G = Discrete Countable

<u>Def:</u> $(G, \leq)$ total order is left invariant if $f<g \Rightarrow hf<hg$
& bi-invariant if Left + right inv

<u>Ex:</u> $\mathbb{Z}^n$ lexicographical

<u>Ex:</u> F_2 is L-orderable & in fact bi-orderable. (harder)

<u>Exercise:</u> If $\leq$ is L-order $f>g \not\Rightarrow f^{-1}<g^{-1}$ give example

$$f>g \Leftrightarrow f^{-1}f>f^{-1}g \Leftrightarrow 1>f^{-1}g.$$

<u>Exercise:</u> $\pi_1(\text{Surface}_+)$ Bi-orderable.
MCG(surface w/ ∂) L-order
Braid Groups L-Order
Thompsons groups

<u>Observe:</u> G has LO $\Rightarrow$ G torsion free
<u>Indeed:</u> $f \neq id \Rightarrow f<id$ (negative) or $f>id$ (positive)
If $f>id \Rightarrow f^n > f^{n-1} > \cdots > id$
$\Rightarrow f^n \neq id.$

<u>Exercise:</u> Give example of G torsion free, not L-orderable
(Must be non-abelian)

<u>Exercise</u>: Give example of group $(G, \leq)$ & $f > id$ st $gfg^{-1} < id$ some $g \in G$.

$\boxed{\text{TH:}}$ (Dedekind ⇝ Hölder ⇝) G admits an Archimedean order $\leq$ $\Leftrightarrow$ $(G, \leq) \hookrightarrow (\mathbb{R}, \leq)$, where order is Archimedean means $\forall f > id$ & $g \in G$ $\exists n$ $f^n > g$.

But this imposes a cardinality restriction on which groups have an Archimedean order.

<u>Paul Conrad '50's</u>:

<u>Def</u>: $\leq$ a L-order on G is Conradian if $\forall f > id$, $\forall g > id$ $\exists n \in \mathbb{Z}, fg^n > g$.
($\Leftrightarrow \exists n \in \mathbb{N}$)

$n > 0$

$fg^{-n} > g$ & $g > g^{-n} \Rightarrow fg > fg^{-n} > g$

<u>Exercise</u>: $\Leftrightarrow$ G is Conradian for $n = 2$

$\boxed{\text{TH:}}$ (Conrad - Brodzky) G is Conrad orderable
$\Leftrightarrow$ G is locally indicable, i.e. $\forall G_0 < G$ finitely gen'ed $\exists \phi : G_0 \to \mathbb{R}$ hom, nontrivial.

$\boxed{\text{TH}}$ (Dave Witte Morris) Assume G amen

G is L-orderable $\Leftrightarrow$ G is loc. indicable.

"Exercise": Give example of L-orderable not locally indicable

$\boxed{\text{TH:}}$ (Hyde-Lodha 2018, Matteban-Triestino)

$\exists G$ fin'ly gen'ed, L-orderable & simple.

The dynamical approach:

G is L-orderable $\Leftrightarrow$ acts on $(\Omega, \leq_\Omega)$ by order pres. bij's (faithfully)

$\Rightarrow$ trivial ; $\Leftarrow$: do on an orbit & use lexicographical

i.e. $\Omega = \{\omega_1, \dots \omega_n \dots\}$ or fix well order & define

$f > g$ if $f(\omega_1) > g(\omega_1)$ or if $=$ then $f(\omega_2) > g(\omega_2)$

or if $=$ then $\cdots$ get total order

Remark: by changing the well order can change $\leq$

$\boxed{\text{TH:}}$ (Folklore) G countable & L-orderable then

$G \hookrightarrow \text{Homeo}_+(\mathbb{R})$

conversely if $G \hookrightarrow \text{Homeo}_+(\mathbb{R}) \Rightarrow G$ is L-order

"Exercise": $x \mapsto x+1$, $x \mapsto x^3$ generate copy of $F_2 \leq \text{Homeo}_+\mathbb{R}$.

