MSRI Summer School on Geometric Group Theory, Oaxaca Mexico, July 1–12, 2019
Exercises for the Minicourse by Spencer Dowdall
Geometry and topology of free group automorphisms: hyperbolic extensions

1 – Free groups and Folding

Recall that the *free group* F_X on a set X consists of all freely reduced finite (or empty) words in the alphabet $X \sqcup X^{-1}$, where the group operation is "concatenate and freely reduce

1. Convince yourself that the definition above indeed defines a group. (Check that multiplication is well-defined (independent of choices in reducing), that multiplication is associative, and the axioms regarding inverses and the identity).

2. The free group F_X is alternately defined by the *universal property* that: For any group G and any set map $\varphi_0 \colon X \to G$ there is a unique group homomorphism $\varphi \colon F_X \to G$ extending φ_0.
   (a) Convince yourself that our definition of F_X above satisfies the universal property.
   (b) Prove that group homomorphisms $F_X \to G$ are in bijective correspondence with set maps $X \to G$.

A *graph* is a 1–dimensional cell complex. This is formally defined as a tuple $(V, E, -, \iota)$ where V is the set of vertices, E is the set of oriented edges, $- \colon E \to E$ is a free involution (i.e $\bar{e} \neq e$ and $\bar{\bar{e}} = e$ for all $e \in E$), and $\iota \colon E \to V$ is a function recording the initial vertex of each oriented edge.

A graph *morphism* is a cellular map $G \to G'$ that sends each open edge of G homeomorphically onto an open edge of G' (where we consider morphisms equivalent if they differ by a homeomorphism (of G and/or G') that is isotopic to the identity rel vertices). Formally, a morphism $(V, E, -, \iota) \to (V', E', -, \iota)$ is a pair of maps $V \to V'$ and $E \to E'$ that commute with $-$ and ι. You should convince yourself these are the same concept.

3. Show that a graph morphism $G \to G'$ is an *immersion* (i.e., locally injective) iff it satisfies:
$$\iota(e_1) = \iota(e_2) \text{ and } f(e_1) = f(e_2) \implies e_1 = e_2 \text{ for any edges } e_1, e_2 \text{ of } G,$$

4. Let $f \colon X \to Y$ be an immersion of finite graphs. Show that it is possible to attach finitely many 0 and 1 cells to X to obtain a graph $\tilde{X}$ to which f extends to a morphism $\tilde{f} \colon \tilde{X} \to Y$ that is a covering map. Moreover, if Y has only one vertex, it is possible to build $\tilde{X}$ by only attaching edges. Conclude that any immersion is a composition of an embedding and a covering map.

5. An *elementary homotopy* of an edge path $e_1, \dots, e_k$ is a move that inserts or deletes a consecutive pair of edges of the form $e, \bar{e}$.
   (a) Show that every edge path can be transformed into a reduced edge path via a sequence of elementary homotopies. (This is called *tightening*.)
   (b) Show that two reduced edge paths are homotopic rel endpoints iff they are equal.
   (c) Show that edge paths are related by an elementary homotopy iff they are homotopic rel endpoints
   (d) Show that for a graph G and vertex v, elements of $\pi_1(G, v)$ are in bijective correspondence with reduced edge paths that start and stop at v.

6. Show that an immersion between graphs is π_1–injective. (That is, if $f\colon G \to G'$ is an immersion and $v \in G$ is a vertex, then $f_*\colon \pi_1(G,v) \to \pi_1(G', f(v))$ is injective.)
7. If e_1, e_2 are edges of a graph G such that $\iota(e_1) = \iota(e_2)$ and $e_2 \neq e_1 \neq \bar{e_2}$, we may *fold* (i.e. identify e_1 with e_2 and $\tau(e_1)$ with $\tau(e_2)$) to quotient graph morphism $G \to G'$. Prove that:
   (a) if $\tau(e_1) \neq \tau(e_2)$ in G, then the fold $G \to G'$ is a homotopy equivalence.
   (b) if $\tau(e_1) = \tau(e_2)$ in G, then the fold $G \to G'$ is π_1–surjective but not π_1–injective.
8. Prove Stalling's Theorem that every morphism $G \to G'$ of finite graphs factors as
$$G = G_0 \to G_1 \to \cdots \to G_k \to G',$$
   where each map $G_i \to G_{i+1}$ is a fold and the last map $G_k \to G'$ is an immersion.
9. Prove/convince yourself that the fundamental group of any graph is a free group. Here are several routes you might take:
   (a) Use a spanning tree T in G to show that elements γ of $\pi_1(G,v)$ can be put in a normal form that agrees with our definition of F_X where X is the set of edges of $G \setminus T$.
   (b) Use van Kampen's theorem and the fact $\pi_1(S) \cong \mathbb{Z}$ to conclude that $\pi_1(G,v)$ satisfies the universal property of free groups.
   (c) Directly show that $\pi_1(G,v)$ satisfies the universal property of free groups. (Use topology: Choose an appropriate subset $X \subset \pi_1(G,v)$ to serve as the free basis. For any other group H and set map $X \to H$, take a space Y with $\pi_1(Y,y) \cong H$ and build a map $G \to Y$ that induces the desired homomorphism $\pi_1(G,v) \to H$.)
10. Prove the Nielsen–Schreier Theorem: Every subgroup of a free group is free.
11. Let $F_1 = F_{\{a,b\}}$ and $H = \langle a^3 b, \bar{a}bab, a^2\bar{b}a \rangle$. In class we saw how to used Stalling's folds to find a free basis of H.
   (a) Try different factorizations / folding sequences. Check that the resulting graph is always the same.
   (b) How do we know the result of this process indeed gives a free basis for H?

