Geometry + Topology of free group automorphisms: hyperbolic extensions

5-part minicourse in MSRI summer school in Oaxaca, Mexico, July 1-12, 2019

Course consists of five 75-minute lectures, plus problem session for each lecture

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Lecture I - Free groups & Folding

For any set X, the free group on X consists of:

- all freely reduced, finite words (plus "empty word") that can be written in alphabet $X \sqcup \underbrace{X^{-1}}$
  - set of symbols $\{x^{-1} \mid x \in X\}$
  - (don't allow letter & its inverse to be adjacent
    - ↳ replace any occurrence xx^{-1} or $x^{-1}x$ w/ empty word.
- group operation is: concatenate then freely reduce.

Ex $X = \{a, b, c, d\}$, $w_1 = abc^{-1}a^{-1}a^{-1}d$, $w_2 = d^{-1}a\overbrace{bbb}^{\text{shorthand } b^3}cca \in F_X$

$$w_1 w_2 \underset{\text{concat}}{=} abc^{-1}a^{-1}\cancel{a^{-1}}\cancel{d}\cancel{d^{-1}}\cancel{a}bbbcca \underset{\text{reduce}}{\rightsquigarrow} abc^{-1}a^{-1}bbbcca$$

Exercise: Convince yourself this is a group:

1) Check multiplication well defined (independent of choices in reducing)
2) multiplication is associative
3) identity element - what is it?
4) Inverses - how to find them?

Universal Property of free groups

For any group G & set map $X \xrightarrow{\varphi_0} G$, there exists unique group homomorphism $F_X \xrightarrow{\varphi} G$ s.t. diagram commutes

$$\begin{tikzcd} X \arrow[r, "\text{inclusion}"] \arrow[dr, "\varphi_0"'] & F_X \arrow[d, dashed, "\exists!\,\varphi_0"] \\ & G \end{tikzcd}$$

That is, have bijection

$$\left(\begin{array}{c}\text{Group homomorphisms}\\ F_X \to G\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{Set maps}\\ X \to G\end{array}\right)$$

Exercise: prove this bijection (← this comes from universal property).

In language of category theory, the assignment $X \rightsquigarrow F_X$ is a functor Sets → Groups, & Univ. Property says this functor is (left) adjoint to the forgetful functor Groups → Sets.

②

Easy to see (from Univ. Prop!) that up to isomorphism F_X is determined by the cardinality of X.

Notation $F_n = F_{\{x_1,\dots,x_n\}}$ free group on n letters = free group of rank n.

Application: Generating sets (one reason to care about free groups)

for any group G have correspondence $\left\{\begin{matrix}\text{generating sets}\\ \text{of } G \text{ of size } n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{epimorphisms}\\ F_n \longrightarrow G\end{matrix}\right\}$

Topological Perspective

Def A graph is a 1-dimensional cell complex

(Not necessarily ~~[illegible]~~ simplicial: can have more than one edge w/ same endpts, + can have edges that are loops)

- we label oriented edges $a, b, c, \dots$,
  denote same edge w/ ~~the~~ reverse orientation by $\bar{a}$ or A
- $\iota(e)$ = initial vertex of oriented edge e
- $\tau(e)$ = terminal ——"—— so ~~[illegible]~~ $\iota(\bar{e}) = \tau(e)$

Formally, a graph G is a triple $(V, E, \bar{\ }, \iota)$ where:

- V & E are sets (~~[illegible]~~ of vertices & oriented edges)
- $\bar{\ }: E \to E$ a free involution (so: $\bar{\ } \circ \bar{\ } = id$, & $\bar{e} \neq e \ \forall e \in E$)
- $\iota: E \to V$ a function

Def a Morphism of graphs is a cellular map that sends each open edge homeomorphically onto an open edge

— adjusting by a homeomorphism that is isotopic to identity rel vertices is regarded as same morphism (i.e: don't care ~~[illegible]~~ about precise map on edges)

Formally a morphism $(V, E, \bar{\ }, \iota) \to (V', E', \bar{\ }, \iota)$ is a pair of maps $V \to V'$, $E \to E'$ that commute with $\bar{\ }$ and ι

Exercise: Convince yourself these are the same concepts

Def a graph morphism is an ~~[illegible]~~ immersion if it is locally injective (i.e, each pt has a neighborhood on which map is injective)

— always loc. inj at edge pts, so only need to check at vertices!

