ex</u>: tree; hyp plane (from student talk)

<u>Non-example</u>:

Euclidean plane



for γ a geodesic in X, & $y \in X$

$$d(y,\gamma) = \inf \{ d(y,p) \mid p \in \gamma \}$$

closest point projection $\pi_\gamma : X \to \mathcal{P}(\gamma)$

$$\pi_\gamma(y) = \{ p \in \gamma \mid d(y,p) = d(y,\gamma) \} \subset \gamma$$

Exercise: • $\pi_\gamma(y)$ always nonempty

• $\operatorname{diam}(\pi_\gamma(y)) < \infty$ (bdd in terms of $d(y,\gamma)$).

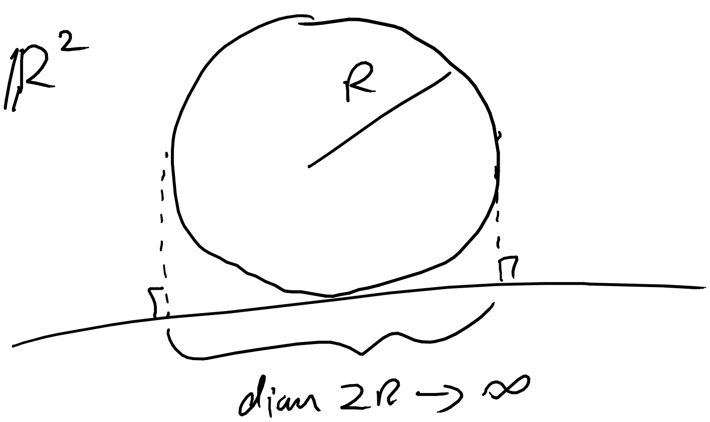
Def Call γ D-strongly contracting (D-SC) if

$$d(y,y') \leq d(y,\gamma) \Rightarrow \operatorname{diam}(\pi_\gamma(y) \cup \pi_\gamma(y')) \leq D$$

(i.e: $\operatorname{diam}(\pi_\gamma(B)) \leq D$ for any metric ball $B = B_r(y)$ that is disjoint from γ)

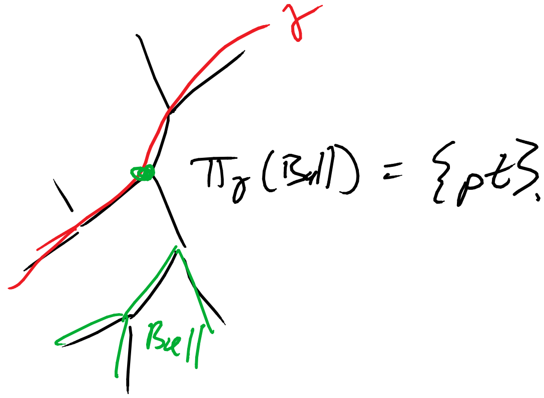
Rmk: any finite-length geodesic is D-SC for some D. But an infinite length geodesic may not be!

NonEx: $\mathbb{R}^2$



infinite geods Not SC.

Ex · tree

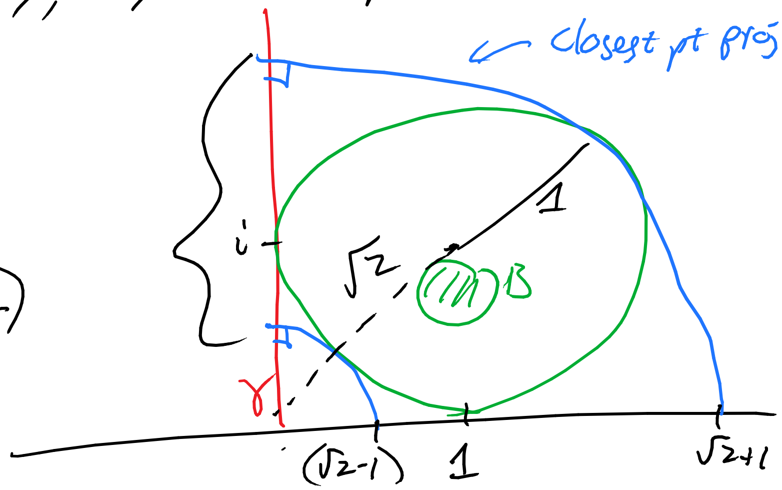


hyp plane $\mathbb{H}^2$ any disjoint ball lives in horoball.

