Geometry + Topology of free group automorphisms: hyp extns

Tues July 9, 2019

Lecture 2 – $Out(F_n)$ + Outer Space

Study $Out(F_n) = Aut(F_n) / Inn(F_n)$

- $F_n \curvearrowright F_n$ via conjugation $\quad F_n \to Aut(F_n)$, $\quad w \mapsto (x \mapsto wxw^{-1})$
- $Inn(F_n) = \text{image} \leq Aut(F_n)$ $\quad$ Exercise: $Inn(F_n) \cong F_n$ (ι injective)
- Think: $Inn(F_n)$ is trivial/understood part; $Out(F_n)$ interesting part

Topological Interpretation: (symmetries of graphs)

G finite graph $\rightsquigarrow \pi_1(G, v)$ free $\cong F_n$

Exercise: $\left\{ \begin{matrix} \text{homotopy equivalences} \\ G \to G \end{matrix} \right\} \Big/ \text{homotopy} \cong Out(F_n)$

Analogous to mapping class groups:

Σ surf (finite-type) $\quad MCG^{\pm}(\Sigma) = Homeo^{\pm}(\Sigma) / Homeo_0(\Sigma)$ $\quad$ path component of id

natural map $\downarrow$

$Out(\pi_1(\Sigma))$

Dehn-Nielsen-Baer Theorem:

when Σ closed, this is an isomorphism
(injectivity is basic algebraic topology,
surjectivity uses hyperbolic geometry)

If Σ not closed, $\pi_1(\Sigma)$ is free ($\cong F_{2g}$ if genus g, 1 puncture)

& the map $MCG^{\pm}(\Sigma) \to Out(\pi_1(\Sigma))$

injective but not surjective.

$\rightsquigarrow$ view $MCG(\Sigma) \leq Out(F_n)$ ("small" subgroup)

so $Out(F_n)$ generalizes MCG

• $Out(F_n)$ theory motivated via analogy/relationship with MCG

Eg: (Hamenstädt–Hensel) $MCG(\Sigma_{g,1})$ undistorted in $Out(F_n)$

Exercise: $\Sigma_{g,1}$ is a $K(F_{2g},1)$ space, so

$\underbrace{H.E}_{\text{homotopy equivalences}}(\Sigma_{g,1})/htpy \cong Out(F_{2g})$

Compare: $Homeo \lneq H.E.$

Study $Out(F_n)$ via actions on things (the mantra of GGT)

(eg: last time basically used action of $Aut(F_n)$ on space of labelled graphs to prove finite generation)

$Out(F_n) \curvearrowright$ · {conj classes of elts of F_n}

· {conj classes of $\underbrace{\text{free factors}}_{A \leq F_n \text{ s.t. } F_n = A * B}$ F_n}

Basic invariant of $\phi \in Out(F_n)$: stretch factor $\lambda(\phi) \geq 1$

$$\log \lambda(\phi) = \sup_{\alpha \in F_n} \lim_{n \to \infty} \frac{\log \| \phi^n(\alpha) \|}{n}, \quad \|\cdot\| = \text{conjugacy length wrt free basis}$$

Exercise: independent of free basis

limit exists for each α (because seq is subadditive)

ϕ <u>exponentially growing</u> if $\lambda(\phi) > 1$, else <u>polynomially growing</u>

↳ some conj class grows exponentially

↳ every conj class grows at most polynomially

<u>Issue</u>: How to calculate λ?

need to account for <u>cancellation</u> arising during iteration

<u>Solution</u>: a good representative.

Recall: a core graph Γ is a finite graph, valence ≥ 2 each vertex.

Fix n-petal rose R_n and $\pi_1(R_n) \cong F_n$

a <u>marking</u> of Γ is a homotopy equivalence $f : R_n \to \Gamma$

self map $\Gamma \xrightarrow{\sigma} \Gamma$ of marked core graph <u>represents</u> $\phi \in Out(F_n)$

if $\sigma_* = \phi$ (via identification $F_n \overset{f_*}{\cong} \pi_1(\Gamma)$)

<u>Def</u> ϕ is <u>reducible</u> if it has a rep $\Gamma \xrightarrow{\sigma} \Gamma$ that leaves

a homotopically nontrivial proper subgraph invariant (up to htpy)