Exercise $f: G \to G'$ immersion $\Leftrightarrow (\iota(e_1) = \iota(e_2)$ & $f(e_1) = f(e_2) \Rightarrow e_1 = e_2)$

Exercise $f: X \to Y$ immersion b/w finite graphs.
Show it is possible to attach finitely many 0 + 1 cells to X to yield a graph $\tilde{X}$ to which f extends to a covering map graph morphism $\tilde{f}: \tilde{X} \to Y$.

- ~~Then~~ Moreover: If X has only one vertex then can build $\tilde{X}$ just by attaching edges
- Conclude: any immersion is a composition of an embedding (= injective ~~map~~ morphism) + a covering map

Def an edge path in a graph G is a (or just a pt) seq of edges $e_1, e_2, \dots, e_k$ st $\tau(e_i) = \iota(e_{i+1})$ $\forall i$.
equivalently: a morphism $I \to G$ where I is a graph homeomorphic to $[0,1]$ (w/ orientation) or to a point (for empty edge path)
The edge path is tight/reduced if $I \to G$ is an immersion,
equiv: no consecutive edges of form $e, \bar{e}$.
An elementary homotopy of an edge path is a move that deletes or inserts such consecutive edges

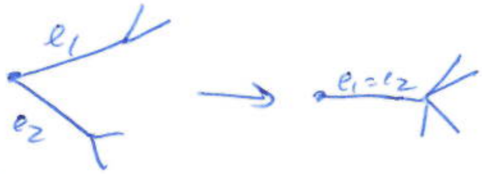
Exercises

- edge paths related by elem homotopy $\Rightarrow$ homotopic rel endpts
- Every edge path can be transformed into reduced edge path via seq of elem. homotopies. (called tightening)
- two reduced edge paths are homotopic rel endpts iff they are equal! (htpic rel endpts $\Leftrightarrow$ related by elem. homotopies)
- elements of $\pi_1(G, *)$ (basept vertex) are in bijection with reduced edge paths that start + stop at $*$

Exercise: An immersion b/w graphs is π_1-injective.

Folding: If e_1, e_2 are edges of a graph G with $e_2 \neq e_1 \neq \overline{e_2}$ and $\iota(e_1) = \iota(e_2)$, can form a new graph G' with (quotient) morphism $G \to G'$ by identifying e_1 with e_2 and $\tau(e_1)$ with $\tau(e_2)$.

Type 1: $\tau(e_1) \neq \tau(e_2)$:



Exercise: $G \to G'$ is htpy equivalence.

Type 2 $\tau(e_1) = \tau(e_2)$

not htpy equiv (π_1-surj, but not π_1 inj) auspicious year

Thm (Stallings, 1983) Every morphism $G \to G'$ of finite graphs factors as

$$G = \underbrace{G_0 \to G_1 \to \cdots \to G_k}_{\text{folds}} \underbrace{\to G'}_{\text{immersion.}}$$ where

where

Moreover, factorization can be found by fast algorithm

Exercise: Proof

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edge path starting & stopping at same pt

Def a tree is a graph for which: every reduced loop is trivial/degenerate

equiv: · exists a unique reduced edge path b/w any two vertices.

= maximal tree

Def a spanning tree in a graph G is a subtree T that contains every vertex of G

Facts:
- For any subtree $T \subset G$, the quotient map $G \to G/T$ (identify T to a vertex) is a homotopy equiv.
- If $T \subset G$ is a spanning tree, then G/T is a wedge of circles (i.e., a graph w/ only one vertex)
- For any graph G, $\pi_1(G)$ is a free group.