Recall that the *core* of a based graph Y is the smallest subgraph that contains the base vertex and to which Y deformation retracts.

12. Let G be a finite graph and $v \in G$ a vertex, so $\pi_1(G,v)$ is free. Let $H \leq \pi_1(G,v)$ be any finitely generated subgroup and $(Y_H, \tilde{v}) \to (G,v)$ the corresponding cover of this subgroup.
   (a) Prove that the following are equivalent:
      1. The core of $(Y_Y, \tilde{v})$.
      2. The largest connected finite subgraph of Y_H that contains $\tilde{v}$ and has no valence 1 vertices (except possibly at $\tilde{v}$).
      3. The union of images of all reduced edge paths of Y_H that start and end at $\tilde{v}$.
      4. The union of the finitely many reduced edge paths representing the generators of H.
   (b) Show that Y_H can be built from the core by attaching trees at the vertices.

(c) Show the core of Y_H can be calculated via the Stalling's fold method described in lecture. (That is, prove the folding method always terminates with the core of Y_H.)

13. If $H \leq F_n$ is finitely generated and normal, prove that either $H = \{1\}$ or else $[F_n : H] < \infty$.

14. Given a generators of a finitely generated subgroup $H \leq F_n$:
    (a) Can you compute its normalizer $N(H) = \{w \in F_n \mid wHw^{-1} = H\}$?
    (b) What can you say about $[N(H) : H]$?
    (c) Given $w \in F_n$, can you algorithmically decide if $w \in H$?
    (d) Given $w \in F_n$, can you algorithmically decide if w is conjugate to an element of H?
    (e) Can you decide if H is normal in F_n?
    (f) Can you decide if $[F_n : H] < \infty$?

15. Let $F_2 = F_{\{a,b\}}$ and $H = \langle abab^{-1}, ab^2, babba^3b^{-1} \rangle$.
    (a) What is the rank of the free group H? Find a free basis of H.
    (b) Is $ab^2a^{-2}ba63b^{-1}$ and element of h? What about b?
    (c) Does H have finite index in F_2?
    (d) Is H normal in F_2?

16. Consider the subgroup $H = \langle a^2, b^2, aba^{-1}, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$ of $F_2 = F_{\{a,b\}}$. Try to compute it's normalizer $N(H) = \{w \in F_2 \mid wHw^{-1} = H\}$. What is the index $[N(H) : H]$?

17. Show that the subgroup $H = \langle b^n ab^{-n} \mid n \in \mathbb{Z} \rangle$ of $F_2 = F_{\{a,b\}}$ is normal but not finitely generated. (Hint: try to build a covering that represents this subgroup).

18. Given a homomorphism $F_n \to F_m$, can you decide when h is injective/surjective/bijective?

19. Show that for any homomorphism $h \colon F_n \to F_m$, there is a free factorization $F_n = A * B$ such that h kills A (i.e., $h(A) = \{1\}$) and h is injective on B.

20. Prove Marshall Hall's Theorem: For every finitely generated $H \leq F_n$, there exits a finite-index subgroup $H' \leq F_n$ so that $H \leq H'$ with H a free factor of H'.

21. Prove that F_n is *Hopfian*, meaning that every epimorphism $F_n \to F_n$ is an automorphism. This says that F_n is not isomorphic to a proper quotient of itself. Conclude that any generating set of F_n of size n is a free basis.

22. Prove that F_n is *residually finite*: For any nontrivial $w \in F_n$, there exists a finite-index normal subgroup $H' \lhd F_n$ so that $w \notin H'$. (Hint: Build a smart covering of the rose.)