Proof of TH: $G = \{g_1, \dots, g_n \dots\}$ enumeration

$\Rightarrow$ Look @ $g_1 \mapsto p(g_1) = 0$, if $g_2 > g_1$ then $p(g_2) = p(g_1)+1$

$g_2 < g_1 \Rightarrow p(g_2) = p(g_1) - 1$.

For g_3 do same unless in middle & then $p(g_3)$ is midpoint

Exercise

$\Rightarrow G \curvearrowright \{p(g) : g \in G\}$ & this extends to a continuous action

Q 1: Assume $\leq$ is bi-inv. $\Rightarrow$ what do you get?

Q2: Same for Conradian

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Space of Orders: (Chabauty, Ghys, Sakora)

$\mathcal{LO}(G) = \{$ Left-orders on $G\}$

$\mathcal{BO}(G) = \{$ bi-orders $\}$; $\mathcal{CO}(G) = \{$ Conrad orders $\}$.

Chabauty Topology: $\leq \in \mathcal{LO}(G)$ Assume $f_1 > g_1, \dots, f_n > g_n$

$\Rightarrow N(\leq, f_1 > g_1, \dots, f_n > g_n) = \{ \leq' : \leq'$ satisfies all these ineq's $\}$

Ex: $\mathcal{LO}(G)$ w/ this topology is totally disconnected

<u>Remark</u>: topology same as $\leq \mapsto \{-1,1\}^{G\times G\setminus\Delta}$
according to how f,g are $\leq$ or $\geq$.

$G = \langle g_1, \dots, g_n \rangle$; define $\mathcal{B}(n)$ ball radius n
& $d(\leq, \leq') = \frac{1}{n}$ if n smallest radius
s.t. $\leq, \leq'$ do not coincide on $B(n)$.

If G is L-orderable & amenable $\Rightarrow$ G is locally indicable.

<u>Proof</u>: $G \curvearrowright \mathcal{LO}(G)$ by conj. compact metriz. space
G amen $\Rightarrow \exists \mu$ inv. measure

<u>Lemma</u>: $\mu(\mathcal{C}(G)) = 1$

* Uses Poincaré Recurrence!

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<u>DAY 2</u>:

G is L-O $\Leftrightarrow$ $G = P \sqcup P^{-1} \sqcup \{id\}$, both P, P^{-1} semigroups

<u>Proof</u>: $\Rightarrow$: Let $P = \{g \in G : g > id\}$, $P^{-} = \{g \in G : g < id\}$ be the positive & negative cones, respectively.

P is a semigroup: $g>id,\ h>id \Rightarrow gh>g>id$

P^- is too: $g<id,\ h<id \Rightarrow gh<g<id$ $\blacksquare$

<u>Conversely</u>: Given the decomposition define

$$g<h \Leftrightarrow 1<g^{-1}h \Leftrightarrow g^{-1}h \in P$$

<u>Total order</u>: $g \neq h \Rightarrow$ either $g^{-1}h \in P$ or $g^{-1}h \in P^-$

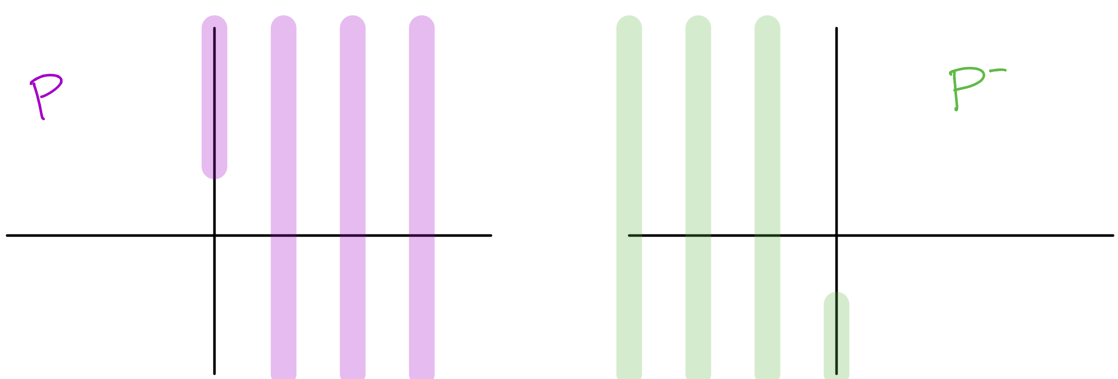
$\hookrightarrow g<h \qquad \hookrightarrow h<g$