upto isometry, may assume picture is :

all geods are 2-s.c

$\text{diam} = \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$
$= \log(3+2\sqrt{2})$
$\leq \log(6)$



Recall a (K,C) quasi-geodesic in X is a map (or image)

$\gamma : I \to X$, $I \subset \mathbb{R}$ interval, st

$$\frac{1}{K}|s-t| - C \leq d(\gamma(s), \gamma(t)) \leq K|s-t| + C.$$

Def : A geodesic is Morse if $\forall K \geq 1, C \geq 0$ $\exists N(K,C)$ s.t

for any (K,C) quasi-geod $\sigma : [a,b] \to X$ w/ $\sigma(a), \sigma(b) \in \gamma$,

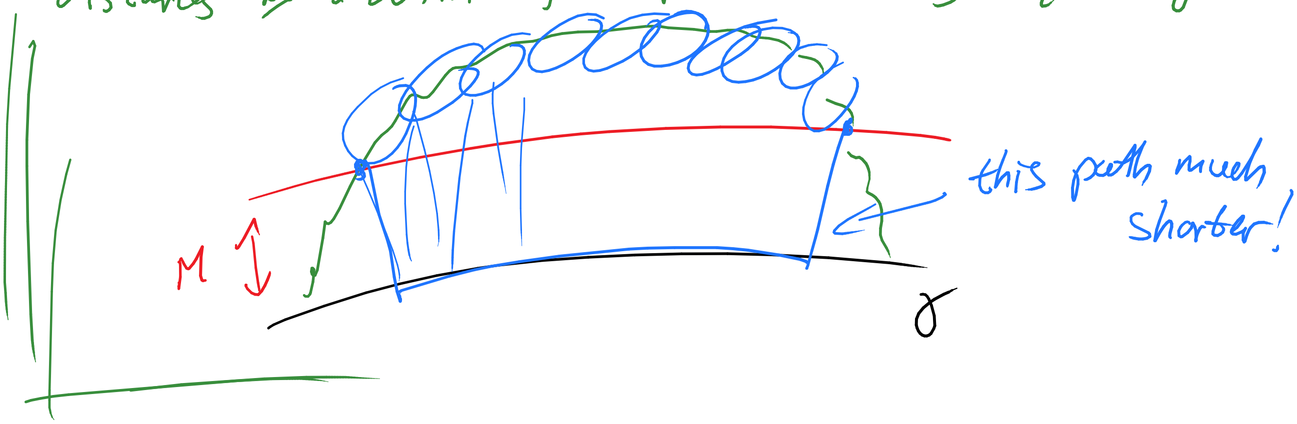
have $d_{Haus}(\sigma, \gamma) \leq N(K,C)$

(i.e $\sigma \subset N$-nbhd of γ,
$\gamma \subset N$-nbhd of σ)

call $N : [1,\infty) \times [0,\infty) \to \mathbb{R}$ Morse gauge for γ.

Exercise: $\forall D\ \exists$ Morse guage N s.t. every D-S.C. geodesic is N-Morse } in any metric space.

Idea: If β gets too far away, projecting to γ shrinks distances by a definite factor!! violate being a quasi-geo



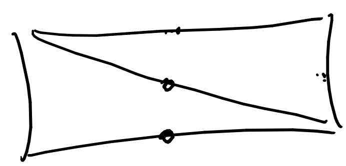
Thm: for X a geod metric space, TFAE:

1) $\exists \delta$ st X δ-hyp
2) $\exists D$ st any geodesic is D-SC
3) $\exists$ guage N st every geod is N-Morse.