Specifically: for any spanning tree T, vertex v, & choice of orientation for edges of $G \setminus T$ (i.e., write $E_G \setminus E_T = X \sqcup X'$ where $-$ gives bijection $X \leftrightarrow X'$) get natural isomorphism $\pi_1(G,v) \xrightarrow{\cong} F_X$

γ closed loop at v $\rightsquigarrow$ reduced loop at v $\longmapsto$ word in $X \sqcup X^{-1}$ obtained by reading label of edges of γ outside T.

Exercise · Prove this is an isomorphism (can you use univ property?):

Alternate perspectives on π_1(graphs) is free

1) From G htpy equiv to wedge of circles, know $\pi_1(S^1) = \mathbb{Z}$
   - use van Kampen to get $\pi_1(G) \cong$ free product $\mathbb{Z} * \cdots * \mathbb{Z}$
   - use univ property of free product to conclude this is free
2) Show univ property of free groups is satisfied by $\pi_1(G)$:
   - for any grp H, let Y be space w/ $\pi_1(Y,y) \cong H$, the set map $X \to H$, defines topological map $\bigvee_X S^1 \to Y$ & gives $\pi_1(\bigvee_X S^1) \to \pi_1(Y) = H$
3) Prove directly that any elt of $\pi_1(G,v)$ or $\pi_1(\bigvee_X S^1)$ can be given normal form & this is same as our construction of F_X

Nielsen - ~~[illegible]~~ Schreier Thm : Every subgroup of a free group is free.

Proof : Exercise (For $H < F_X$, let G be graph w/ $\pi_1(G,v) \cong F_X$
+ let Y be covering space corresp to H.
Check that Y also a graph, + hence $H \cong \pi_1(Y)$ is free)

Subgroups + Covers

Basic problem: given $g_1, \dots, g_k \in F_n$, find a basis (= free gen set) of the subgroup $H = \langle g_1, \dots, g_k \rangle \leq F_n$. (we know H is free!)

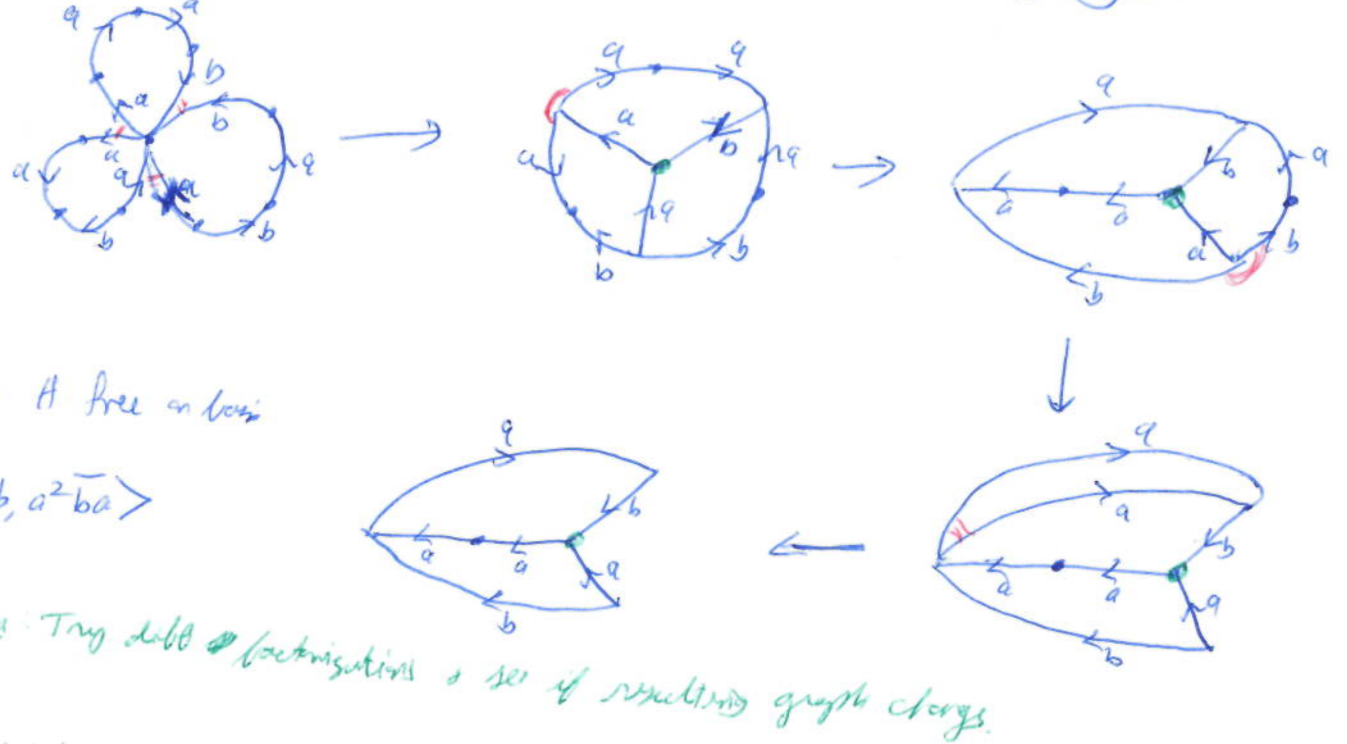
Ex $F_2 = \langle a, b \rangle$, $H = \langle a^3 b, \bar{a} b a b, a^2 \bar{b} a \rangle$

