Amenability:

Ref's: "Geometric Group Theory" Bestvina-Sageev-Vogtmann

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Bekka-de la Harpe-Valette [BHV]

Def. Γ discrete countable group, $p \in [1,\infty)$

$p \neq \infty$ $\ell^p(\Gamma) = \{ f : \Gamma \to \mathbb{C} \mid \sum_{\gamma \in \Gamma} |f(\gamma)|^p < \infty \}$

$p = \infty$ $\ell^\infty(\Gamma) = \{ f \ \cdots \mid \sup_{\gamma \in \Gamma} |f(\gamma)| < \infty \}$

* These are metric spaces

Fact: $\Gamma \curvearrowright X$ discrete countable

$\rightsquigarrow \Gamma \curvearrowright \ell^p(X)$ by isom's $(\gamma f)(x) = f(\gamma^{-1} x)$

Ex: Show this is an action

Example: $X = \Gamma$, Left reg. rep

$X = \Gamma/H$ Left quasi-reg. rep.

Ex: Show $\ell^p(X) \cong \bigoplus_{s \in S}^{p} \ell^p(\Gamma/H_s)$ some set S

The isometry is Γ-equivariant.

Ex: Show $\ell^p(X)$ has (nonzero) Γ-inv. vector

$\iff [\Gamma : H_s] < \infty$ some $s \in S$

Def $\Rightarrow$ TH: Γ is amenable iff one of the following equivalent conditions hold:

1. ★ $\exists$ Left-inv. finitely add. meas on 2^Γ
2. ★ $\exists$ sequence of Følner sets
3. ★ $\ell^2(\Gamma) \succ 1_\Gamma$ ($\ell^2(\Gamma)$ has alm. inv. vectors)
4. ★ K compact metriz. & $\Gamma \to \mathrm{Homeo}(K)$ $\Rightarrow$ $\exists$ Γ-inv. prob. measure on K.
5. ★ Γ does not admit a Ponzi Scheme.
6. ★ Γ not paradoxical.

Some Def's. (1) $m(\Gamma) = 1$, $m(\bigsqcup_{i=1}^{n} E_i) = \sum m(E_i)$, $m(\gamma E) = m(E)$

(2) $\exists F_n \subseteq F_{n+1} \subseteq \cdots$ $\bigcup_n F_n = \Gamma$, & $\dfrac{|F_n \Delta \gamma F_n|}{|F_n|} \longrightarrow 0$

(3) $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ has alm inv vects if $\exists v_n \in \mathcal{H}$

st $\dfrac{\|\pi(\gamma)v_n - v_n\|}{\|v_n\|} \xrightarrow{n\to\infty} 0$

⚠ — $\|v_n\|$

(5) $P : \Gamma \to \Gamma$ st $|P^{-1}(\gamma)| \geq 2$ $\forall \gamma \in \Gamma$ & $\exists S \subset \Gamma$ finite: $P(\gamma) \in \gamma S$ $\forall \gamma$.
(Ponzi scheme)

(6) Paradoxical. $\exists X$, $\Gamma \curvearrowright X$, $X = A \sqcup B$, $A = \bigsqcup_{i=1}^{n} A_i$, $B = \bigsqcup_{j=1}^{m} B_j$

$\exists \gamma_i, \eta_j \in \Gamma$ $\bigsqcup \gamma_i A_i = X = \bigsqcup \eta_j B_j$

Ex (1) If Γ amen & $H < \Gamma \Rightarrow H$ amen (not true if Γ loc. compact & H not closed)

(2) If $1 \to N \to \Gamma \to Q \to 1$ exact

Γ amen $\Leftrightarrow Q$ & N amen.

(3) $\{\Gamma_i\}_{i \in I}$ directed fam'y, $\Gamma = \bigcup_{i \in I} \Gamma_i$ is amen $\Leftrightarrow \Gamma_i$ amen $\forall i$

(4) Finite groups, $\mathbb{Z}$, finitely generated abelian groups, & finally all discrete countable abelian groups amen

(5) Solvable groups

(6) $\Gamma \curvearrowright X$, $E \subset X \Rightarrow \exists$ finitely add. measure on X that is Γ-inv. w/ $\mu(E) = 1 \Rightarrow E$ not Γ-paradoxical

(7) Subexp. growth $\Rightarrow$ amenable

(8) How many directions can you prove for equivalences of def.?

(9) Prove F_2 not amenable.

Def. Γ is linear if $\Gamma \hookrightarrow GL_n\mathbb{R}$ some n.

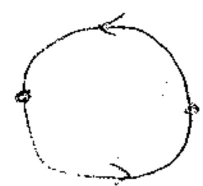
<u>Question</u> Which linear groups are amenable?

