Relative (T) & Semidirect Products

Semidirect Products A group, Aut(A).

$\varphi: \Gamma \to \mathrm{Aut}(A) \rightsquigarrow \Gamma \ltimes_\varphi A$;

Set $(\Gamma \ltimes_\varphi A) = \Gamma \times A$

Group structure

$$\gamma a \eta b = \gamma\eta \underbrace{(\eta^{-1} a \eta)}_{} b$$

only makes sense if set $= \varphi(\eta^{-1})(a)$

Observe: $N \triangleleft \Gamma$ then conjugation yields $\Gamma \to \mathrm{Aut}(N)$. (May be trivial, does not mean $\Gamma = \Gamma' \ltimes N$.)

Recall: $\pi: G \to \mathcal{U}(\mathcal{H})$ <u>has almost inv. vectors</u> (π strong cts) if $\exists v_n \in \mathcal{H}$ s.t. $\forall K \subset G$ compact

$$\max_{g \in K} \frac{\|\pi(g)v_n - v_n\|}{\|v_n\|} \xrightarrow{n\to\infty} 0 \quad ; \pi \succ I_G$$

& π has <u>invariant vectors</u> if $\exists v \in \mathcal{H} \setminus \{0\}$

$$\pi(g)v = v \quad \forall g \in G$$

Def: G LCSC, $H < G$ closed; (G,H) has rel(T) if $\pi \succcurlyeq I_G \Rightarrow \pi|_H \geq I_H$.

(ie. any unitary G-rep with almost inv. vectors has non-zero H-inv. vectors)

Ex: Assume $H = N \lhd G$. Prove for $\pi : G \to \mathcal{U}(\mathcal{H})$

$$\mathcal{H}_N = \{ v \in \mathcal{H} : \pi(n)v = v \ \forall n \in N \}$$

is $\pi(G)$ invariant & hence

so is $\mathcal{H}_N^{\perp} = \{ v \in \mathcal{H} \mid \langle v, v' \rangle = 0 \ \forall v' \in \mathcal{H}_N \}$

Conclude if (G,N) has rel(T)

setting $\pi^{\perp} : G \to \mathcal{U}(\mathcal{H}_N^{\perp})$ restriction

$\Rightarrow \pi^{\perp} \not\succcurlyeq I_G$. i.e. $\pi^{\perp}$ does not have alm. invariant vectors.

Example: $(SL_2\mathbb{Z} \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ will prove

Nonex: Γ amenable $H < \Gamma$, $|H| = \infty$

$\Rightarrow (\Gamma, H) \notin \mathrm{Rel}(T)$

Ex: Prove this.

$\boxed{\text{Ex:}}$ G amen (LCSC) $H < G$ closed
& (G,H) has rel (T) $\Rightarrow$ H compact.
Warning! $\text{Haar}_G(H)$ could be 0!

Def: G has (T) $\Leftrightarrow$ (G,G) has rel (T).

Lemma: $1 \to A_0 \to A \to A_1 \to 1$ short exact.
$\Gamma \to \text{Aut}(A, A_0) \leq \text{Aut}(A)$ preserves A_0.
$(\Gamma \ltimes A_0, A_0)$ & $(\Gamma \ltimes A_1, A_1)$ both
have rel (T) $\Longleftrightarrow$ $(\Gamma \ltimes A, A)$ has rel (T).

$\boxed{\text{Ex:}}$ Prove this lemma

Lemma: Compact groups have (T)
Proof: Let Haar be Haar measure on G
compact. Then $\text{Haar}(G) < \infty$,
$\Rightarrow \mathbb{1}_G \in L^2(G, \text{Haar}) = \{f : G \to \mathbb{C} \mid \int_G |f(x)|^2 d\text{Haar} < \infty\}$

$\boxed{\text{Ex}}$ Prove $H < G$ closed & $H \in (T)$
$\Rightarrow$ (G,H) has rel (T).

Cor. Assume $1 \hookrightarrow A \to V \twoheadrightarrow T \to 1$ short exact, T with property (T)

$(\Gamma \ltimes V, V)$ has rel(T) $\Leftrightarrow$ $(\Gamma \ltimes A, A)$ has rel(T).