Step 1 : Build a graph morphism whose π_1-image is H:
Identifying $F_n = \pi_1($  $= R_n)$, such morphism given by graph G whose oriented edges labelled by ~~[illegible]~~ : $a_1, \dots, a_n$

Step 2 : Apply Stallings fold factorization; all graphs G_i map to Y w/ same π_1-image

Step 3 : From last graph, read off free basis for H:

Ex : $F_2 = \langle a, b \rangle$, $H = \langle a^3 b, \bar{a} b a b, a^2 \bar{b} a \rangle$, $F_2 = \pi_1($  $)$



So H free on basis

$\langle a^3 b, a^2 \overline{ba} \rangle$

Exercise : Try diff factorizations + see if resulting graph changes.

Exercise : How do we know result gives a free basis?

Fact: The resulting graph does not depend on ~~folded~~ folding choices!

(Exercise: check some other factorizations in example)

· only ~~depends~~ depends on subgroup H, & is equal to the following
└ Exercise!

Def: The core of a graph Y is the smallest subgraph that contains the base vertex & to which Y deformation retracts.

Prop: For H a f-gen subgroup of $F_n \cong \pi_1(G,v)$ & $(Y_H,\tilde{v}) \to (G,v)$ the cover corresp to H, TFAE:

1) The core of $(Y_H,\tilde{v})$
2) The largest connected finite subgraph of Y_H that contains $\tilde{v}$ & has no valence 1 vertices (except possibly at $\tilde{v}$)
3) The union of images of all reduced edge paths of Y_H that start & end at $\tilde{v}$.
4) The union of all the finitely many reduced edge paths that represent generators of H.

Exercise · Prove these equivalences

- Show Y_H can be constructed from ~~core~~ the core by attaching (typically infinite) trees at the vertices.

☆ The core is a canonical topological representative associated to a subgroup H (given identification $F_n \cong \pi_1(G,v)$) & can be computed algorithmically using Folds.

Exercise:

- If $H \leq F_n$ is f-gen & normal, then either $H = \{1\}$ or $[F_n : H] < \infty$
- Given $H < F_n$ – can you compute $N(H) = \{\gamma \in F_n \mid \gamma H \gamma^{-1} = H\}$?
  - what can you say about $[N(H):H]$?
- given $w \in F_n$, can you algorithmically decide if $w \in H$?
  [~~Soln~~ $w \in H \Leftrightarrow$ reduced word lifts to path in core starting & stopping at base vertex]
- given $w \in F_n$, can you decide if w conjugates to an elt in H?
  [Yes ⇔ cyclically reduced version of w lifts to a closed reduced loop at some vertex of the core]

Exercise: Given $H \le F_n$ f.gen., can you decide if H is normal in F_n?

