Geometry and Topology of Mapping Class Groups; Jing Tao

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Lecture 1

The following sections will be a reminder of Topology of Surfaces.

0.1 Classification of Surfaces

Any compact oriented connected surface, possibly with boundary is homeormophic to $S^2\#T^2\#\cdots\#T^2$ of genus g ($g-$copies of T^2) with b open disk removed.

Definition 1. A finite-type surface: is a connected, compact, oriented surface S with n points removed (punctures).

Put surface image.

Remark 2. Sometimes we can think in marked points instead of removed points.

Remark 3.

- Surfaces can be endowed with a differential structure.
- $\mathfrak{X}(S)\leq 0$, which means $\partial S=\varnothing$.
- Any S can be endowed with a Riemannian metric (complete of finite area) with constant curvature.

S	Euler Characteristic	k
S^2	$\mathfrak{X}(S)>0$	$+1$
T^2	$\mathfrak{X}(S)=0$	0
else	$\mathfrak{X}(S)<0$	-1

We have to remark that in this classification we are thinking that S is not the annulus or the disk.

Definition 4. A closed curve on S is a continuous map $\alpha : S' \to S$. We will say that α is simple if it has no self-intersections.

We often work with free homotopy classs of α. We will say that α is essential if its not null-homotopic.

Some facts: Every continuous curve is freely homotopic to a smooth curve. Transversallity, can put curves in general position. We understand general position for $\alpha_1, \cdots, \alpha_n$, if their intersections are transverse, are isolated and no triple intersections allowed.

Definition 5. Let a, b two free homotopic class of α and β. The geometric intersection number of a and b is defined as:

$$0 \leq i(a,b) = \min\{\alpha \cap \beta : \alpha \in a, \beta \in b\} < \infty$$

It is clear by definition that $i(a,a) = 0$.

Definition 6. Let $\overrightarrow{a}, \overrightarrow{b}$ two oriented curves. The algebraic intersection number is defined as

$$\hat{i}(\overrightarrow{\alpha}, \overrightarrow{\beta}) = \sum_{p \in \overrightarrow{\alpha} \cap \overrightarrow{\beta}} \text{index}(p)$$

where index(p) is equal to 1 if the oriented pair of velocity vectors agree with surface orientation and -1 otherwise. For free homotopic classes $\hat{i}(a,b) = \hat{i}(\overrightarrow{\alpha}, \overrightarrow{\beta})$ and is independent of choice of representatives.

Exercise 1. Prove the following equations

1. $i(a,b) = i(b,a)$.
2. $\hat{i}(a,b) = -\hat{i}(b,a)$.
3. $i(a,b) \geq |\hat{i}(a,b)|$.
4. $i(a,b) = \hat{i}(a,b) \mod 2$.

Definition 7. Let α, β representatives of a and b. We say that α and β are in minimal position if $i(\alpha, \beta) = i(a,b)$

How ca tell if two curves are in minimal position? The Bigon criterion is one who tells.

Put image of Bigon criterion

Lemma 8. *Let α, β two curves, they are in minimal position if and only if there are no bigons.*

Definition 9 (Change of Coordinate Principle)**.** Suppose we are given 2 sets of curves $A = \{\alpha_1, \cdots, \alpha_n\}$ and $B = \{\beta_1, \cdots, \beta_n\}$. Suppose after cutting S along A and cutting S along B. We see that the complementary regions "match up" Then there exists an orientation-preserving homeomorphism of S taking the set A to B.

put image of 3-genus surface cutting in one handle equator and other handle anulli.

Remark 10.

- Modulo the following examples: disk, annulus, disk with one puncture, open disk, open disk with one puncture. Homotopic homeomorphisms are isotopic [Baer].
- Every homeomorphism is homotopic to a diffeomorphism [Munkres, Smale,...].

Let $S = S(g, n, b)$ of finite-type. Let

$$\text{Homeo}^+(S, \partial S) = \{g : S \to S : \text{ is a orientation-preserving homeomorphism} g(\partial S) = \partial S\}$$

It is a group under composition and a topological group with the compact-open topology.

