Conrads Property

So i will talk about the Conrad's property for a left-ordenable group.

The definition it's the following

Def: We say that the left-orderable group $(\Gamma, \leq)$ has the Conrad's property (or is a C-order) if for all positive elements $f, g \in \Gamma$ there exists a $n \in \mathbb{N}$ such that

$$gf^n > f$$

A firts proposition it's an alternative definition

(Jimenez)

Proposition 1.- If $(\Gamma, \leq)$ it's a C-order then for all positive elements $f, g \in \Gamma$

$$gf^2 > f$$

holds.

We will proof this for contrapositiva

Proof: Suppose that two positive elements f, g are such thar

$$gf^2 \leq f$$

then $(f^{-1}gf)f \leq id \Rightarrow f^{-1}gf < id \Rightarrow gf < f$

because f it's positive

So let's take $h = gf^{>id}$, then for $n \in \mathbb{N}$

$$g(gf)^1 < gf \quad (\ gh < h)$$

$$g(gf)^2 = ggfgf < ggff = ggf^2 \leq gf \ (gh^2 < h)$$

and for all $n > 2$

$$gh^n = gh^{n-2}(gf)(gf) < gh^{n-2}(gf)f$$
$$\leq gh^{n-2}f$$
$$= gh^{n-3}gf^2$$
$$\leq gh^{n-3}f$$
$$\vdots$$
$$\leq g(gf)f$$
$$\leq gf = h$$

So the order $<$ does not satisfy the Conrad's property.

This implies the following fact about the Topology of C-orders in the space $\mathcal{O}(\Gamma)$

Corollary 2 The family of C-orders $<$ in a left-ordenable group Γ is a closed set of $\mathcal{O}(\Gamma)$.

Proof: Let's see the next equation for all $f, g \in \Gamma$

$$\mathcal{U}_{id,f} \cap \mathcal{U}_{id,g} \cap \mathcal{U}_{f^{-1}gf^2, id} = \{\leq \mid id \leq f,\ id \leq g,\ f^{-1}gf^2 \leq id\}$$

$$= \{\leq \mid f, g \text{ are positive and } gf^2 \leq f\}$$

$$= N(f,g)$$

This set is open (because it's finite intersection of open set's), then the unión or

$$\bigcup_{f,g\in\Gamma} N(f,g)$$

is open, and his complement (the $\mathcal{C}$-orders for proposition 1) is a closed set.

The next step is stablish a charectization of $\mathcal{C}$-orders. First we define:

Def.- For a $S \subseteq \Gamma$ we define

$$\mathcal{C}<S>$$

as the smalles semi-group that satisfy

i) $S \subseteq \mathcal{C}<S>$

ii) $\forall f,g \in \mathcal{C}<S>$, then $f^{-1}gf^{2} \in \mathcal{C}<S>$

Thm.- Γ admits a $\mathcal{C}$-order iff $\forall \{g_1,\dots,g_k\} \subseteq \Gamma \setminus \{id\}$ there exists a colection of $\eta_i \in \{-1,+1\}$ for $1 \leq i \leq k$ such that

$$id \notin \mathcal{C}<\{g_1^{\eta_1},\dots,g_k^{\eta_k}\}>$$

Proof: $\Rightarrow$] If $(\Gamma, \leq)$ is a $\mathcal{C}$-order and $\{g_1, \dots, g_k\} \subseteq \Gamma \setminus \{id\}$ Let's take n_i such that $g_i^{n_i} > id$, for Proposition $\underline{1}$, P (positive elements) we have

$$\text{i) } \forall i \quad g_i^{n_i} \in P$$
$$\text{ii) } \forall f, h \in P \Rightarrow \underline{f^{-1} h f^{2}} \in P$$

so $\mathcal{C}\langle\{g_1^{n_1}, \dots, g_k^{n_k}\}\rangle \subseteq P \subseteq \Gamma \setminus \{id\}$

$\Leftarrow$] Let's take $\{g_1, \dots, g_k\}$ and $n_i \in \{-1, +1\}$ such that

$$\underline{id \notin \mathcal{C}\langle\{g_1^{n_1}, \dots, g_k^{n_k}\}\rangle}$$

and define

$$\underline{\chi\mathcal{C}(g_1, \dots, g_k, n_1, \dots, n_k)} \quad --- (\star)$$

the set of functions $\operatorname{sing}: \Gamma \setminus \{id\} \to \{-, +\}$ such that

$$\operatorname{sing}(h) = + \quad \forall h \in \mathcal{C}\langle\{g_1^{n_1}, \dots, g_k^{n_k}\}\rangle$$