(<u>equiv</u>: ϕ fixes the conj class of a proper free factor (exercise))

otherwise ϕ is <u>irreducible</u>

<u>Def</u> <u>metric</u> on a core graph Γ is choice of positive lengths

$\ell(e)$ for each edge

- view Γ as geodesic metric space
- volume = sum of edge lengths

ℓ(path or loop α) = length of geodesic / reduced representative (rel endpts)

a <u>direction</u> at $x \in \Gamma$ is a germ of isometric emb. $[0,\varepsilon) \to \Gamma$ with $0 \mapsto x$

— think "tangent vector" 

(2 directions at edge pts; # directions at vertex = valence)

· a <u>turn</u> at x is a pair $\{d, d'\}$ of distinct directions.

· <u>Illegal Turn Structure</u> (ITS) on Γ is an equiv relation

on set of directions at each point (or just vertices)

- equiv classes called <u>gates</u>
- turn $\{d, d'\}$ is <u>illegal</u> if $d \sim d'$; else <u>legal</u>
- path/loop is legal if it only takes legal turns

★ any map $\sigma : \Gamma \to \Gamma$ (say immersion on edges) induces <u>derivative</u>

$$D_x\sigma : \left\{ \begin{matrix} \text{directions} \\ \text{at } x \end{matrix} \right\} \to \left\{ \begin{matrix} \text{directions} \\ \text{at } \sigma(x) \end{matrix} \right\}$$

$\rightsquigarrow$ ITS: declare $d \sim d' \iff D\sigma(d) = D\sigma(d')$

· a <u>Train Track Structure</u> is an ITS w/ ≥ 2 gates at each point.

· a <u>train track rep</u> of $\phi \in Out(F_n)$ is a map $\Gamma \xrightarrow{\sigma} \Gamma$ equipped

w/ a TTS s.t

- edges map to immersed legal paths
- legal turns map to legal turns

(equiv $\sigma^k(e)$ immersed for each edge e + $k \geq 1$)

<u>UPSHOT</u>: never get any cancellation!

$\rightsquigarrow$ can calculate $\lambda(\phi)$ from transition matrix $M(\sigma)$:

$$M_{ij} = \#\text{times } \sigma(e_i) \text{ crosses } e_j \text{ (in either direction)}$$

(Legal loop $\alpha \iff$ vector v, + you can use linear algebra to find growth rate of $M^k v$)

How can we find good representatives? ($\exists$ algorithm! student talk Thurs)

Def The Culler–Vogtmann Outer Space

$$X_n = \left\{ (\Gamma, f, \ell) \;\middle|\; \begin{array}{l} \cdot\ \Gamma \text{ a core graph} \\ \cdot\ f : R_n \to \Gamma \text{ a marking} \\ \cdot\ \ell \text{ metric on } \Gamma \text{ of volume} = 1 \end{array} \right\} \Big/ \sim$$

where $(\Gamma, f, \ell) \sim (\Gamma', f', \ell')$ if $\exists$ isometry $\Gamma \xrightarrow{\sigma} \Gamma'$ st $\sigma f \simeq f'$

$$\begin{tikzcd} R_n \arrow[r, "f"] \arrow[dr, "f'"'] & \Gamma \arrow[d, dashed, "\exists \sigma \text{ isom.}"] \\ & \Gamma' \end{tikzcd}$$

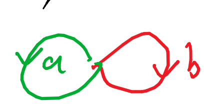
(Rmk: isometry need not be graph morphism, since topological edges could be subdivided
Rmk: can always unsubdivide to assume Γ has no valence 2 vertices)