23. Show that F_n has the following property: If $H \leq F_n$ is a finitely-generated subgroup such that for every $w \in F_n$ there is some $k = k(w) > 0$ such that $w^k \in H$, then H has finite index in F_n. Note that this is not true for arbitrary groups! Indeed, there exists infinite finitely generated groups (e.g Burnside groups) where every element has finite order.

24. Show that for every finitely generated $H \leq F_n$ and elements $g_1, \ldots, g_k \in F_n \setminus H$, there is a finite-index subgroup $H' \leq F_n$ such that $H \leq H'$ with H a free factor of H' and so that $g_1, \ldots, g_k \notin H'$.

25. Show that if H is a finite index subgroup of F_n, then $\text{rank}(H) - 1 = [F_n : H](n-1)$.

26. In the statement of Nielsen's theorem that $\text{Aut}(F_n)$ is finitely generated, verify that each of the maps $F_n \to F_n$ in the indicated generating set are in fact automorphisms of F_n. (Use the universal property.)

27. Prove that $\text{Aut}(F_4)$, where $F_4 = F_{\{a_1,\ldots,a_4\}}$ is generated by the following 4 elements:

    - Φ_1, which sends $a_i \mapsto a_{i+1}$ (with indices taken mod 4)
    - Φ_2, which sends $a_1 \mapsto a_2$, $a_2 \mapsto a_1$, $a_3 \mapsto a_3$, and $a_4 \mapsto a_4$.
    - Φ_3, which sends $a_1 \mapsto a_1^{-1}$ and $a_i \mapsto a_i$ for $i = 2, 3, 4$.
    - Φ_4, which sends $a_1 \mapsto a_1 a_1$ and $a_i \mapsto a_i$ for $i = 2, 3, 4$.

28. If T and T' are two spanning trees of a finite graph G, show there is a sequence $T = T_0 \to \cdots \to T_k = T'$ of spanning trees in G such that each move $T_i \to T_{i+1}$ is a single edge swap (i.e the symmetric difference of T_i and T_{i+1} is exactly 2 edges).

2 – $\mathrm{Out}(F_n)$ and Outer Space

29. Show that the center $Z(F_n) = \{w \in F_n \mid wx = xw \text{ for all } x \in F_n\}$ is trivial. Conclude that $\mathrm{Inn}(F_n)$ is isomorphic to F_n.

30. For G a finite graph, let $\mathrm{HE}(G)$ be the set of homotopy equivalences $G \to G$. Put an equivalence relation on $\mathrm{HE}(G)$ by declaring elements to be equivalent iff they are homotopic. Show that the quotient $\mathrm{HE}(G)/\sim$ is naturally a group isomorphic to $\mathrm{Out}(\pi_1(G, v))$ for any $v \in G$.

If X is a free basis of F_n, define the *word length* and *conjugacy length* with respect to X by

$$|w|_X = \min\left\{n \mid w = x_1 \cdots x_n \text{ with each } x_i \in X \cup X^{-1}\right\} \quad \text{and} \quad \|w\|_X = \min_{g \in F_n} \left|gwg^{-1}\right|$$

The *stretch factor* of an element $\phi \in \mathrm{Out}(F_n)$ is defined as $\log \lambda(\phi) = \sup_{\alpha \in F_n} \lim_{n \to \infty} \dfrac{1}{n} \log \|\phi^n(\alpha)\|_X$.

31. Show that the definition of the stretch factor $\lambda(\phi)$ is independent of the free basis X.

32. In lecture we defined Outer space X_n to be the space of equivalence classes of marked metric graphs of volume 1. Show that Outer space X_n may alternately be defined as the space of metric trees equipped with a minimal, isometric F_n action that has covolume 1, up to equivalence given by F_n–equivariant isometry.

33. For an automorphism $\phi \in \mathrm{Out}(F_n)$ and point $\Gamma \in X_n$, suppose that $\sigma\colon \Gamma \to \Gamma \cdot \phi$ is an optimal difference of markings whose tension graph $\Delta = \Delta(\sigma) \subset \Gamma$ and associated illegal turn structure on Δ satisfy the conditions:

    - $\sigma(\Delta) \subset \Delta$,
    - σ sends each edge of Δ to a legal path, and
    - σ sends legal turns to legal turns.

    Prove that Γ realizes $\tau(\phi)$ (i.e., $d(\Gamma, \Gamma \cdot \phi) = \inf\{d(\Gamma', \Gamma' \cdot \phi) \mid \Gamma' \in X_n\}$) and that $\lambda(\phi) = e^{\tau(\phi)}$.

