Orders on Groups; Andrés Navas

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Lecture 1

Consider G be a countable group, and a total order $\leq$ on it. We can define some properties on the order.

Definition 0.1. Let $(G, \leq)$ be an orderable group. We will say that $\leq$ is a *left-invariant* if for $f < g$ implies $hf < hg$ for all elements $h \in G$.

Example 0.2. Some examples of countable ordered groups are:

- $\mathbb{Z}^n$ with the lexicographic order.
- $\mathbb{F}_2$ is a left-orderable group, even more it is bi-orderable.
- $\pi_1(\Sigma)$ is bi-orderable, where Σ is an orientable surface.
- $MCG(\Sigma)$ is left-orderable.
- Braid groups are left-orderable.
- Thompson's groups are left-orderable.

Exercise 0.3. If $\leq$ is a left-invariant total order, then $f > g$ doesn't not imply $f^{-1} < g^{-1}$. Give an example of this.

Remark 0.4. If G is left-orderable, then G is torsion-free. We take an element $f \neq e$, either $f > e$ or $f < e$. If happens that f has finite order, in some point $e = f^m > e$ or $e = f^m < e$.

Exercise 0.5. Give an example of a torsion-free group which is NOT left-orderable.

Exercise 0.6. Give an example of a left-orderable group $(G, \leq)$ and $f > e$ such that $gfg^{-1} < e$ for some $g \in G$.

Definition 0.7. The *Archimedean Property*: We say that $(G, \leq)$ has the Archimedean property if for all $f > e$ and for all $g \in G$ there exists $n > 0$ such that $f^n > g$.

Theorem 0.8 (Hölder Theorem)**.** *G admits an Archimedean order if $G \hookrightarrow (\mathbb{R}, +)$ ordered embedding.*

Definition 0.9. A left-orderable $\leq$ on G is conradian if for all $f > e$, $g > e$ there exists n such that $fg^n > g$.

Exercise 0.10. If this happens then for all $f > e$ and $g > e$ it holds $fg^2 > g$.

Theorem 0.11 (Conrad-Brodsky)**.** *G is Conrad ordenable if and only if G is locally indicable, i.e., for all $G_0 \subset G$ finitely generated there exists $\phi : G_0 \to (\mathbb{R}, +)$ non-trivial group homomorphism.*

Theorem 0.12 (D Withi Morris)**.** *Let G be an ordenable group. G is left-orderable if and only G is locally indicable.*

Exercise 0.13 (*)**.** Give an example of a left-orderable group which is NOT locally indicable.

Theorem 0.14 (Hyde-Lodhie; Mattebon-Triestino)**.** *There exists G a finitely generated, left-orderable and simple group.*

0.1 The dynamical approach

Afirmation 0.15. *We will say G is left-orderable if and only if acts faithfully on a totally ordered space $(\Omega, \leq_\Omega)$ by order-preserving bijections.*

Proof. One implication is obtained by the action of the group on itself with the left-order of the hypothesis.

The converse consider $\Omega = \{\omega_1, \omega_2, \ldots\}$. We will say that $f > g$ if $f(\omega_1) > g(\omega_1)$ or $f(\omega_1) = g(\omega_1)$ and $f(\omega_2) > g(\omega_2)$ and so on. □

Theorem 0.16. *If G is countable and left-ordered then $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$. Conversely, if $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ then G is left-orderable.*

Proof. Let G whit list of elements is $\{g_1, g_2, g_3, \ldots\}$. Define $p(g_1)$ as a fixed point in $\mathbb{R}$ and define $p(g_2)$ depending on the orders on g_1 and g_2 by adding o substracting one. In the case that some point of the list be between two predecesors of the list we define $p(g_i)$ as the middle-point of the segment $p(g_{i-2})$ and $p(g_{i-1})$.

By this process, we can assure that G have an action on the set of points obtained by the process in $\mathbb{R}$. We can extend continuously this action to $\mathbb{R}$. □

Q 1. Assume that $\leq$ is bi-invariant. What kind of action you get?

Definition 0.17. The space of left-orders for a group G will be denoted by $\mathrm{LO}(G)$. This space is a topological space with the Chabauty topology whose basis elements are defined in the following way:

Consider $\leq$ be a left-order on G and assume $f_i > g_i$ for some $1 \leq i \leq n < \infty$. The set $N_{\leq,(f_i,g_i)} = \{\leq' : f_i >' g_i \, \forall i\}$.

Exercise 0.18. $\mathrm{LO}(G)$ is totally disconnected and compact. **HINT:** use Tychonov's theorem.

When the group $G = \langle g_1, \cdots, g_k \rangle$. We can define balls over the group as $B(m) = \{g : g = g_1^l g_2^l \cdots g_m^l, m \leq n, l = \pm 1\}$. Using this balls we can define a distance function over the set of left-orders by the following:

$$d(\leq, \leq') = \frac{1}{n},$$

if n is the smallest number such that the orders does not coincide in $B(n)$.

Lecture 2

We began this lecture with a characterisation of the orderable groups.

Theorem 0.19. *Let G be group, G is left-orderable if and only if $G = P \sqcup P^{-1} \sqcup \{1\}$, where P is a semigroup.*

Proof. If G is left-orderable with order $\leq$. Consider the set $P = \{g : g > 1\}$ is a semigroup that holds the conditions.

If for G there exists a semigroup P such that $G = P \sqcup P^{-1} \sqcup \{1\}$. Define the order $\leq$ as follows, we will say that $g > h$ if and only if $h^{-1}g \in P$. It is easy to prove that $(G, \leq)$ is a total ordered group. □

From this characterisation, it is clear that the cones on a group define the orders on the group.