<u>Locally Compact Groups</u>

- $\star$ SO_n is compact (O_n) or $SO(V)$
- $\star$ Soluble groups (virtually solv.) conjugate into upper Δ'ar $\begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}$

$\star$ Amenable groups have invariant measures when acting on compact metriz. spaces

$\mathbb{P}(\mathbb{R}^n)$ = equiv classes of lines through $\mathcal{O}$ = compact metriz

$\boxed{Ex}$ $\mathbb{P}(\mathbb{R}^1) = \{\star\}$, $\mathbb{P}(\mathbb{R}^2) \cong S^1$, $\mathbb{P}(\mathbb{R}^3) \cong$ 

Grassman Varieties also compact, metriz

$Gr(k,V)$ = equiv. classes of k-dim'l subspaces in V.

Example of amenable group in $GL(V)$

$$\begin{pmatrix} SO(k) & \star \\ & D(n-k) \end{pmatrix}$$

Corresponds to Soluble $\triangleleft$ G.

& Those have invariant measures on $\mathbb{P}(V)$

Lebesgue & anything. (We will see)

Furstenberg Lemma $\mu \in \mathbb{P}(\mathbb{R}^N)$, $\Gamma \leq PGL_N\mathbb{R}$

Assume μ is Γ-invariant

$\Rightarrow$ either Γ is precompact or $\exists\, 0 \neq V \subsetneq \mathbb{R}^N$ w/ $\mu[V] > 0$

Proof Assume Γ not precompact $\Rightarrow \exists\, \gamma_n \xrightarrow{\text{in } PGL_N\mathbb{R}} \infty$ fixed ($\&\ \gamma_n\mu = \mu$)

Lift $\gamma_n \in PGL_N\mathbb{R}$ to $\tilde{\gamma}_n \in GL_N\mathbb{R}$ st

$$\|\tilde{\gamma}_n\| := \max|\text{entries}| = 1 \quad \forall n.$$

Since $M_N(\mathbb{R})$ is locally compact $\exists\, \tilde{g} \in M_N(\mathbb{R})$

st $\|\tilde{g}\| = 1$ & $\tilde{\gamma}_n \to \tilde{g}$

$\Rightarrow \operatorname{Im}(\tilde{g}) \neq 0$ & $\gamma_n \to \infty \Rightarrow \operatorname{Ker}(\tilde{g}) \neq 0$

$\Rightarrow \forall x \in \mathbb{P}(\mathbb{R}^N) \setminus [\operatorname{Ker}\tilde{g}] \quad \gamma_n x \to gx \in [\operatorname{Im} g]$

Grassmann Varieties compact

$\Rightarrow \exists\, V$ st $\gamma_n[\operatorname{Ker}\tilde{g}] \longrightarrow V$ (up to pass subseq)

Let $A_0 = [V] \cup [\operatorname{Ker}\tilde{g}]$ Note: A_0 closed!

Claim $\mu(A_0) = 1$

Fix metric on $\mathbb{P}(\mathbb{R}^N)$ let $D_{A_0}(x) = \mathrm{dist}(x, A_0)$

<u>Since</u> $\gamma_n[\ker \tilde{g}] \to [V]$ &

$\gamma_n(x) \to g(x) \in [W] \quad \forall x \notin [\ker \tilde{g}]$

$\Rightarrow D_{A_0}(\gamma_n x) \xrightarrow[n\to\infty]{} 0$ <u>Note</u> D_{A_0} bounded continuous

<u>Consider</u> $\mathbb{P}(\mathbb{R}^N) \setminus A_0 = \bigcup_{m\in\mathbb{N}} \underbrace{\{x \;\; D_{A_0}(x) > \frac{1}{m}\}}_{A_m}$

$$\Rightarrow \frac{1}{m}\mu(A_m) \le \int_{A_m} D_{A_0}(x)\,d\mu(x) \le \int_{\mathbb{P}(\mathbb{R}^N)} D_{A_0}(x)\,d\mu(x)$$

$$\overset{\mu \text{ is } \gamma_n\text{-inv}}{=} \int_{\mathbb{P}(\mathbb{R}^N)} D_{A_0}(x)\,d\mu(\gamma_n^{-1}x) = \int_{\mathbb{P}(\mathbb{R}^N)} D_{A_0}(\gamma_n x)\,d\mu(x) \longrightarrow \int 0\, d\mu$$

Dominated Convergence Th

$\Rightarrow \mu(A_0) = 1$

<u>To finish</u> Let V_0 be minimal dim st $\mu[V_0] > 0$

(<u>Note</u> exists by above) Then $\gamma \in \Gamma$ $\mu[\gamma V_0] = \mu[V_0] > 0$

$\Rightarrow \mu([\gamma V_0] \cap [V_0]) = 0$ or $\mu([\gamma V_0] \cap [V_0]) > 0$

& by minimality of dim $\Rightarrow [\gamma V_0] = [V_0]$

$\Rightarrow \Gamma[V_0] = [V_0] \cup \cdots \cup [V_k]$ since dim $< \infty$ & μ prob. $\blacksquare$