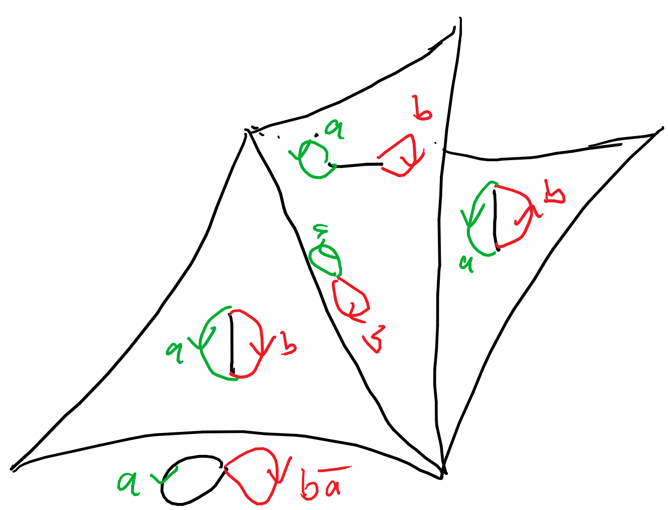
Exercise: X_n could alternately be defined as space of simplicial metric trees equipped w/ minimal, isometric F_n-action of covolume 1, up to equivalence given by F_n-equivariant isometry.

Simplicial Structure:

X_n decomposes into open simplices:
each marked graph $R_n \to \Gamma$ gives open simplex of metrics.

Ex: X_2,

$R_2 =$ 



Length Functions

$\Gamma \in X_n$, + elt/conj class $\quad \alpha \in F_n \cong \pi_1(R_n) \overset{f_*}{\cong} \pi_1(\Gamma)$

$\ell_\Gamma(\alpha)$ = length of geodesic / reduced representative in corresp. free htpy class of loops.

Topology: Give X_n weakest topology st each length func.
$\ell_\cdot(\alpha): X_n \to \mathbb{R}$ is continuous

(i.e: subspace topology from length function embedding $X_n \to \mathbb{R}^{\{\text{conj classes}\}}$)

Lipschitz Distance

a difference of markings from Γ to Γ' is any map $\sigma: \Gamma \to \Gamma'$ st

- $\sigma f \simeq f'$ (in right htpy class

$$\begin{tikzcd} & \Gamma \arrow[dd, "\sigma"] \\ R_n \arrow[ur, "f"] \arrow[dr, "f'"'] \arrow[r, phantom, "\simeq"] & \\ & \Gamma' \end{tikzcd}$$

)

- σ is linear on edges

Set $L(\sigma)$ = Lipschitz const of σ = max edge slope

tension graph $\Delta(\sigma) \subset \Gamma$ is union of edges of maximal slope

Prop for $\Gamma, \Gamma' \in X_n$,

$$\inf\left\{L(\sigma) \mid \sigma: \Gamma \to \Gamma' \text{ diff of markings}\right\} = \sup_\alpha \frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)}$$

and both are realized.

proof: always have $\ell_{\Gamma'}(\alpha) \le L(\sigma)\ell_\Gamma(\alpha)$,

This proves $\ge$ (for any σ, $L(\sigma)$ is upper bound on $\frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)}$)

- Arzela-Ascoli $\Rightarrow$ infimum is realized (set of maps w/ bounded Lipschitz const is equicontinuous)

say by $\sigma: \Gamma \to \Gamma$

· If ITS on Δ induced by $D\sigma$ is not TTS, then some vertex v has only one gate (eg if Δ not core graph)

$\rightsquigarrow$ may homotope σ (push v in direction $D_v\sigma(d)$) any dir d at v

so that $\Delta(\sigma)$ decreases

$\Rightarrow$ may assume σ induces TTS on $\Delta = \Delta(\sigma)$ & that Δ is a core graph

(exercise: why is Δ homotopically nontrivial?)

· For α any immersed loop in Δ,

$\sigma(\alpha)$ is immersed $\Longleftrightarrow$ α is legal!

· each vertex of Δ has ≥ 2 gates

$\Rightarrow \exists$ legal immersed loop α in Δ (since legal paths may be extended indefinitely)

· $\alpha \subset \Delta \Rightarrow \ell_{\Gamma'}(\alpha) \leq L(\sigma)\,\ell_\Gamma(\alpha)$

Hence conclude $\leq$ & that sup realized (by this α) ∎

Rmk: legal loop α can be chosen to cross each edge of Γ at most twice

$\rightsquigarrow$ $\sup_\alpha$ may be calculated as max over finite (& identifiable) set of loops in Γ (without finding or realizing infimum)

Def a difference of markings $\sigma: \Gamma \to \Gamma'$ is optimal if:

· realizes $L(\Gamma,\Gamma') = \inf\{L(\sigma) \mid \sigma \text{ diff markings}\}$

· $\Delta(\sigma)$ a core graph & $D\sigma$ induces a TTS on $\Delta(\sigma)$

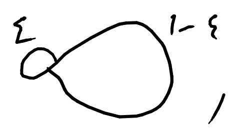
Def The Lipschitz dist on X_n is $d(\Gamma,\Gamma') = \log L(\Gamma,\Gamma')$.