<u>Transitivity</u>: $f>g$ & $g>h \Rightarrow g^{-1}f \in P$ & $h^{-1}g \in P$

$\Rightarrow h^{-1}gg^{-1}f \in P$ i.e $h^{-1}f \in P$

$\mathbb{Z}^2$: $P = \{(a,b) : a>0 \text{ or } a=0 \,\&\, b>0\}$

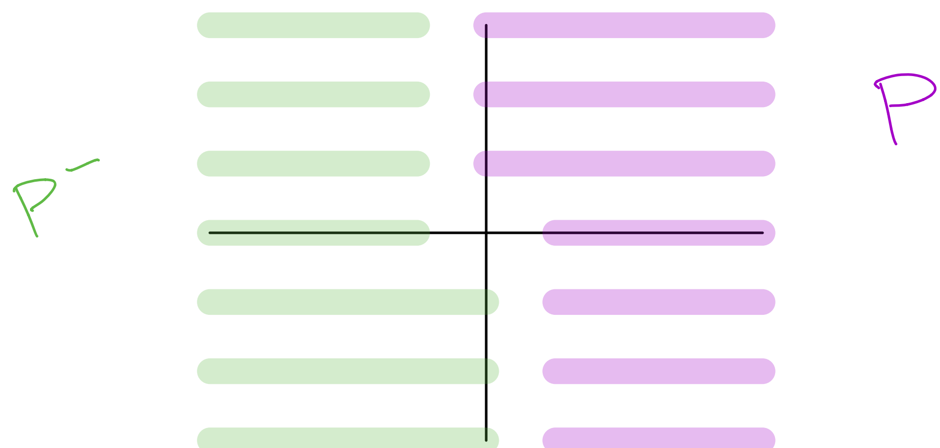
$P^- = \{(a,b) : a<0 \text{ or } a=0 \,\&\, b<0\}$



<u>Alternatively</u>: Let $\alpha \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow \mathbb{Z}^2 \cong \langle 1,\alpha\rangle$

<u>Ex</u>: $G = \pi_1(\text{Klein Bottle}) = \langle a,b \mid bab=a\rangle$
$= \langle a,b\rangle^+ \sqcup \langle a,b\rangle^- \sqcup \{id\}$

$\langle g,h\rangle^\pm$ $\leftrightarrow$ means words using only $\pm$ powers



<u>Alternative</u>: $\langle a,b^{-1}\rangle^+ \sqcup \langle a,b^{-1}\rangle^- \sqcup \{id\}$

$\star$ Can always swap $P^+ \leftrightarrow P^-$ get new order

<u>Claim</u>: $\pi_1(\text{Klein})$ has only these 4 orders

"<u>Proof</u>": Choice $\leftrightarrow$ to whether $a,b \gtrless id$.

Ex: $BS(1,-2) = \langle a,b : aba^{-1} = b^{-2} \rangle$ has only 4 orders

$\boxed{\text{TH:}}$ (Tororin) Complete Classif. of groups w/ fini many Left orders have solvable structure where "conjugation" reverses orient. on previous element.

Remark: (Linnell) Let $\leq$ be LO on G s.t. $P_\leq$ is finitely gen'ed $\Rightarrow$ $\leq$ is isolated point in $\mathcal{O}(G)$

Proof: $G = \langle a_1, \dots, a_k \rangle^+ \sqcup \langle a_1, \dots, a_k \rangle^- \sqcup \{id\}$
$= !$ pt in open set $a_i > id$

Ex: $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$
$= \langle a, b \mid ba^2 b = a,\ a = \sigma_1 \sigma_2, b = \sigma_2^{-1} \rangle$

$\Rightarrow B_3 = \langle a, b \rangle^+ \sqcup \langle a, b \rangle^- \sqcup \{id\}$.