34. Consider the automorphism of $F_2 = F_{\{a,b\}}$ defined by $\phi(a) = a$ and $\phi(b) = ab$. Show that ϕ acts parabolically on by finding a sequence $\Gamma_k \in X_2$ such that $d(\Gamma_k, \Gamma_k \cdot \phi)$ tends to 0.

35. Let us call an automorphism of F_n "positive" if it maps each generator a_i to a positive word in the alphabet $\{a_1, \ldots, a_n\}$ (i.e., without using any inverses a_j^{-1}). Show that if ϕ is positive, then obvious representative on the rose $\Phi\colon R_n \to R_n$ is a train track representative.

36. Consider the automorphism ϕ of $F_{\{a,b,c\}}$ defined by $\phi(a) = ac^2$, $\phi(b) = c$ and $\phi(c) = ab$. The previous problem says the obvious map $f\colon R_3 \to R_3$ of the rose is a train track representative.
    (a) Verify that ϕ is an automorphism.
    (b) Find the appropriate train track structure on R_3. That is, define the illegal turns so that f maps edges to legal paths and legal turns to legal turns.
    (c) Find the transition matrix M of f
    (d) Compute the largest eigenvalue of M and its associated eigenvector λ.
    (e) Put a metric on R_3 so that f stretches every edge by λ, and thus that the tension graph of f is all of R_3.

(f) What are the stretch factor $\lambda(\phi)$ and translation length $\tau(\phi)$? How do you know?

(g) (Bonus thing to consider: Compute the inverse ϕ^{-1} and carry out this analysis for it.)

(h) (More bonus: Now let ϕ_n be given by $\phi(a) = ac^n$, $\phi(b) = c$, and $\phi(c) = ab$. What happens to $\lambda(\phi_n)$, $\tau(\phi_n)$ and the metric on R_3 as $n \to \infty$? What about for ϕ_n^{-1}?)

37. Suppose Γ is a finite core graph and that $\sigma\colon \Gamma \to \Gamma$ is a map that sends vertices to vertices and edges to nondegenerate immersed paths. Show that finding an invariant train track structure on Γ is algorithmic (when it exists). (Hint: Build a finite directed graph whose vertices are the directions at the vertices of Γ and whose edges record the action of the derivative $D\sigma$. How does this graph help to find a train track structure or else show that none exists?)

38. Here is a more elaborate example: Let ϕ be the automorphism of $F_3 = F_{\{x_1,x_2,x_3\}}$ given by $\phi(x_1) = x_2$, $\phi(x_2) = x_2^{-1}x_1^{-1}x_2x_1x_3$ and $\phi(x_3) = x_1$. Let Γ be the graph with (oriented) edge set $E^+ = \{a, b, c, d\}$, vertex set $V = \{v_0, v_1\}$, and attaching maps $\iota(a) = \iota(b) = \iota(d) = v_0$ and $\iota(\bar{a}) = \iota(\bar{b}) = \iota(\bar{d}) = \iota(c) = \iota(\bar{c}) = v_1$. Let $f\colon \Gamma \to \Gamma$ be a map sending vertices to vertices and edges to immersed edge paths as follows: $f(a) = d$, $f(b) = a$, $f(c) = \bar{b}a$ and $f(d) = b\bar{a}d\bar{b}ac$.

(a) Draw a picture of Γ with a labeling so that you can see the map f.

(b) Show that f represents ϕ: Find an identification $F_3 \cong \pi_1(\Gamma)$ for with $f_* = \phi$.

(c) Verify that ϕ is an automorphism. (Maybe try the folding method!)

(d) Find an illegal turn structure on Γ so that f becomes a train track map.

(e) Find the transition matrix M of f.

(f) Put a metric on Γ so that the tension graph of f is all of Γ.

(g) Find the stretch factor $\lambda(\phi)$ and translation length $\tau(\phi)$.

3 – Hyperbolicity

39. Let $K \geq 1$ and $C, B \geq 0$ be given. Prove there exist constants $K' \geq 1$ and $C', B', A \geq 0$ such that the following holds: If $f\colon (X,d) \to (Y,\rho)$ is any (K,C)–quasi-isometry whose image is B dense (meaning $\forall y \in Y\ \exists x \in X$ so that $\rho(y, f(x)) \leq B$), then there exists a (K', C')–quasi-isometry $g\colon Y \to X$ that is B'–dense and such that $d(x, g(f(x))), \rho(y, f(g(y))) \leq A$ for all $x \in X$ and $y \in Y$. (That is, prove every quasi-isometry has quasi-isometry coarse inverse).