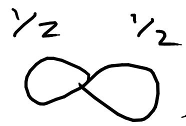
Prop d is an asymmetric metric on X_n:

i) $d(\Gamma,\Gamma') \geq 0$ with equality iff $\Gamma = \Gamma'$

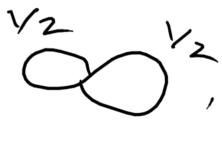
ii) $d(\Gamma,\Gamma'') \leq d(\Gamma,\Gamma') + d(\Gamma',\Gamma'')$ $\quad (\forall \Gamma,\Gamma',\Gamma'')$

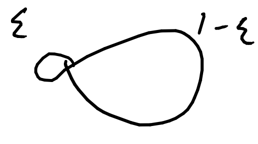
pf: asymmetry easy to see:

$\cdot\; d\big($ 

, 

$\big) = \log\left(\frac{1/2}{\varepsilon}\right) \to \infty$ as $\varepsilon \to 0$

$\cdot\; d\big($ 

, 

$\big) = \log\left(\frac{1-\varepsilon}{1/2}\right) = \log(2-2\varepsilon) \to \log 2$

i) if $L(\Gamma,\Gamma') < 1$, then image of optimal $\sigma: \Gamma \to \Gamma'$ has volume <1
$\Rightarrow \sigma$ not surjective $\Rightarrow \sigma$ not h.e. $\lightning$

If $L(\Gamma,\Gamma') = 1$, σ must be isometry
(since must have slope ≥ 1 on each edge or again vol(image) <1)

ii) is immediate from composition of Lipschitz maps. $\blacksquare$

$Out(F_n) \curvearrowright X_n$ by changing marking (right action)

$$\phi \cdot [(\Gamma, f, \ell)] = [\Gamma, f \circ \Phi, \ell],$$

$R_n \xrightarrow{\Phi} R_n$ htpy equiv induced by $\phi \in Out(F_n)$.

Action is isometric for Lipschitz distance!

Goal: use X_n to study $Out(F_n)$

Thm (Culler–Vogtmann '87) X_n is contractable,

application: $vcd(Out(F_n)) = 2n-3$

(they find $2n-3$ dim'l spine, & know $2n-3$ is max rank of an abelian subgroup)

Set $\tau(\phi) = \inf_{\Gamma \in X_n} d(\Gamma, \Gamma\cdot\phi)$, translation length

3 possibilities
- (elliptic) $\tau(\phi) = 0$ + realized
- (hyperbolic) $\tau(\phi) > 0$ + realized
- (parabolic) $\tau(\phi)$ not realized.

What do they tell us about ϕ?

1) Elliptic If $\Gamma = \Gamma\cdot\phi$, there is an isometry $\sigma : \Gamma \to \Gamma\cdot\phi$ st $\sigma f \simeq f\Phi$

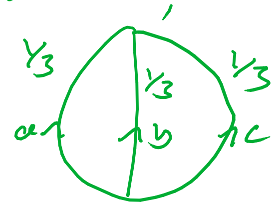
Isometry of finite graph must have finite order $\Rightarrow \exists\, k \geq 1$ st

$$\sigma^k = Id \Rightarrow f = \sigma^k f \simeq f\Phi^k \Rightarrow \Phi^k \simeq id_{R_n}$$

So: ϕ elliptic $\Rightarrow$ ϕ has finite order in $Out(F_n)$.

Example F_3, $\phi(a) = \bar{b}$, $\phi(b) = \bar{c}$, $\phi(c) = \bar{a}$,

ϕ has order 6 + fixes pt Γ



2) hyperbolic

say $d(\Gamma, \Gamma\phi) = \log\lambda > 0$ realizes $\tau(\phi)$.

take $\sigma : \Gamma \to \Gamma\cdot\phi$ optimal map w/ TTS on core tension graph Δ

(assume optimal throughout construction)

Prop After arb small perturbation of Γ (maintaining $d(\Gamma, \Gamma\phi) = \log\lambda$) may assume σ satisfies.