Definition 11 (Mapping Class Groups)**.**

$$\begin{aligned} \text{MCG}(S) &= \pi_0\left(\text{Homeo}^+(S, \partial S)\right) \\ &= \text{Homeo}^+(S, \partial S)/\text{Homeo}_0(S, \partial S) \\ &= \text{Homeo}^+(S, \partial S)/\text{isotopy} \\ &= \text{Homeo}^+(S, \partial S)/\text{path component of identity} \end{aligned}$$

Remark 12. Can replace Homeo by Diffeo in the definition of MCG(S). Also MCG(S) is discrete.

Example 13. Some examples of elements of the MCG(S) are:

1. Any homeomorphism, i.e., change of coordinate homeomorphism.
2. Symmetries
3. Dehn twists
4. Pseudo-Anosovs.

Example 14.

1. $S = \mathbb{R}^2, S^2, \mathbb{R}^2 \setminus \{p\}$, $\text{MCG}(S) = 1$.
2. $S = D^2, D^2 \setminus \{p\}$, $\text{MCG}(S) = 1$ by Alexander lemma.

3. $S = S^2 \setminus \{p, q\}$, $\mathrm{MCG}(S) = \mathbb{Z}/2$.

4. $S = A$, $\mathrm{MCG}(S) = \mathbb{Z}$.

5. $S = T^2, T^2 \setminus \{p\}$, $\mathrm{MCG}(S) = \mathrm{SL}_2\mathbb{Z}$.

Definition 15. Consider the annulus $A = I \times \mathbb{R}$. Let $T : A \to A$ the map given by $T(a)$ intersecting b transversally will give a "turn". We will call this map T the left Dehn twist about b.

Let S be a surface and b a curve in S. Let N_b be a cylindrical neighbourhood and $\phi : A \to N_b$ an homeomorphism. Define $T_b(x) = \phi T \phi^{-1}(x)$ if $x \in N_b$ and identity otherwise.

Remark 16. 1. T_α is non-trivial in $\mathrm{MCG}(S)$.

2.

Theorem 17. $\mathrm{MCG}(T^2) \to \mathrm{Aut}^+(H_1(T^2)) \simeq \mathrm{SL}_2\mathbb{Z}$.

Remark 18. Some elements in $\mathrm{MCG}(T^2)$ are of the form

Type	Example	Trace
Finite order	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Change-orientation	$\lvert\mathrm{tr}(A)\rvert < 2$
Reducible	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ Dehn Twist	$\lvert\mathrm{tr}(A)\rvert = 2$
Anosov	$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $\lambda^\pm$, $e^\pm$ eigenvalues and eigenvectors	$\lvert\mathrm{tr}(A)\rvert > 2$

Lecture 2

We will remember that for a surface $S_{g,n,b}$, for $\mathrm{Homeo}^+(S, \partial S)$ the set with elements $f : S' \to S'$ preserving orientation such that $f(x) = x$ for every $x \in \partial S'$ where S' is the surface $S_{g,0,b}$ and the set of puncture is preserved.

Exercise 2. Prove that there exists a short exact sequence of the form

$$1 \to P\mathrm{MCG}(S) \to \mathrm{MCG}(S) \to \Sigma_n \to 1$$

where is the permutation group of $n-$elements and $P\mathrm{MCG}(S)$ is the pure mapping class group.

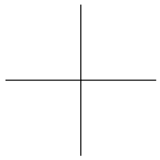

Figure 1: Iteration of Dehn twists

0.2 Basic Facts about Dehn Twists

Let a, b curves and $f \in \mathrm{MCG}(S)$.

1. For $k \in \mathbb{Z}$, $i(T_a^k(b), b) = |k| i(a,b)^2$.

   Corollary 19. $\langle T_a \rangle = \mathbb{Z}$.

   This image present the idea of the proof.

   draw picture of Dehn twist iteration

   Remark 20. It follows that $\hat{i}(T_a^k(b), b) = k\hat{i}(a,b)$.

2. $T_a = T_b$ if an only if $a = b$

3. $fT_af^{-1} = T_{f(a)}$.

4. $fT_a = T_af$ if and only if $f(a) = a$.

   Proof. We have that $fT_af^{-1} = T_a$ and by the previous, $T_{f(a)} = T_a$ and by (1), $f(a) = a$. □

5. $i(a,b) = 0$ if and only if $T_aT_b = T_bT_a$. Equivalently $T_a(b) = b$.

6. If $i(a,b) = 1$ then $T_aT_bT_a = T_bT_aT_b$.

   Proof. It is easy to convince that $T_aT_b(a) = b$, from the fact that the intersection number is 1 we have that the curves are in a subsurface which is a torus.