Application: $A = \mathbb{Z}^N$, $V = \mathbb{R}^N$, $T = \mathbb{R}^N/\mathbb{Z}^N$
N-torus.

OR: $A = \mathbb{Z}[\frac{1}{p}]^N \leq V = \mathbb{R}^N \times \mathbb{Q}_p^N$,

$T = V/A$ is p-Solenoid.

$\boxed{\text{Ex:}}$ Prove: $\mathbb{Z}[\frac{1}{p}] \hookrightarrow \mathbb{R} \times \mathbb{Q}_p$ the diagonal embedding has discrete image & precompact fund. domain.

OR. $A = \mathcal{O}$ ring of integers in $K/\mathbb{Q}$ finite & $V = \mathbb{R}^? \times \mathbb{C}^?$ & $V/A \cong$ N-torus.

OR Can Mix these; ie invert primes in $\mathcal{O}$

$\boxed{\text{TH:}}$ (Cor) $\Gamma \to SL_N\mathbb{Z} \underset{\text{fin. ind.}}{\leq} \mathrm{Aut}(\mathbb{Z}^N) \leq \mathrm{Aut}(\mathbb{R}^N)$

$(\Gamma \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has Rel(T)

$\Leftrightarrow$ $(\Gamma \ltimes \mathbb{R}^N, \mathbb{R}^N)$ has rel(T)

Ref's: BHV, Shalom, Burger, My Thesis

Rel(T) for: $SL_N\mathbb{Z} \ltimes \mathbb{R}^N$

Recall: $\mathbb{R}$ = local field (i.e. locally compact union compact sets)

Then $\hat{\mathbb{R}}^N = \{\pi : \mathbb{R}^N \to \mathcal{U}(\mathcal{H}) \mid \pi \text{ is irred}\}$
$\cong \mathrm{Hom}(\mathbb{R}^N, S^1) \cong \mathbb{R}^N.$

$\hat{\mathbb{R}}^N = \{\varphi : \mathbb{R}^N \to \mathbb{R} \mid \varphi \text{ linear}\}.$

Fix basis $\langle v_1, \dots, v_N\rangle$ & $\langle \cdot, \cdot\rangle \Rightarrow \hat{\mathbb{R}}^N = \langle \varphi_1, \dots, \varphi_N\rangle$

where: $\varphi_j(v) = \langle v_j, v\rangle = v_j^t v = (v^t v_j)^t$

& $\pi_j(v) = e^{i\varphi_j(v)}$ $\quad i^2 = -1,$

Observe: $GL(\mathbb{R}^N) \to GL(\hat{\mathbb{R}}^N)$ by $M \mapsto (M^{-1})^t$

$M \mapsto M^*$ Dual action.

$(M\varphi_j)(v) = \varphi_j(M^{-1}v) = \langle v_j, M^{-1}v\rangle$

$= \langle (M^{-1})^t v_j, v\rangle.$

Will study Rel(T) for $\Gamma \ltimes \mathbb{R}^N$ by studying

$\pi|_{\mathbb{R}^N}$: A lot of structure theory for that

Proj. Valued Measure

Ref: BHV Appendix D

Let $\pi : \Gamma \ltimes \mathbb{R}^N \to \mathcal{H}$ be unitary rep.

$$\Rightarrow \exists P : \underbrace{\mathcal{B}(\hat{\mathbb{R}}^N)}_{\substack{\text{Borel} \\ \sigma\text{-alg.}}} \longrightarrow \underbrace{\mathrm{Proj}(\mathcal{H})}_{\text{projections } \mathcal{H} \to \mathcal{H}}$$

s.t. $\star$ $P(\hat{\mathbb{R}}^N) = Id_{\mathcal{H}}$.

$\star$ $\forall v \in \mathcal{H}$ $B \mapsto \langle P(B) v, v \rangle$ is a positive Borel measure on $\hat{\mathbb{R}}^N$, mass $= \|v\|^2$.