By a *geodesic* in a metric space (X,d) we mean a map $\gamma\colon I \to X$ where $I \subset \mathbb{R}$ is an interval and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in I$. We often confuse the map γ with its image in X. For any point $y \in X$, the *distance* to γ and the *closest-point-projection* to γ are defined by

$$d(y,\gamma) = \inf\{d(y,p) \mid p \in \gamma\} \qquad \text{and} \qquad \pi_\gamma(y) = \{p \in \gamma \mid d(y,p) = d(y,\gamma)\} \subset \gamma.$$

40. Let $\gamma \subset X$ be a geodesic and $y \in X$ any point.
    (a) Prove that the infimum $d(y,\gamma) = \inf\{d(y,p) \mid p \in \gamma\}$ is always realized. Thus $\pi_\gamma(y) \neq \emptyset$.
    (b) Prove that $\operatorname{diam}(\pi_\gamma(y))$ is finite.

A geodesic γ in a metric space X is called D*–strongly contracting*, where $D \geq 0$, if for all $y, y' \in X$ with $d(y,y') \leq d(y,\gamma)$ one has $\operatorname{diam}(\pi_\gamma(y) \cup \pi_\gamma(y')) \leq D$.

A geodesic γ in a metric space is *Morse* if for all $K \geq 1$ and $C \geq 0$ there exists $N = N(K,C)$ such that for any (K,C)-quasi-geodesic $\rho\colon [a,b] \to X$ with $\rho(a), \rho(b) \in \gamma$, the Hausdorff distance $d_{\mathrm{Haus}}(\rho,\gamma)$ between the sets γ and $\rho = \rho([a,b]) \subset X$ is at most N. We call the requisite function $N\colon [1,\infty) \times [0,\infty) \to \mathbb{R}$ a *Morse gauge* for γ and say that γ is "N–Morse."

41. Prove that for every $D \geq 0$ there exists a Morse gauge N such that any D–strongly contracting geodesic in any geodesic metric space X is N–Morse. (*Hint:* Pick some large constant M to be determined later. Argue that if ρ contains a long subsegment $\rho([c,d])$ that lies outside the M–neighborhood of γ, then closest-point-projection to γ contracts this subpath by a definite amount. If M is sufficiently large compared to K, then the geodesic concatenation from $\rho(c)$ to $\pi_\gamma(\rho(c))$ to $\pi_\gamma(\rho(d))$ to $\rho(d)$ will be significantly shorter than the quasi-geodesic $\rho([c,d])$. This will violate the fact that ρ is a quasi-geodesic, unless $|d - c|$ is uniformly bounded.)

42. Let X be a δ–hyperbolic geodesic metric space.
    (a) Prove that all geodesic quadrilaterals in X are 2δ–thin.
    (b) Prove that if $y, y' \in X$ are such that $\operatorname{diam}(\pi_\gamma(y) \cup \pi_\gamma(y')) \geq 10\delta$, then $d(y,y') \geq d(y,\gamma) - 2\delta$. (We sketched this in lecture.)
    (c) Show that for any number r there exists a bound B such that $d(y,y') \leq r$ implies $\operatorname{diam}(\pi_\gamma(y) \cup \pi_\gamma(y')) \leq B$.
    (d) Prove there exists $D \geq 0$ such that every geodesic in X is D–strongly contracting.

43. Let X be any geodesic metric space, γ be a geodesic, and $y \in X$ any point. Choose a point $z \in \pi_\gamma(y)$. Let ρ be a geodesic joining y to z. Let β be the concatenation of ρ with the subgeodesic of γ traveling away from z (either to the left or right); parameterize β by arclength. Prove that β is a $(3,0)$-quasi-geodesic.

44. Prove that for every $D \geq 0$ there exists $\delta \geq 0$ such that if X is a geodesic metric space in which every geodesic is D–strongly contracting, then X is δ–hyperbolic.

45. Cannon proved the amazing fact that if G is any hyperbolic group with finite generating set S, then the formal power series $p(x) = \sum_{k=0}^{\infty} \sigma_k x^k$ with coefficients $\sigma_k = \#\{g \in G : |g|_S = k\}$ is a rational function. Verify this in the case of the free group F_n with S its standard basis:

    (a) Calculate the cardinality $\sigma_k = \#\{w \in F_n : |w|_S = k\}$ of the k–sphere in F_n.

    (b) Form the formal power series $\sum_{k=0}^{\infty} \sigma_k x^k$ and determine what rational function it is. If the general case is too hard, try it for $n = 2$ or 3.