   If we take $(T_aT_b)T_a(T_aT_b)^(- 1) = T_b$. □

   Exercise 3. Prove that the converse is true.

7. Alexander's Method

   Recall that $\mathrm{MCG}(D^2, \partial D) = \{1\}$ and $\mathrm{MCG}(D^2 \setminus \{p\}, \partial D) = \{1\}$.

Theorem 21. *Let S be a surface of finite type and $\{\alpha_1, \cdots, \alpha_n\}$ a collection of simple closed curves and proper arcs on S that fill S (complementary regions are disk or once-puncture disks) which are all in minimal position (no triangles allowed). Suppose $f \in \text{MCG}(S)$ that fixes the oriented homotopy class of each α_i. Then $f = Id$.*

Remark 22. Real application often set oriented homotopy for free.

8. Center of $\text{MCG}(S)$

Proposition 23. *If $g \geq 3$, then $Z(\text{MCG}(S_g))$ is trivial.*

Proof. By the property 4, if $f \in Z(\text{MCG}(S_g))$, then $f(a) = a$ for all unoriented class of curves. Assume that there is a set of essential curves where α belongs, even more, assume that we can set that α be have more curves on one side of α.

Up to homotopy, f fixes a graph (made with a set of essential curves, acting as an automorphism of the graph. But this graph has no automorphism that change the orientation of α where alpha is a curve in the graph.

Propagating this, f fixes the oriented orientations of every curve. Now apply Alexander's method and $f = Id$.

If f is the hyperelliptic involution such that $f(a) = a$ but $f(\overrightarrow{a}) = \overleftarrow{a}$. In this case the center is non-trivial and is of the form

$S = S_{g,b,n}$	$Z(\text{MCG}(S)$
$b = 0$ then $(g, n) = (0, 2), (1, 0), (1, 1), (1, 2), (2, 0)$	$\mathbb{Z}/2$
$(0, 4)$	$\mathbb{Z}/2 \times \mathbb{Z}/2$

□

0.3 Finite Generation

Observe that a Dehn twist cannot permute punctures.

Theorem 24. *$P\text{MCG}(S)$ is generated by finitely many Dehn twists.*

Corollary 25. *$\text{MCG}(S)$ is countable and is generated by finitely many Dehn twists and half twists.*

Remark 26. Using the capping homeomorphism we can forget about ∂S

$$1 \to \prod_{c \in \partial S} \langle T_c \rangle \to P\text{MCG}(S_{g,n,b}) \to P\text{MCG}(S_{g,n+b}) \to 1$$

Lemma 27. *Suppose that G acts on a connected graph X by automorphisms transitively on edges and vertices. Let $v, w \in X^{(\cdot)}$ is connected by an edge and if $h \in G$ such that $h(w) = v$. Then $G = \langle G_v, h \rangle$.*

Definition 28. The curve complex $\mathcal{C}(S)$, is the graph with set of vertices the homotopy classes of simple closed curves, it will be an edge between two vertices is $i(a, b) = 0$. The $n-$simplex is the set of $n + 1$ disjoint curves.

Theorem 29 (Harvey)**.** *Let $\zeta(S_{g,n}) = 3g-3+n \geq 2$ then $\mathcal{C}(S)$ is connected. Where $\zeta(S_{g,n})$ is the complexity of the surface.*

Definition 30. S is a surface with a marked point $*$ and possibly with other punctures. There is a homomorphism $\mathrm{MCG}(S, *)$.

$push : \pi_1(S, *) \to \mathrm{MCG}(S, *)$.

Exercise 4. $push(\alpha) = T_{\alpha_1} T_{\alpha_2}^{-1}$.

Theorem 31 (Birman)**.** *Let $forget : \mathrm{MCG}(S, *) \to \mathrm{MCG}(S)$ the map given by f maps to the homotopy class of f not fixing $*$. Then the following is a short exact sequence*

$$1 \longrightarrow \pi_1(S, *) \xrightarrow{push} \mathrm{MCG}(S, *) \xrightarrow{forget} \mathrm{MCG}(S) \longrightarrow 1$$

Lecture 3

Proof of theorem 31. Base Steps:

1. For $g = 0, n = 3$, we have

   $$1 \longrightarrow \pi_1(S_{0,3}) \longrightarrow P\mathrm{MCG}(S_{0,4}) \longrightarrow P\mathrm{MCG}(S_{0,3}) \longrightarrow 1\ ,$$

   but $\pi_1(S_{0,3}) = F_2$ and $P\mathrm{MCG}(S_{0,3}) = 1$. So we claim the $P\mathrm{MCG}(S_{0,4})$ is finitely generated and apply Birman exact sequence assure the claim for $n \geq 4$.