Remark: If every cone is finitely gen'ed then there are finitely many LO's.

$\pi_1(K)$: $P_\leq = \langle a, b \rangle^+ \Rightarrow ba < a$ since $b^{-1} = a^{-1} ba < id$

Exercise: Find $x > y$ s.t. $x^2 < y^2$

Ex: $x = a$, $y = ab^{-1}$ check $x > y$ & $x^{-1} > y^{-1}$

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Free Groups:

Ping Pong Lemma: (Klein) ...

Suppose: $\leq$ is bi-inv on $G \rightsquigarrow$ action on $\mathbb{R}$

s.t: $p(h) \to p(gh)$ (to right)

$\Rightarrow gh > h \Leftrightarrow g$ positive

$g(x) \geq x \quad \forall x \in \mathbb{R}$ if $g > id$

$g(x) \leq x \quad \forall x \in \mathbb{R}$ if $g < id$.

* F_2 is biorderable: in particular has action as above

* $PL_+[0,1]$ is bi-orderable have $+, -$ as above.

in Particular Thompson's grp $\leq PL_+[0,1]$

Ex: Conversely if $G \leq Homeo_+(\mathbb{R})$ s.t. $g \neq id \Rightarrow g(x) \geq x$

or $g(x) \leq X \Rightarrow G$ is Bi-orderable.

Day 3:

$\boxed{\text{TH:}}$ (Vinogradov) G_i bi-orderable $\Rightarrow$ so is $G_1 * G_2$

Dynamical Proof for L-O:

Each G_i with LO $\rightsquigarrow$ $G_i \leq \mathrm{Homeo}_+(\mathbb{R})$
Use Baire Category to find φ st $\langle G_1, \varphi G_2 \varphi^{-1} \rangle \cong G_1 * G_2$

$\boxed{\text{TH:}}$ (Rivas) $\mathcal{LO}(G_1 * G_2)$ is a Cantor Set, in particular $\mathcal{LO}(F_2)$

Open Problems:

* $\exists$? $G \in (T)$ & Left orderable?
* is $\mathcal{BO}(F_2)$ a Cantor Set?

$\boxed{\text{TH:}}$ (Linnell) If $\mathcal{LO}(G)$ is ∞ $\Rightarrow$ uncountable. contains cantor set.

*Some remarks on Compactness:

G is LO $\iff$ $\forall g_1, \dots, g_n$ all $\neq id$ $\exists$ $K_1, \dots, K_n \in \{-1, +1\}$ s.t. $id \notin \langle g_1^{K_1}, \dots, g_n^{K_n} \rangle^+$ (semigroup)

Proof: $\leq$ LO on G take: $K_i = \begin{cases} +1 & \text{if } g_i > id \\ -1 & \text{if } g_i < id \quad (\Leftrightarrow g_i^{-1} > id) \end{cases}$

$\Rightarrow \langle g_1^{k_1}, \dots, g_n^{k_n} \rangle \subseteq P \not\ni \{id\}$

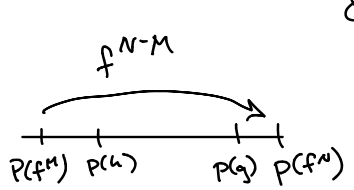
Ex: $\Leftarrow$ Use Tychonov's Th.

G is BO $\Leftrightarrow$ $\forall g_1, \dots, g_n$ all not id $\exists k_1, \dots, k_n$
s.t. $id \notin \langle g_1^{k_1}, \dots, g_n^{k_n} \rangle^{N(+)}$
where $N(+)$ = Smallest Normal sub-semigp

Ex: Find Group that is not Left orderable
Hint: must have finitely many elmts can't be ordered.

$(G, \leq) \rightsquigarrow G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$, w/o global fixed pt
Archimedean $\rightsquigarrow$ Free action $\circledast$
Conradian $\rightsquigarrow$?