(There exist because $id \notin \mathcal{C}\langle\{g_1^{n_1}, \dots, g_k^{n_k}\}\rangle$

and Exercise

$$\mathcal{X}\mathcal{C}(g_1, \dots, g_k) \Rightarrow \text{closed set}$$

the unión of sets (☆) for all the compatible collection $\eta_1, \dots, \eta_k$

If we have

$$\{ \mathcal{X}_j = \mathcal{X}\mathcal{C}(g_{j,1}, \dots, g_{j,p_j}) \mid j = 1, \dots, \ell \}$$

We have that

$$\underbrace{\mathcal{X}\mathcal{C}\left(\bigcup_{j=1}^{\ell}\{g_{j,1}, \dots, g_{j,p_j}\}\right)}_{\neq \emptyset} \subseteq \bigcap_{j=1}^{\ell} \mathcal{X}_j$$

because $\{-,+\}^{\Gamma\setminus\{id\}}$ is compact

There exist a sing: $\Gamma\setminus\{id\} \rightarrow \{-,+\}$

such that belong to all

$$\mathcal{X}\mathcal{C}(g_1, \dots, g_k)$$

and if we take $P = \{h \in \Gamma \mid \text{sing}\, h = +\}$

we have $\forall f, h \in P$

$$f^{-1}hf^2 \in \mathcal{C}\langle\{f,h\}\rangle$$

$$\Rightarrow \text{sing}(f^{-1}hf^2) = + \Rightarrow$$

$$f^{-1}hf^2 > id$$

$\Rightarrow h f^2 > f \Rightarrow$ this sing function

[illegible]
defines a $\mathcal{C}$-order.

The last result will be

Thm: If Γ is locally indicable, then Γ admits a $\mathcal{C}$-order.

Proof: Let's take $\{g_1, \dots, g_k\} \subseteq \Gamma \setminus \{id\}$ and (by locally indicability)

$$\Phi_1 : \langle g_1, \dots, g_k \rangle \longrightarrow (\mathbb{R}, +)$$

non-trivial, Let's $i_1, \dots, i_{k'}$ the index (if any) such that

$$\Phi_1(g_{i_j}) = 0$$

Let's take

$$\Phi_2 : \langle g_{i_1}, \dots, g_{i_{k'}} \rangle \rightarrow (\mathbb{R}, +)$$

non-trivial and again $i'_1, \dots, i'_{k''}$ the index such that

$$\Phi_2(g_{i'_l}) = 0$$

and this process must finish in at most k-steps.

For all g_i we take the index $j(i)$ such that

$$\Phi_{j(i)}(g_i) \neq 0$$

$\Phi_{j(i)}(\ldots)$

and tak n_i s.t.

$$\Phi_{j(i)}(g_i^{n_i})>0$$

and take $\mathcal{C}<g_1^{n_1},\dots,g_k^{n_k}>$

if $f,h\in \mathcal{C}<g^{n_1},\dots,g_k^{n_k}> \subseteq <g_1^{n_1},\dots,g_k^{n_k}>$

and Φ_j is define at f, h then

i) $\Phi_j(fh)=\Phi_j(f)+\Phi_j(h)\geq 0$

ii) $\Phi_j(f^{-1}hf^2)=\Phi_j(h)+\Phi(f)\geq 0$

so if we have $h\in \mathcal{C}<g^{n_1},\dots,g_k^{n_k}>$

and $\Phi_1(h)=0 \Rightarrow h\in\mathcal{C}<g_{i_1}^{n_{i_1}},\dots,g_{i_{k'}}^{n_{i_{k'}}}>$

so $\Phi_2(h)\geq 0$ (for the same argument)

and if $\Phi_2(h)=0\Rightarrow h\in\mathcal{C}<g_{i_1'}^{n_{i_1'}},\dots,g_{i_{k''}}^{n_{i_{k''}}}>$

and eventually we find a index p s.t.

$$\Phi_p(h)>0$$ so $h\neq id$

as we want.

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