2. For $g = 1, n = 0$, we have $P\mathrm{MCG}(T^2) = \mathrm{MCG}(T^2) = SL_2(\mathbb{Z})$ which is finitely generated. For the case $g = 1, n = 1$, $P\mathrm{MCG}(T^2, *)$ is also equal to $SL_2(\mathbb{Z})$.

Now we can induct for g and n with all$g \geq 2$. □

Remark 32. The previous proof is constructive. In particular there are Dehn twists and half-twists on $\mathrm{MCG}(S_{g,n,b})$.

For a closed surface of genus g, there are $3g-1$ non-separating curves known as the Lickorish generators. We can reduce this set of generators to $2g-1$, known as Humphries generators.

0.4 Some Relations

1. If $f, g \in \mathrm{Homeo}^+(S)$ with disjoint support, then their classes commute.

2. If $i(a,b) = 1$, then $T_a T_b T_a = T_b T_a T_b$

3. Lantern Relation For a, b, c, d, x, y, z we have that $T_x T_y T_z = T_a T_b T_c T_d$.

   Put the Lantern relation picture

   We can apply the Lantern relation to prove that $\mathrm{MCG}(S_g)^{ab} = 1$ for $g \geq 3$ and $G^{ab} = G/[G,G]$.

4. Root of Dehn Twists: Assume that $i(a,b) = 1$. This means that we can obtain T^2 with a boundary component inside our surface. Then $(T_a T_b)^6 = T_c$.

   The proof is inspired in the fact that if we change the boundary component by a once-punctured torus. T_a and T_b are triangular matrices such that $T_a T_b$ has order six.

Theorem 33 (McCool)**.** $\mathrm{MCG}(S)$ *is finitely presentable.*

0.5 Nielsen-Thurston Classification of Mapping Classes

Our goal is find a nice normal from for an element of Mapping class group.

0.5.1 Structures on Surface

Definition 34. An hyperbolic structure on a surface S_g with $g \geq 2$ is:

1. an atlas of charts to $\mathbb{H}^2$ which a transitions maps isometries of $\mathbb{H}^2$.

2. There are a complete Riemann metric with constant curvature $\kappa = 1$.

3. is homeomorphic to $\mathbb{H}^2/\Gamma$ where Γ is a discrete group of isometries of $\mathbb{H}^2$ and $\Gamma \simeq \pi_1(S)$.

Remark 35. The previous are equivalent definitions.

Definition 36. A complex structure on a surface S is an atlas of charts to $\mathbb{C}$ with transitions maps are biholomorphisms.

Remark 37. The Hyperbolic structure on a surface induce a complex structure because $\mathbb{H}^2 \subset \hat{\mathbb{C}}$ and isometries of $\mathbb{H}^2$ are biholomorphism of the Riemann sphere.

Definition 38. A measured foliation on a surface S is an atlas of charts on $S \setminus P$ to $\mathbb{R}^2$ which transitions maps preserve vertical lines and spacing between them, where P are "singular points".

By this we mean, that the pull back of $x = a$ gives a foliation F on S. Pull back of $|dx|$ give a measure on arcs transverse to F. At a point $p \in P$ the foliation have a $k-$prong singularity with $k \geq 3$.

Definition 39. A half-translation structure on a surface S are charts on $S \setminus P$ to $\mathbb{R}^2 = \mathbb{C}$, which transitions maps are $z \mapsto \pm z + c$.

By this we mean, that the transitions maps preserve a pair of measured foliation (vertical-horizontal). The singularities look like two singular points of measured foliations. The previous definition is often known as a quadratic differential on S.

Some connections about the previous structures over surfaces. There is an equivalence between hyperbolic structures and complex structure. In the case to obtain an hyperbolic structure based on a complex structure is due to the Riemann uniformization theorem. Also given an measured lamination we can obtain a measured foliation and viceversa.