- $\sigma(\Delta) \subset \Delta$
- σ sends edges of Δ to legal edge paths
- σ sends legal turns to legal turns

* perturbation changes Γ + $\Gamma\cdot\phi$!

* Since at minimum, $d(\Gamma, \Gamma\phi) = \log\lambda$ automatic as long as we don't increase $L(\sigma)$

Proof: $D\sigma$ moves decrease Δ

(Lower complexity $\left(\operatorname{rank}(H_1(\Delta)),\ -\operatorname{rank}(H_1\Delta),\ \sum_{\substack{v\in\Delta \\ \text{vertex}}} \max\{0, \#\text{gates at } v - 2\}\right)$

Δ core graph & $\Delta' \subsetneq \Delta \Rightarrow$ complexity Δ' strictly less)

1) Say $\sigma(\Delta) \not\subset \Delta$ so $\sigma(e) \not\subset \Delta$ some edge e of Δ

Scale Δ by $\mu > 1$ & shrink $\Gamma \setminus \Delta$ to maintain vol 1

new $\sigma' : \Gamma' \to \Gamma'$ has $\Delta' \subsetneq \Delta$ since $e \not\subset \Delta'$,

& $L(\sigma') \leq L(\sigma)$

2) Say σ maps edge e of Δ over an illegal turn.

⇝ Fold the illegal turn:

Identify initial length $\varepsilon > 0$ segments & rescale to maintain volume 1

turn illegal $\Rightarrow$ σ descends to $\sigma' : \Gamma' \to \Gamma'$ with $L(\sigma') \leq L(\sigma)$

after optimizing, e drops out of Δ'

3) Say $D\sigma$ maps a legal turn to an illegal turn.

Fold the illegal turn.

⇝ converts the legal turn into illegal one

& lowers complexity

(either lowers $\sum_v \max\{0, g(v)-2\}$, or induces 1-gate vertex which is removed by optimizing)

repeat steps as needed ∎

Cor: If Γ realizes $\tau(\phi) = \log\lambda > 0$,
then $d(\Gamma, \Gamma\phi^k) = k\log\lambda \quad \forall k \geq 1$

pf: exercise:
- triangle ineq gives $\leq$
- after perturbing as in Prop, $\sigma: \Gamma \to \Gamma$ sends legal loops in Δ to legal loops. So α legal in $\Delta \Rightarrow$

$$\ell_{\Gamma\phi^k}(\alpha) = \ell_\Gamma(\sigma^k(\alpha)) = \lambda^k \ell_\Gamma(\alpha)$$

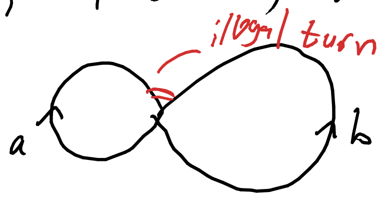
$$\Rightarrow d(\Gamma, \Gamma\phi^k) \geq k\log\lambda$$

Now by continuity, also holds before perturbing Δ ∎

Obsv: If $\sigma: \Gamma \to \Gamma\cdot\phi$ satisfies conclusion of prop, then Γ realizes $\tau(\phi)$

pf: Exercise

Example F_2, $\phi(a) = ab$, $\phi(b) = bab$ (exercise: check auto! find inverse)

take $\Gamma =$ 

$$\left.\begin{array}{l} \ell(a) + \ell(b) = 1 \\ \lambda\ell(a) = \ell(a) + \ell(b) \\ \lambda\ell(b) = \ell(a) + 2\ell(b) \end{array}\right\} \Rightarrow \begin{array}{l} \lambda = \frac{3+\sqrt{5}}{2} \\ \ell(a) = \frac{3-\sqrt{5}}{2} \\ \ell(b) = \frac{\sqrt{5}-1}{2} \end{array}$$

then obvious map $\sigma: \Gamma \to \Gamma\cdot\phi$ has

$L(\sigma) = \lambda$ & $\Delta(\sigma) = \Gamma$ with $\{\bar{a}, \bar{b}\}$ only illegal turn.