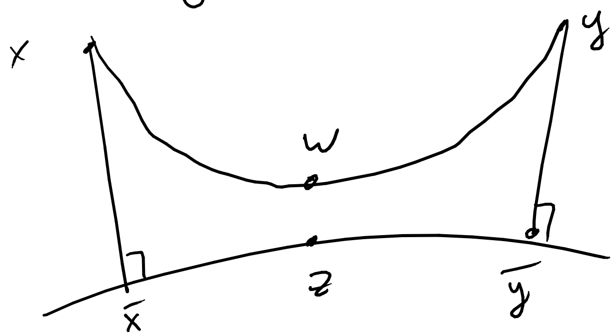
pf ideas: (Exercise: details)

$(1) \Rightarrow (2)$:

- geodesic quadrilaterals are 2δ-thin



- say γ geod + $\mathrm{diam}(\pi_\gamma(x), \pi_\gamma(y)) \geq 10\delta$



If $d(z, \bar{x}) \geq 5\delta$, then z cannot be within 2δ of a pt on $[x \bar{x}]$! Hence z is 2δ from a pt w on $[x,y]$

thus $d(x,y) \geq d(x,w) \geq d(x,z) - 2\delta \geq d(x,\gamma) - 2\delta$

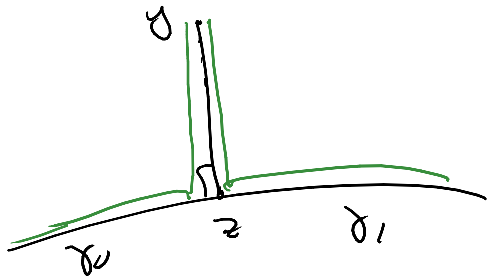
so: $d(x,y) < d(x,\gamma) - 2\delta \Rightarrow \mathrm{diam}(\pi_\gamma(x), \pi_\gamma(y)) \leq 10\delta$

which is basically the claim.

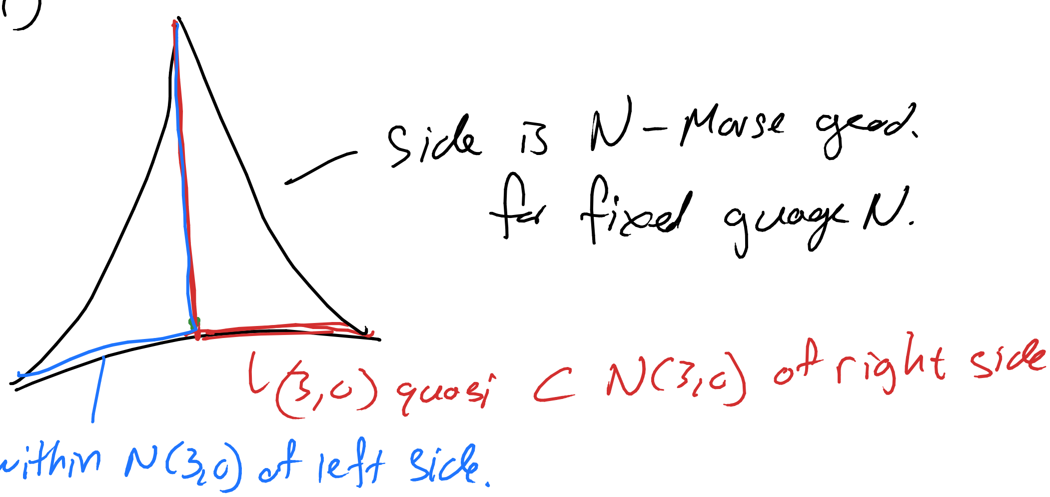
$(2) \Rightarrow (3)$ ✓

<u>Lem</u> in any geod metric space. For γ a geod, $y \in X$ & $z \in \pi_\gamma(y)$, break $\gamma = \gamma_0 \cdot \gamma_1$ at z.

then concatenation $[y,z] \cup \gamma_1$ is a $(3,0)$ quasi-geodesic.



$(3) \Rightarrow (1)$



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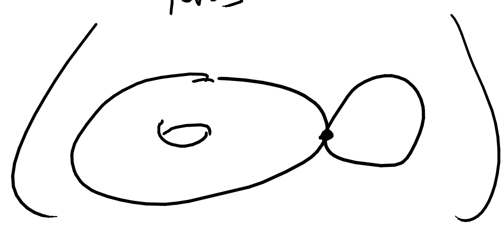
☆ <u>Upshot</u>: S.C - geodesics behave like geods in a hyp space.