Theorem 40 (Nielsen-Thurstons)**.** *Let $f \in \mathrm{MCG}(S)$. Then one of the following situations occurs:*

1. *f is periodic / finite order.*
2. *f is reducible, i.e., there exists a multicurve C on S such that $f(C) = C$ setwise.*
3. *f is homotopic to a pseudo-Anosov homeomorphism ϕ, that is, there exists $\lambda > 1$ and 2 tranverse measured foliation F^s, F^u on S such that:*
   (a) $\phi(F^s) = \frac{1}{\lambda} F^s$.
   (b) $\phi(F^u) = \lambda F^u$.

Remark 41. If f is of type 3, then cannot be of types 1 and 2.

Lecture 4

0.6 Teichmüller Space

Teichmüller space or Deformation space of hyperbolic structures. We have to think in deformation of a "fundamental domain".

Definition 42. Let S_g a surface of $g \geq 2$. The Teichmüller space of S_g is the set

$$\mathcal{T}(S_g) = \{\text{Hyperbolic structure}\}/\text{Diffeo}_0(S)$$

An equivalent definition

$$\mathcal{T}(S) = \{(X, f) : X \text{ hyperbolic surface}, f : S \to X \text{ homeomorphism}\}/\sim$$

where $(X, f) \sim (Y, g)$ if there exists an isometry $I : X \to Y$ such that $I \sim gf^{-1}$.

If we ask that X be a Riemann surface and I be a biholomorphism, we obtain the Deformation of Riemann structure on S.

The mapping class group MCG(S) acts on $\mathcal{T}$ by change of marking, i.e., $\phi \cdot (X, f) = (X, f\phi^{-1})$.

Definition 43. The Riemann's Moduli Space is the set

$$\mathcal{M}(S) = \mathcal{T}(\mathcal{S})/\text{MCG}(S).$$

Remark 44. $\mathcal{T}(S)$ is manifold, MCG$(S)-$action is properly discontinuous, not-free, and $\mathcal{M}(S)$ is an orbifold.

Example 45. Let $S = T^2$, $\mathcal{T}(S) = \mathbb{H}^2$, MCG$(S) = \text{SL}_2\mathbb{Z}$. and the action is by linear fractional transformations.

The space $\mathcal{M}(S)$ is the modular orbifold.

0.6.1 Topology on $\mathcal{T}(S)$

Let $\mathscr{S}$ = {set of simple closed curves on S up to free homotopy}. For $a \in \mathscr{S}$, define $\ell_\alpha : \mathcal{T}(S) \to \mathbb{R}_{>0}$ given by $(X, f) \mapsto \ell_\alpha(X)$ which is the length of the geodesic representing of $f(\alpha)$ in the hyperbolic metric on X.

We can obtain an embedding from $\mathcal{T}(S) \to \mathbb{R}^{\mathscr{S}}_{>0}$ with the weak topology.

Theorem 46. *The image of the embedding is homeomorphic to* $\mathbb{R}^{6g-6}$.

In the case of measured foliations structure.

Definition 47.

$$\mathcal{MF}(S) = \{\text{Measured foliations on } S\}/\{\text{isotpy and whitehead moves}\}.$$

As in the hyperbolic case, we can embed $\mathcal{MF}(S)$ on $\mathbb{R}^{\mathscr{S}}_{>0}$ given by $F \mapsto (\alpha \to i(\alpha, F)$, where $i(\alpha, F)$ is the arc length measured with transverse measure of F.

Theorem 48 (Thurston)**.** *$\mathcal{MF}(S) \cong \mathbb{R}^{6g-6}$ and $\mathrm{P}\mathcal{MF}(S) \cong \mathbb{S}^{6g-7}$, where $\mathrm{P}\mathcal{MF}(S)$ is the projectivization of $\mathcal{MF}(S)$.*

Theorem 49 (Thurston)**.** *The following are true:*

1. *The closure $\overline{\mathcal{T}(S)} = \mathcal{S} \cup \mathrm{P}\mathcal{MF}(S)$.*
2. *The action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ extends continuously to $\overline{\mathcal{T}}(S)$.*

Corollary 50. *Every $\phi \in \mathrm{MCG}(S)$ must have a fixed point in $\overline{\mathcal{T}(S)}$.*

In higher genus, Thurston: $\varphi \in \mathrm{MCG}(S)$ is not periodic or reducible if and only if φ has exactly 2 fixed points $[F^+], [F^-]$ such that F^s and F^u together give a half-translation on S.