Exercise: Verify this satisfies conclusion of the propn. Hence realizes $\tau(\phi)$

3) **Parabolic**: say $\tau(\phi)$ not realized.

take $\Gamma_k \in X_n$ st $d(\Gamma_k, \Gamma_k\phi) \to D = \tau(\phi)$

Let $X_n^\varepsilon = \{\Gamma \mid \ell_\Gamma(\alpha) \geq \varepsilon \ \forall \text{nontrivial } \alpha \in F_n\}$ "ε-thick part"

Fact: $\mathrm{Out}(F_n)$ acts cocompactly on X_n^ε

only finitely many marked graphs up to action, & in thick part set of allowable metrics is compact

Prop $\forall \varepsilon$, have $\Gamma_k \in X_n^\varepsilon$ for only finitely k.

pf: If not, pass to subseq st $\Gamma_k \in X_n^\varepsilon$ $\forall k$.

Compactness $\Rightarrow$ choose $\psi_k \in \mathrm{Out}(F_n)$ st $\Gamma_k \psi_k \to \Gamma$ (after subseq)

$d(\Gamma \psi_k^{-1}, \Gamma \psi_k^{-1} \phi)$

$\leq d(\Gamma\psi_k^{-1}, \Gamma_k) + d(\Gamma_k, \Gamma_k \phi) + d(\Gamma_k \phi, \Gamma_k \psi_k^{-1} \phi)$

$\to 0 \quad + \quad D \quad + 0$

Hence $d(\Gamma, \Gamma \psi_k^{-1} \phi \psi_k) \to D$

Arzela-Ascoli $\Rightarrow$ only finitely many $\psi \in \mathrm{Out}(F_n)$ st $d(\Gamma, \Gamma\psi) \leq D+1$

Set of e^{D+1}-Lipschitz maps is equicontinuous, & nearby maps are htpic, hence only get finitely many homotopy classes

$\rightsquigarrow$ after subseq $\psi_k^{-1} \phi \psi_k$ is constant

$\Rightarrow d(\Gamma\psi_k^{-1}, \Gamma\psi_k^{-1}\phi) = D = \tau(\phi)$, contradicting parabolic $\boxtimes$

Prop For large k, any optimal map $\sigma: \Gamma_k \to \Gamma\phi$ leaves a proper core subgraph invariant up to homotopy.

Hence ϕ is reducible!

pf: Take $\varepsilon > 0$ small, h large st $\Gamma = \Gamma_h \notin X_n^{\varepsilon}$

& $d(\Gamma, \Gamma\phi) \leq D+1$, $\sigma : \Gamma \to \Gamma.\phi$ optimal.

Set $\delta_i = \varepsilon \cdot (e^{D+1})^i$ &

$\Gamma^i \subset \Gamma$ union of all loops in Γ (not nec embedded) length $\leq \delta_i$

obsv. $\sigma(\Gamma^i) \subset \Gamma^{i+1}$ by defn & $L(\sigma) \leq e^{D+1}$.

- Γ^0 htpically nontriv (since $\Gamma \notin X_n^{\varepsilon}$)
- only boundedly many core subgrps, so for some unif bdd i, Γ^i & Γ^{i+1} will have same core
- for ε suff small, Γ^i is neccessarily a <u>proper</u> subgraph Γ. ☒

<u>Exercise</u>: Do this carefully by picking constants!

- $\exists B_n$ bound on # distinct core subgraphs of any $\Gamma \in X_n$
- $\exists \mu > 0$ st $\Gamma^{\mu} \subsetneq \Gamma$ $\forall \Gamma \in X_n$
  - ↳ show Γ must always have edge of length $\geq \frac{1}{3n-3}$

Consequence of all this:

<u>Thm</u> (Bestvina–Handel '92 Annals of Math)

Every irreducible $\phi \in Out(F_n)$ admits a train track representative.

<u>pf</u>:
- ϕ cannot be parabolic.
- If ϕ elliptic, fixed pt gives simplicial rep $\Gamma \to \Gamma.\phi$ (clearly satisfies TT condition)
- If ϕ hyperbolic, found $\sigma : \Gamma \to \Gamma.\phi$ that restricts to TT map on tension subgraph $\sigma : \Delta \to \Delta$, but irreducible $\Rightarrow$ Δ must be all of Γ ☒