Even if space is <u>not</u> δ-hyperbolic, it may still have <u>some</u> S.C. geodesics & exhibit aspects of hyperbolicity!

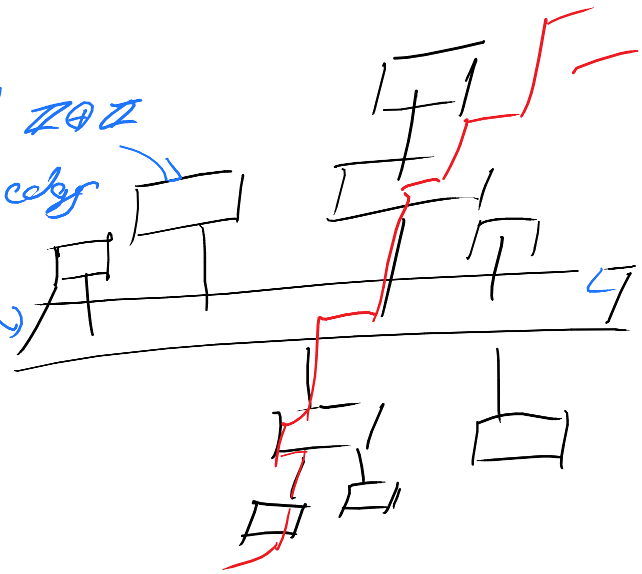
Example

$G = \langle a, b, c \mid ab = ba \rangle = \pi_1$ Torus $\vee S^1$



univ. cover

many copies of $\mathbb{Z} \oplus \mathbb{Z}$ attached by edges

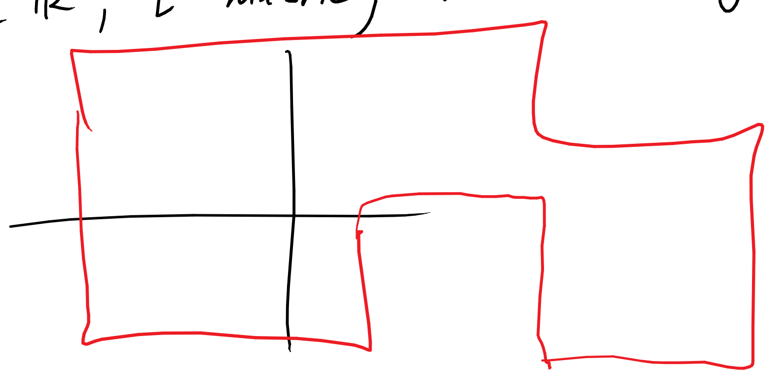


acacacacacac....
gives a geodesic ray
that is 1-s.c.
(for any disjoint ball, projection c single a edge)

Def a map $\gamma : I \to \mathbb{R}$ is an L-local geodesic if restriction to each length $\leq L$ sub int is geodesic.
i.e. if $d(\gamma(s), \gamma(t)) = |s-t|$ whenever $|s-t| \leq L$.

Lem In a hyp space, local geodesics are quasi-geodesics.

Non Ex: In $(\mathbb{R}^2, L^1\text{-metric})$ have local geodesics like



Not a quasi-geod.

pf: given δ, let $N = N(2,0)$ be Morse const for $(2,0)$ quasi-geodesics. Fix $\frac{L}{4} > 2N + 2\delta$

Claim: Every L local geod $\gamma : I \to X$ is $(2,0)$ q.g.