0.6.2 Tiechmuller Metric on $\mathcal{T}(S)$

Let X, Y two Riemann surfaces, and let $h : X \to Y$ an orientation preserving diffeomorphism. Let $p \in X$, we have that $(D_h)_p : T_p(X) \to T_{h(p)}(Y)$ is an $\mathbb{R}-$linear map. By the Sigular Value decompositon, we can rewrite $(D_h)_p$ as $r\mathrm{Diag}(a, b)s$ where r, s are rotations of $\mathbb{R}^2$.

Definition 51. For h and p as in the previous paragraph. Let

$$(K_h)_p = \frac{\max\{a, b\}}{\min\{a, b\}}.$$

The dilatation of h is defined as $K_h = \sup(K_h)_p \geq 1$.

Remark 52.

1. $K_{h^{-1}} = K_h$.
2. $K_{hg} \leq K_h K_g$. The equality holds if and only if $s = (r')^{-1}$ in the SVD.

Definition 53. Let $(X, f), (Y, g) \in \mathcal{T}(S)$. Define

$$d(X, Y) = \frac{1}{2} \inf \left\{K_h : h \sim gf^{-1}\right\}$$

Definition 54. Let $h : X \to Y$ is called a Teichmüller map if there exists a half-translation structure q_X on X and q_Y on Y such that

$$x + iy \mapsto Kx + \frac{1}{K}y.$$

Theorem 55 (Bers)**.** *Given* $(X, f), (Y, g) \in \mathcal{T}(S)$

1. *There exists a Teichmüller map* $h : X \to Y$ *such that* $h \sim gf^{-1}$*.*
2. *For all* $h \sim h'$, $K_{h'} \geq K_h$ *and equality holds if and only if* $h = h'$*.*

Lecture 5

Theorem 56 (Grölsch, Baby Case of Teichmüller's Extremal Thm)**.** *Suppose* R *and* R' *are rectangles, with sides* (a, b) *and* (a', b')*, and* $h : R \to R'$ *taking the sides of* R *to sides of* R'*. Then* $K_h \geq K_{h'}$ *where* $h' = \mathrm{Diag}(\frac{a}{a'}, \frac{b}{b'})$ *and equality holds if and only if* $h = h'$*.*

Definition 57. Let X be a metric space, $g \in \mathrm{Isom}(X)$ and $\tau_g = \inf_{x \in X} d_X(x, gx)$. Then:

1. g is called elliptic if τ_g is realized.
2. g is called parabolic if τ_g is not realized.
3. g is called hyperbolic if τ_g is positive and realized.

Theorem 58 (Bers)**.** *For* $g \in \mathrm{MCG}(S)$*, we have that the following are true:*

1. *if* g *is elliptic, then* g *is of finite order.*
2. *if* g *is parabolic, then* g *is reducible.*
3. *if* g *is hyperbolic, then* g *is pseudo-Anosov.*

The following are preliminary facts that we will need in order to prove the classification theorem.

Theorem 59 (Collar Lemma)**.** *There exists* ϵ_0 *such that for all hyperbolic surface* X *and all single closed curves* α, β *on* X*. If* $\ell_x(\alpha)$ *and* $\ell_x(\beta)$ *are less or equal than* ϵ_0*, then* $i(\alpha, \beta) = 0$*.*

Theorem 60 (Wolpert's Lemma)**.** *Let $h : X \to X$ a $k-$casi-conformal map, then*

$$\frac{1}{k} \leq \frac{\ell_Y(\alpha)}{\ell_X(\alpha)} \leq k$$

for all α single closed curve on X.

Definition 61. Let $\mathcal{M}(S)$. The $\epsilon-$thick part $\mathcal{M}_\epsilon(S) = \{X \in \mathcal{M}(S) :$ the length of the shortest s.c.c. is $\geq \epsilon\}$.

Theorem 62 (Mumford's Compactness Theorem)**.** $\mathcal{M}_2(S)$ *is compact.*

Proof of Theorem 58. (2).

Suppose g is parabolic, and $X_n \in \mathcal{T}(S)$ such that $d(X_n, gX_n) \to \tau_g$ but not realizing.