$\star$ $\gamma \in \Gamma$:

$$\pi(\gamma^{-1}) P(B) \pi(\gamma) = P(\gamma^* B)$$

$\star$ projection onto $\mathcal{H}^{\mathbb{R}^N}$ = subspace of $\mathbb{R}^N$-inv. vectors

$$= P(\{0\})$$

Remark: P is telling you how much weight π assigns to each of the irreducible components

Example: $\pi : \mathbb{R} \to \underset{V_1}{\mathbb{C}} \times \underset{V_2}{\mathbb{C}}$, $\pi(x)(v_1, v_2) = (e^{ix} v_1, e^{i2x} v_2)$

$$\Rightarrow P(\varphi) = \begin{cases} \mathrm{Proj}(V_j) & \text{if } \varphi = j \\ 0 & \text{otherwise} \end{cases}$$

$\star$ Normally P defined on $\mathcal{B}(\hat{\mathbb{R}})$ simpler

<u>Burger's Criterion</u>: $\varphi: \Gamma \to GL_N\mathbb{R}$ such that there are no inv. prob meas on $\mathbb{P}(\hat{\mathbb{R}}^N)$ (proj space)

$\Rightarrow (\Gamma \ltimes_\varphi \mathbb{R}^N, \mathbb{R}^N)$ has Rel(T)

<u>Proof</u>: (Combined Burger & Shalom Bounded Gen. See My Thesis)

<u>Proof</u> Let $\pi: \Gamma \ltimes_\varphi \mathbb{R}^N \to \mathcal{U}(\mathcal{H})$ be unitary rep w/ almost Γ-inv. vectors (don't Need $\mathbb{R}^N$!)

$\Rightarrow \pi|_{\mathbb{R}^N} \rightsquigarrow P: \mathcal{B}(\hat{\mathbb{R}}^N) \to \mathrm{Proj}(\mathcal{H})$.
proj valued meas.

If $\{v_n\}$ sequence of <u>unit</u> vect's alm. Γ-inv.

$\Rightarrow \mu_n(B) := \langle P(B)v_n, v_n\rangle$ is almost Γ-inv. prob. measure on $\mathbb{P}(\hat{\mathbb{R}}^N)$

<u>i.e.</u>: $\|\gamma_*\mu_n - \mu_n\| := 2 \sup_{B \in \mathcal{B}(\hat{\mathbb{R}}^N)} |\gamma_*\mu_n(B) - \mu_n(B)|$

Ex

$\leq 2\|\pi(\gamma)v_n - v_n\| \xrightarrow[n\to\infty]{} 0$

Shalom p153 Bdd Gen

Suppose $(\Gamma \ltimes_\varphi \mathbb{R}^N, \mathbb{R}^N) \notin \mathrm{rel}(T)$

$\Rightarrow$ Assume $\pi|_{\mathbb{R}^N} \not\supset I_{\mathbb{R}^N}$ ie $\mu_n\{0\} = 0 \ \forall n$.

$\Rightarrow$ Can push $\mu_n \mapsto \bar{\mu}_n \in \mathrm{Prob}(\mathbb{P}(\hat{\mathbb{R}}^N))$

$\Rightarrow \|\gamma_*^* \bar{\mu}_n - \bar{\mu}_n\| = \|\gamma_* \mu_n - \mu_n\| \xrightarrow{n \to \infty} 0$

Banach-Alaoglu $\mathrm{Prob}(K)$, K compact metrizable is w-* compact (w-* top $\mu_n \to \mu$ means $\int f d\mu_n \to \int f d\mu \ \forall f$ meas)

Let $\bar{\mu} \in \mathrm{Prob}(\mathbb{P}(\hat{\mathbb{R}}^N))$ be a w-* limit point of $\{\bar{\mu}_n\}$.

$\Rightarrow \bar{\mu}$ is Γ-inv ※.

Cor $\varphi: \Gamma \to GL_N\mathbb{R}$ be strongly irreducible (i.e. no finite index subgroup reducible). If $(\Gamma \ltimes \mathbb{R}^N, \mathbb{R}^N) \notin \mathrm{rel}(T)$ then $\overline{\varphi(\Gamma)}$ compact.

Cor. $(SL_N\mathbb{Z} \ltimes \mathbb{R}^N, \mathbb{R}^N)$ has $\mathrm{rel}(T)$ (& hence so does $(SL_N\mathbb{Z}^N, \mathbb{Z}^N)$)