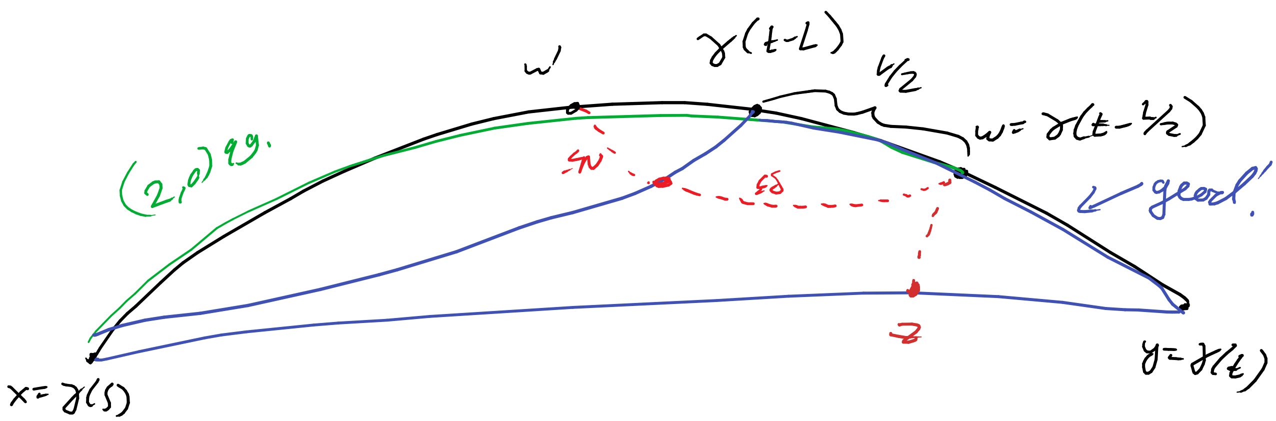
pf: $d(\gamma(s), \gamma(t)) \le |s-t|$ is immediate

NTS $d(\gamma(s), \gamma(t)) \geq \frac{1}{2}(t-s)$ $\forall s<t$. $(*)$

Induct on $(t-s)$:

Suppose $(*)$ holds for all $t-s \le R$. (true for $R = L$!)

show holds when $R < t-s \le R+L$



$\forall\, s \le r \le t-L$, $d(w, \gamma(r)) \geq \frac{t - \frac{L}{2} - r}{2} \geq \frac{L}{4} > N + \delta$. Hence

$d(w, z) \le \delta$ some $z \in [x,y]$. So:

$$d(x,y) = d(x,z) + d(z,y) \geq d(x,w) + d(w,y) - 2N - 2\delta$$

$$\geq \frac{t-s-L/2}{2} + \left(\frac{L}{4} + \frac{L}{4}\right) - 2N - 2\delta$$

$$= \frac{t-s}{2} + \frac{L}{4} - (2N - 2\delta) \geq \frac{t-s}{2}$$

∎

pf:

$N = N(2,1)$

Set $L_0 \geq \delta N + 4\delta$

Clean $(2,c)$-qgs, just need

$d(\gamma(s),\gamma(t)) \geq \frac{1}{2}|s-t| - 1$

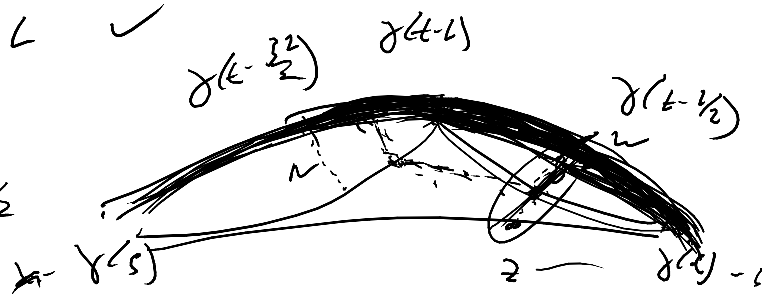
f

(some multiple + by Δ ineq)

Induct on $|s-t|$. $|s-t| \leq L$ ✓

Say true $\forall\, |S-t| \leq R$,

show $|s-t| \leq R + 1/2$



$d(\gamma(t-L/2), z) \leq N+\delta$

$d(x,y) = d(x,z) + d(z,y)$

$\geq d(x,w) - N - \delta + d(w,y) - N - \delta$ $\qquad \frac{L}{4} \geq 2N + \delta$

$\geq \frac{1}{2}|t-w| + |w-1| - 2N - \delta$ $\qquad L \geq \cancel{4N}\ 8N + 4\delta$

$= \frac{1}{2}|t-s| + \frac{L}{4} + \frac{L}{4} - 2N - \delta$