$\circledast$ Let $f \in G$ prove $f(x) \neq x$
Archimedean: $\forall f > id$ $\forall g \in G$ $\exists N$ s.t. $f^N > g$
& $\forall h \in G$ $\exists M$ s.t. $f^M < h$



Free Actions by Homeo of $\mathbb{R}$:

Hölder every action is semiconj. to translations

Semiconjugate means can contract orbit by intervals, "Denjoy"
(not mean can contract ; to get symmetry say you have a common lift)

$(G, \leq)$ Archimedean fix $f > id$.

<u>Hölder's Map</u>: $\varphi(g) = \lim_{n\to\infty} \left\{ \frac{p(g)}{q} : f^{q} \leq g^{p(g)} \leq f^{q+1} \right\}$
$\rightsquigarrow \varphi : G \hookrightarrow (\mathbb{R}, +)$ is order embedding

$\Longrightarrow$ $\varphi(G) \simeq \mathbb{Z}$ boring
$\varphi(G) \simeq$ Dense in $\mathbb{R}$

Dense $\Rightarrow$ $\Psi(x) := \sup \{ \varphi(h) : h(0) \leq x \}$ give semi-conj $\mathbb{R} \circlearrowleft$

<u>Claims</u>: ① Ψ is nondecreasing
② $\Psi(hx) = \Psi(x) + \varphi(h)$ $\forall h \in G$
③ Ψ is continuos

<u>Proof</u>: ② $\Psi(h(x)) = \sup \{ \varphi(g) : \underbrace{g(0) \leq h(x)}_{h^{-1}g(0) \leq x} \}$

$$= \sup \{ \underbrace{\varphi(hg)}_{\varphi(h)+\varphi(g)} : g(0) \leq x \}$$
$$= \varphi(h) + \sup \{ \varphi(g) : g(0) \leq x \} = \varphi(h) + \Psi(x)$$

★ ② Gives that Ψ is (maybe) semi-conj.

③ Proof is exercise: use $\varphi(G)$ dense, if discontinuity $\Rightarrow$ everywhere

Mysterious Conrad Property:

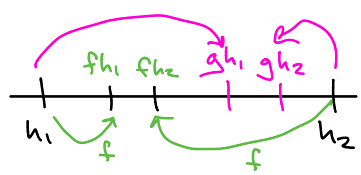
① $\forall f>id\ \forall g>id\ \exists n$ s.t. $fg^n>g$

② $\forall f>id\ \forall g>id$ one has $fg^2>g$.

★ Conrad orders is closed set in $LO(G)$

③ The Following does not happen:

$h_1<fh_1<fh_2<gh_1<gh_2<h_2$



★ This failure is Lexicographical in nature.

~~XXX~~

DAY 4: Start w/ torsion free group, not L-orderable

$\Gamma = \langle a,b : a^2ba^2=b,\ b^2ab^2=a\rangle$ Crystallographic on $\mathbb{R}^3$:

$a(x,y,z)=(x+1,1-y,-z)$, $b(x,y,z)=(-x,y+1,1-z)$

$c=(ab)^{-1}$ $c(x,y,z)=(1-x,-y,z+1)$

Look @ $(a^\epsilon b^\delta)^2(b^\delta a^\epsilon)^2$, $c:=(ab)^{-1}$

$a^2ba^2=b \Rightarrow ba^2b^{-1}=a^{-2} \Rightarrow b^2a^2b^{-2}=ba^{-2}b^{-1}\cdots=a^2$

$b^2ab^2=a$ $\Rightarrow [a^2,b^2]=1.$

Γ torsion free:

$a^2(x,y,z) = a(x+1, 1-y, -z) = (x+2, y, z)$, similarly b, c

$\Rightarrow \langle a^2, b^2, c^2 \rangle \cong \mathbb{Z}^3$

Ex: $\Gamma/\mathbb{Z}^3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$

Let $w \in \Gamma \Rightarrow w = a^{2i} b^{2j} c^{2k} x$, $x = a$ or b

$\Rightarrow w^2 = a^{2i} b^{2j} c^{2k} x a^{2i} b^{2j} c^{2k}$ assume $x = a$

$= a^{2i} b^{2j} c^{2k} a^{2i} b^{-2j} c^{2k} a^2$ using $ab^2a^{-1} = b^{-2}$, $\overset{\text{same}}{b \leftrightarrow c}$