- Given homom $h: F_n \to F_m$, can you decide when h is injective / surjective?

  ⌈Apply Stallings folds: Inj ⟺ no folds type 2
  Surj ⟺ last map a homeo⌋

- Show that for any homomorphism $h: F_n \to F_m$ there is a free factorization $F_n = A * B$ s.t. h kills A & is injective on B

- For every $H \le F_n$ f.gen, $\exists H' < F_n$ st $H < H' < F_n$, H free factor in H', & $[F_n : H'] < \infty$ (Marshall-Hall theorem

  ⌈Use core: Find H' algorithmically by completing core to a cover⌋

- Can you decide if $[F_n : H] < \infty$?

  ⌈YES: core must be a covering⌋

- Prove F_n is Hopfian (every epimorphism $F_n \xrightarrow{\phi} F_n$ is an automorphism; ⟹ F_n not isomorphic to any proper quotient of itself)

  ⌈ $\langle \phi(x_1), \dots, \phi(x_n) \rangle = F_n$, so construction of core cannot have folds of type 2, ⟹ seq folds gives seq of htpy equivalences ⟹ an automorphism⌋

- **Cor**: any gen set of F_n of size n is a free basis.

- Prove F_n is residually finite (for any $1 \ne w \in F_n$, $\exists H' \triangleleft F_n$ finite index & normal st $w \notin H'$)

- Prove F_n is LERF (for any f.g. $H < F_n$, & $g_1, \dots, g_k \in F_n \setminus H$, $\exists H' < F_n$ w $H < H' < F_n$, $[F_n : H'] < \infty$, H free factor of H', & $g_1, \dots, g_k \notin H'$)

⑧

Aut(F_n) = group of automorphisms of F_n

Thm (Nielsen, 1924) Aut(F_n) is finitely generated, in fact by the following 3-types of automorphisms: ($F_n = F_{\{a_1,\dots,a_n\}}$)

1) (Permutation) permute basis elements
2) (Sign change): send each a_i to either a_i or a_i^{-1}
3) (Change one word term):
   For some i: send $a_i \mapsto a_i^{\pm}$

   $\forall j \neq i$, send $a_j \mapsto a_j,\ a_j a_i^{\pm},\ a_i^{\pm} a_j,$ or $a_i^{\pm} a_j a_i^{\mp}$

Exercise Verify each is an automorphism. [Use univ. prop: just find inverse]

Rmk. · First 2 types generate a finite subgroup of size $2^n n!$
· Lot of redundancy in list

Ex Aut(F_4) is gen by the following 4 elts:
- $a_i \mapsto a_{i+1}$ $\forall i$ (indices mod 4)
- $a_1 \mapsto a_2,\ a_2 \mapsto a_1,\ a_3 \mapsto a_3,\ a_4 \mapsto a_4$
- $a_1 \mapsto a_1^{-1},\quad a_i \mapsto a_i$ for $i = 2,3,4$
- $a_1 \mapsto a_1 a_2,\quad a_i \mapsto a_i$ for $i = 2,3,4$

Exercise Prove this assuming theorem

Topological interpretation

Identify $F_n \cong \pi_1(G)$ for some graph G by fixing
- spanning tree $T \subset G$
- orientation on edges of $G \setminus T$
- bijection b/w edges of $G \setminus T$ ⟷ basis elts F_n.

- changing bijection ⟺ type 1 autom.
- changing edge orientation ⟺ type 2 auto
- changing spanning tree $T \to T'$ via edge swap move (add an edge of $G \setminus T$ to T', & remove an edge of T from T') corresp to type 3 auto:

Let $e \in G \setminus T$ be oriented edg, so $T \cup e$ has unique cycle, which corresp. to some basis elt [illegible] a_i of F_n

Let $f \in T$ be an edge in this cycle, (compatibly oriented w/ e)
then $T' = T \cup f - e$ is a spanning tree.