We have that $\overline{X_n} \in \mathcal{M}(S)$ must exit every compact set. Then we can choose for any sufficiently small $\epsilon \ll \epsilon_0$ and $X = X_n$ on which there exists α that is $\epsilon-$short and moreover, the lengths of $\{\alpha, g\alpha, \cdots, g^{3g-3}\alpha\}$ are still bounded and by the Collar lemma we have that they are all simultaneously disjoint. But there are at most $3g-3$ simultaneously disjoint curves on a surface of genus g. So for some i and j we have that $g^i(\alpha) = g^j(\alpha)$ and the set curve $C = \{\alpha, g\alpha, \cdots g^{i-j}\alpha\}$ must be a reducible curve. Therefore $g(C) = C$.

(3).

Let $h : X \to X$ be the Teichmüller map such that $h \sim fgf^{-1}$, i.e., there exists q, q' half translations structures on X on which $h = \mathrm{Diag}(k, \frac{1}{k})$ where $k = e^{2\tau_g}$ that send the $q-$coordinate to $q'-$coordinates.

To finish, we want $q' = q$, i.e., $h_* q = q$.

Let $\mathscr{G}$ be the Teichmüller geodesic through X and gX. Let Y be the morphism between X and gX, and gY between gX and g^2X. From this we have

$$\tau_g \leq d(Y, gY) \leq d(Y, gX) + d(gX, gY) = d(Y, gX) + d(X, Y) = \tau_g.$$

From the previous we can imply that g^2X must be on $\mathscr{G}$, and then h^2 is the Teichmüller extremal map X to X in fg^2f^{-1}.

This shows $q = q'$.

(1).

By definition, if g is elliptic then g fixes a point $X \in \mathcal{T}(S)$. Therefore $g \sim \phi \in \mathrm{Isom}(X)$. It is a fact that $\mathrm{Isom}(X)$ is a finite group of order at most $84(g-1)$. Thus g is represented by a finite order element, hence g is of finite order map class. □

Exercise 5. Prove that if g is pseudo-Anosov then g is irreducible and of finite order, then hyperbolic.

Prove that if g is periodic, then g elliptic.

Theorem 63 (Kerckhoff)**.** *Every finite group if* $\mathrm{MCG}(S)$ *can be realized as a subgroup of isometries of some hyperbolic surface.*

Remark 64. Some historical remarks about the previous paragraphs:

1. In 1959, Kravetz proved Teichmüller metric is negatively curve, and from it he derived Nielsen-Realization for finite subgroups of $\mathrm{MCG}(S)$.

2. Linch discovered the proof was wrong.

3. 1975: Masnr proved Teichmüller metric is in fact NOT negatively curved.

4. 1996: Minsky proved that certain parts of Teichmüller space (Thin part) actually looks like the sup metric over a product of lower dimension Teichmüller spaces. So Teichmüller metric is far from being negatively curved.

Theorem 65 (Tits Alternative, Ivanov, McCarthy)**.** *Every* $G < \mathrm{MCG}(S)$ *either contains* F_2 *or is virtually abelian.*

Q 1. $\mathrm{MCG}(S)$ is linear?

There are fast algorithms to detect the Nielsen-Thurston type of $\phi \in \mathrm{MCG}(S)$, we can refer to [Bestvina-Hendel], [Margalit-Strenner-et.al], [Bell-Webb] and the Bell's Flipper algorithm.

0.6.3 Conjugacy Problem for $\mathrm{MCG}(S)$

Given $f, g \in \mathrm{MCG}(S)$ there is an algorithm to decide if they're conjugate. [Tao, et. al.]

Q 2. Describe complete conjugacy invariants for $\mathrm{MCG}(S)$.

0.6.4 Connections to 3-manifolds

The Mapping Torus $M_f = S_g \to I/\sim$ where $(x, 0) \sim (\phi x, 1)$ and $\phi \in \mathrm{Homeo}^+(S)$. The homeomorphism type only depends on mapping lcass of ϕ (f).

Theorem 66 (Thurston)**.** *M_f can be equipped with a hyperbolic metric if and only f is pseudo-Anosov.*

Theorem 67 (Virtual Fiber Thm, Agol, Wise)**.** *Every hyperbolic closed 3-manifold has a cover which is a mapping torus.*

Q 3. Does MCG(S) have (T)?