$= a^{4i} a^2$ is ∞ order

Claim: Γ not left orderable

Back to: $(a^\epsilon b^\delta)^2 (b^\delta a^\epsilon)^2 = id$ for any choice $\epsilon, \delta \in \{\pm 1\}$

Ex: G is torsion free grup w/ normal subgroup co-cyclic that is also L-orderable is L-orderable

UPP: unique product property G has UPP if $\forall S \subset G$ finite $\exists s \in S \otimes S$ that appears only once where $S \otimes S = \{s = s_1 s_2 : s_i \in S\}$

G has UPP $\Rightarrow$ G is torsion free if $f^n = id$ then take $S = \{id, f, f^2, \ldots, f^{n-1}\}$

The converse is not true (Rips - Segeev) (hyperbolic grps, small cancellation)
Proof by Promislov: computational

Look @ $(u_1,v_1,w_1)\odot(u_2,v_2,w_2)=(u_1\oplus u_2, v_1\oplus v_2, w_1\oplus w_2)$

$u_i,v_i,w_i\in\mathbb{Z}\cup\hat{\mathbb{Z}}$ & $m\oplus n=m+n \quad \hat m\oplus n=\widehat{m-n}$

$m\oplus\hat n=\widehat{m+n} \quad \hat m\oplus\hat n=m-n$

$G=(\mathbb{Z}\cup\hat{\mathbb{Z}},\odot)\leftrightarrow$ crystallographic group above

$a\leftrightarrow(1,\hat0,\hat0)$, $b\leftrightarrow(\hat0,1,\hat1)$

G group $\rightsquigarrow$ $\mathbb{R}G$ group algebra

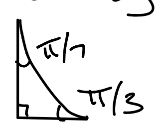
Assume $f^n=id \Rightarrow$ in $\mathbb{R}G$: $(f-1)(f^{n-1}+\cdots+1)=0$

$\Rightarrow$ non-trivial zero divisor

Question: (Kaplanski) Is the converse true?

'73 ~90
Example: (Thurston / Bergman) G finitely gen'ed G s.t.
$\nexists$ nontrivial $\varphi: G\to\mathbb{R}$

$\langle a,b,c : a^2=b^3=c^7=id\rangle \le PSL_2\mathbb{R}$ Acts by reflection



Acts on S^1 can lift to action on $\mathbb{R}$ to $\langle \hat a,\hat b,\hat c \mid \hat a^2=\hat b^3=\hat c^7=\hat a\hat b\hat c\rangle$

<u>Conrad's Property</u>: Will correspond to actions on line with no resilient pairs (Plante, Solodov)
(f, g below)

LO on G is Conradian $\iff$ this can't happen:

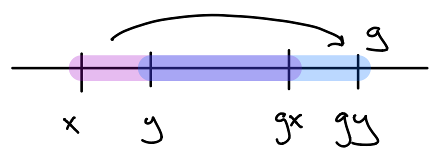
$$h_1 < fh_1 < fh_2 < gh_1 < gh_2 < h_2$$

$\boxed{\text{Th:}}$ (Plante-Solodov) $G \leq \mathrm{Homeo}_+ \mathbb{R}$ is finitely gen'ed & has no resilient pair $\Rightarrow \exists$ G-inv σ-finite meas on $\mathbb{R}$ μ

$\Rightarrow$ consider translation number $\tau: G \to (\mathbb{R}, +)$

$$\tau(g) := \begin{cases} \mu[x, gx) & \text{if } gx > x \\ -\mu(gx, x] & \text{if } gx < x \end{cases}$$

(independent of x)



& $\tau(gh) = \mu[x, gh(x))$
$= \mu[x, hx) + \mu[hx, ghx) = \tau(h) + \tau(g)$

Back to Plante-Solodov Th: Either $\exists$ a discrete orbit or the action semi-conjugates to an action by translation
☆ the semi-conjugacy my result in a kernel ☆

<u>Remark</u>: G is L-O + subexp growth then every order on G is Conradian

Idea of Proof:

Resilient pairs $\Rightarrow \langle f,g\rangle^+$ free semigroup.