Thm: . $e \leftrightarrow f$

. for e_i outside both trees, T-rule for e_i schematically viewed as "Y"

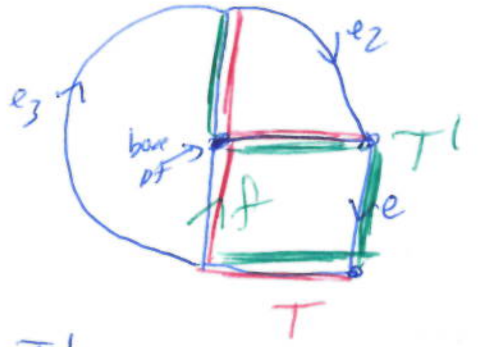
. If $f \notin Y$, then $e_i \mapsto e_i$

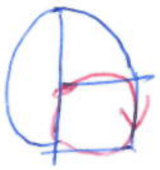
. If $f \in Y$, assume f oriented away from basept

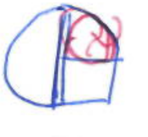
- $f \in$ "base" of Y then $e_i \mapsto f e_i f^{-1}$
- $f \in$ "left branch" of Y then $e_i \mapsto f e_i$
- $f \in$ "right branch" of Y then $e_i \mapsto e_i f^{-1}$

} a type 3 autom.

Ex



$e \xleftrightarrow{T}$  $\xleftrightarrow{T'} f$

$e_2 \xleftrightarrow{T}$  $\xleftrightarrow{T'} e_2$

$e_3 \xleftrightarrow{T}$  $\simeq$  $\xleftrightarrow{T'} f e_3$.

Exercise If T + T' are 2 spanning trees in a graph G, then there is a seq $T = T_0 \to \cdots \to T_k = T'$ of spanning trees in G s.t. each move $T_i \to T_{i+1}$ is a single edge swap.

Exercise: Consider simplicial cplx whose vertices are non closed edges of G, & where collections of edges span a simplex iff union is a forest (= disj union of trees)

- make examples
- what do you think is the homotopy type?

Pf of Thm: Let $\alpha: F_n \to F_n$ be an automorphism.

Let R be n-petal rose corresp to basis $a_1, \dots, a_n$

Let X be rose subdivided & labeled s.t. the petals read words $\alpha(a_1), \dots, \alpha(a_n)$

(So each petal subdivided into length $|\alpha(a_i)|$ labeled & oriented edges)

Get induced ~~[illegible]~~ morphism $X \xrightarrow{g} R$ which π_1-maps $\langle \alpha(a_1), \dots, \alpha(a_n) \rangle = F_n$

Factor as seq of folds $X = X_1 \to X_2 \to \cdots \to X_k = R$

Since g is π_1-isom, each fold of type 1 & last map is a homeo.

Identify $\pi_1(X_1) = \pi_1(X) \cong F_n$ via maximal tree T_1 with appropriate ~~[illegible]~~ identifications s.t. $g_* = \alpha$.

$$\begin{tikzcd}
\pi_1(X_1) \arrow[r, "g_*"] & \pi_1(R) \\
F_n \arrow[u, phantom, "\cong" rotate=90] \arrow[r, "\alpha"] & F_n \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd}$$

Now analyse each fold $X_i \to X_{i+1}$

Case 1: fold 2 ~~edges~~ embedded edges (neither a loop)

- change ~~max tree s.t. both edges~~ $T_i \to T_i'$ s.t. both edges in T_i' (autos of type 3)
  ~~change induced map~~ Then T_i' induces spanning tree of X_{i+1} s.t. $X_i \to X_{i+1}$ induces id $F_n \to F_n$.

Case 2 fold embedded edge onto a loop edge.

- change tree $T_i \to T_i'$ st embedded edge $\subset T_i'$
  now fold $X_i \to X_{i+1}$ induces tree T_{i+1} of X_{i+1} st map $X_i \to X_{i+1}$ induces type 3 autos. (Exercise)

Last map $X_k \to R$ is a homeo, so its induced map on π_1 is a signed permutation.

∎