$\boxed{\text{TH:}}$ $\leq$ Conradian $\Leftrightarrow \nexists$ Resilient (f,g,h_1,h_2)

Proof: Suppose $f>id$, $g>id$ violate Conrad cond:, i.e.

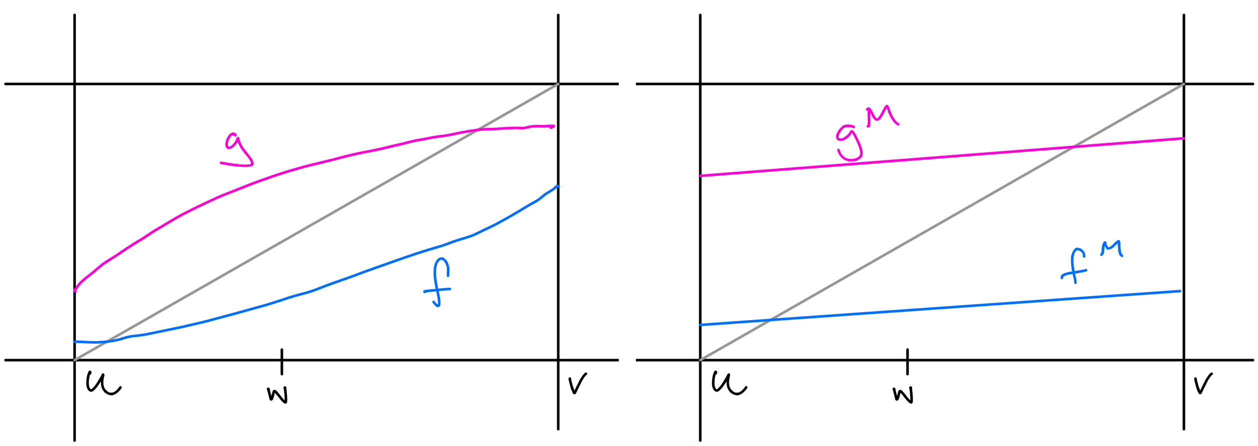
$$fg^n<g \ \forall n$$

set $u:=id$, $v:=f^{-1}g$ $w:=g^2$. the following hold:

i) $u<w<v$

ii) $g^n w \overset{*}{<} v$ & $f^n v \overset{*}{>} u \ \forall n$

iii) $\exists M,N$ s.t. $f^N v<w<g^M u$



Ex: Finish

Day 5:

Ex: G fin'ly gen'ed $|LO(G)|=2 \Leftrightarrow G \cong \mathbb{Z}$

Hint: $\mathcal{LO}$ finite then everyone is Conradian

Rem: G fin'ly gen'ed $\leq$ Conrad $\Rightarrow \exists\, G \xrightarrow{\text{nontrivial}} \mathbb{R}$ order pres.

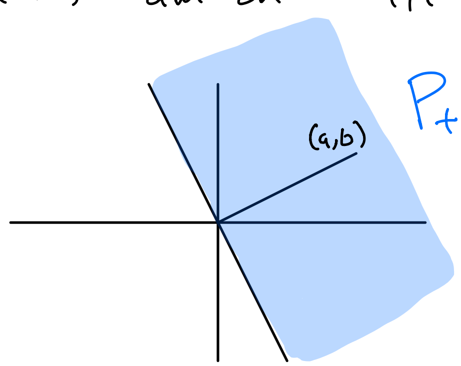
$\hookrightarrow f > id,\ g > id\ \exists\, n\ (\Leftrightarrow n=2)\ gf^2 > f$

Ex: $\mathbb{Z}^2\ \leq\ \Rightarrow$ it's Conrad (obs. biorder)

$\Rightarrow \exists\, \varphi_1 : \mathbb{Z}^2 \to \mathbb{R}$. Ker $\varphi_1 = 0$ ✓

otherwise: restrict to Kernel get $\varphi_2 : \mathrm{Ker}\,\varphi_1 \to \mathbb{R}$

Formula for $\varphi_1(m,n) = am + bn \Rightarrow \varphi_1(m,n) = \langle (a,b), (m,n) \rangle$



$\rightsquigarrow \mathcal{O}(\mathbb{Z}^2) \simeq$ Cantor Set

TH: (Rivas) $\mathcal{CO}(G)$ is finite or Cantor set

Ex: $|\mathcal{LO}(G)| = \infty$, $|\mathcal{CO}| = 4$: $BS(1,2)$

TH: (N, Rivas-Tessera) : G is Solvable $\mathcal{LO}$ & $|\mathcal{LO}(G)| = \infty$

$\Rightarrow \mathcal{LO}(G)$ is Cantor Set

☆ **Q:** true for G amenable? ☆

Already Saw G locally indicable $\Leftrightarrow \exists \leq$ Conradian on G
$\Rightarrow G$ is $\mathcal{LO}$ (converse not true in gen)

$\boxed{\text{TH:}}$ (Dave Witte Morris) G amen. G is $\mathcal{LO} \Rightarrow$ Conrad

Proof:
$G \curvearrowright \mathcal{LO}(G)$, G amen Assume G fin'ly gen'ed

$g : \leq \mapsto \leq_g$; $f >_g id \Leftrightarrow g^{-1}fg > id$

Poincaré Recurrence: $T : (X, \mu) \circlearrowleft$ meas pres. $A \subseteq X$
$\Rightarrow$ a.e. $x \in A$ $\exists n \in \mathbb{N}$, st. $T^n(x) \in A$

Proof: $B := A \setminus \bigcup_{n>0} T^{-n}(A)$; show $\mu(B) = 0$?
Obs: $T^{-i}(B) \cap T^{-j}(B) = \emptyset \ \forall i \neq j$
T meas. pres $\Rightarrow$ All have meas. 0.

Apply to each $g \in G$:

Claim: μ(Conrad Orders) $= 1$

Fix $g \neq id$ $V_g = \{\leq \ : g > id\}$, $f \in G$ define
$B_g(f) := V_g \setminus \bigcup_{n>0} f^{-n}(V_g)$

$B_g := \bigcup_f B_g(f)$; $B = \bigcup_{g \neq id} B_g$.

$\mu(B) = 0$ By Poincaré Recurrence.

<u>Claim</u>: $B^c \subseteq$ Conradian

<u>Proof</u>: Suppose: $\leq \in B^c$. Assume $f > id, g > id \Rightarrow \leq \in V_g \Rightarrow$
$\leq \notin B_g(f) \subseteq B \Rightarrow \leq \in f^{-n}(V_g)$
$\Rightarrow \leq_{f^n} \in V_g$

$\Rightarrow \exists n$ s.t. $g >_{f^n} id \Rightarrow f^{-n} g f^n > id$
$\Rightarrow g f^n > f^n > f > id$, $n > 0$ ∎

<u>Exercise</u>: $G = F_2 \ltimes \mathbb{Z}^2$
i) G has Conrad order
ii) $\forall$ $\mathcal{LO}$ on G $\exists f, g > id$ s.t.
$g f^n < f^n \ \forall n > 0$

<u>Open Problem</u>: Assume G finitely gen'ed, $\mathcal{LO}$ $G \not\supseteq F_2$
$\Rightarrow$ G is Conrad Orderable? i.e. Loc. ind.

☆ Linnell's Conjecture:

<u>Thurston</u> '73 The conjecture for amen. groups is in paper

<u>Open Question</u>: Can $G \curvearrowright \mathcal{LO}(G)$ be minimal?

$\boxed{\text{TH:}}$ (McCleary, Rivas) $\exists$ dense orbit in $\mathcal{LO}(F_2)$.
Denote: $\leq^*$ orbit rep.

$\Rightarrow$ conj. of $\leq^*$ acheives any finite sequence of $\leq$'s

<u>Fix:</u> $f_i > g_i$ $i=1,\dots,n \leftrightarrow g_i^{-1} f_i > 0$.

$\leq$ order on $G \rightsquigarrow$ action on $\mathbb{R}$