Example 0.20. Consider the group $\mathbb{Z}^2$. Consider the set $P = \{(a, b) : a > 0 \text{ or } a = 0, b > 0\}$. It is easy to prove that P is a cone for the lexicographic order.

Another example of cones in $\mathbb{Z}^2$ is take a line passing through 1, and decide a positive side of the line and one "positive" region.

Cayley graph of $\mathbb{Z}^2$

Example 0.21. Consider $G = \langle a, b : bab = a \rangle$ the fundamental group of the Klein bottle.

Remark 0.22. G can be obtained by the following generated semigroups.

$$\langle a, b\rangle \sqcup \langle a, b\rangle^{-} \sqcup \{1\}$$

$$\langle a, b^{-1}\rangle \sqcup \langle a, b^{-1}\rangle^{-} \sqcup \{1\}$$

The Cayley graphs pictures reminiscence the picture of the cones in $\mathbb{Z}^2$ with the lexicographic order.

Cayley graph of G

Exercise 0.23. Prove that in the case of $\mathbb{Z}^2$ with the lexicographic order, P is NOT finitely generated.

Corollary 0.24. *Let G be the fundamental group of the Klein bottle. Then G admit only four orders.*

Proof. Consider that G has a left order $\leq$. From this we know that there are four possible comparisons between the generators and the identity. Assume that $a > 1$ and $b < 1$, it is imply that the semigroup $\langle a, b^{-1}\rangle \subseteq P$ where P is the positive cone of the order $\leq$. And similarly, we can assure that the negatives of the previous semigroup is contained in the negative semigroup. By the remark, we have that P has to be the semigroup and the order coincides with the defined by the semigroup partition. □

Theorem 0.25 (Torosin)**.** *A complete class of groups with finitely many left-orders is*

$$\langle a, b, c : aba^{-1} = b^{-1}, bcb^{-1} = c^{-1}, a = ca\rangle.$$

Remark 0.26 (Linnell). Let $\leq$ be a left-order on G such that $P_{\leq}$ is finitely generated. Then $\leq$ is an isolated point of $LO(G)$.

Example 0.27. Consider the braid group $B_3 = \langle s_1, s_2 : s_1 s_2 s_1 = s_2 s_1 s_2\rangle$.

Exercise 0.28. Let K denote the Klein bottle surface. Find $x > y$ in $\pi_1(K)$ such that $x^2 < y^2$.

0.2 Free Groups

Using the ping-pong lemma, we can prove that $\mathbb{F}_2$ is subset of $\mathrm{Homeo}_+(\mathbb{R})$. From the fact that every countable left-orderable group is inside the homeomorphisms of the real line.

Let $(G, \leq)$ be an ordered group. We can provide a G action on $\mathbb{R}$ induced by the left-order. Consider a ordered morphism $p : G \to \mathbb{R}$ and the action of G will be of the form $g \cdot p(h) = p(gh)$. In the case that $\leq$ is a bi-order, we can prove that for a positive element g, $g(x) \geq x$ for all $x \in \mathbb{R}$ and for a negative element g, $g(x) \leq x$ for all $x \in \mathbb{R}$.

Exercise 0.29. Conversely, assume that $G \subset \text{Homeo}_+(\mathbb{R})$ is such that for all $g \neq id$ either $g(x) \geq x$ or $g(x) \leq x$ for all $x \in \mathbb{R}$. Then G is bi-orderable.

Example 0.30. Some examples of bi-orderable groups are:

- $\mathbb{F}_2$ is bi-orderable.
- $PL_+([0,1])$ is bi-orderable, the group of piecewise-linear automorphims of $[0,1]$.

Lecture 3

Some remarks about left-orderable groups:

1. If G_1, G_2 two left-ordered groups, then $G_1 \times G_2$ is left-orderable. It is also hold for semi-direct products.

   An example about this is the group $\mathbb{F}_2 \ltimes \mathbb{Z}^2 \subset \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, this is a left-orderable group that has Rel(T).

2.

   Theorem 0.31 (Vinagradov)**.** *Let G_1, G_2 two bi-ordenable groups, then $G_1 * G_2$ (the free-product) is bi-ordenable.*

   In the same spirit.

   Theorem 0.32 (Rivers)**.** $\text{LO}(G_1 * G_2)$ *is a Cantor set.*

   Which in particular implies that $\text{LO}(\mathbb{F}_2)$ is a Cantor set.

3.

   Theorem 0.33 (Linnell)**.** *If* $\text{LO}(G)$ *is infinite, then it is uncountable. In particular, contains a Cantor set.*

   Remark 0.34. The interest to determine if the space of left-orders in a group is a Cantor set or not, is based in the fact that if there is an isoleted point in the space of left-orders, this order is determine by a finite number of inequalities which made it a really interesting order.

4. A group G is left-orderable if and only if for a finite set $\{g_i, i = 1, \cdots, m\}$ of elements different to the identity, there exists powers in $k_i \in \{-1, 1\}$ such that the identity does not belong to $\langle g_1^{k_1}, \cdots, g_m^{k_m} \rangle^+$ the semigroup generated by this powered elements.

Exercise 0.35. Prove the converse, **HINT:** use Tychonov's theorem.

There is a similar result, i for a group G we have that is Bi-orderable if an only if there are finite elements such that the identity does not belong to the normal semigroup generated by some powers of the elements.

Search for normal semigroup definition

The dynamical realization

How does the properties of a left-ordered group looks like under the injection of G into $\text{Homeo}_+(\mathbb{R})$?

Consider $(G, \leq)$ a left-ordered group. Remember that, in order to obtain an $\text{Homeo}_+(\mathbb{R})$ of G we need an ordered map $p : G \to \mathbb{R}$ with the tautological action of G.

Remark 0.36. If $\leq$ is Archimedian, then the G action is free.

Theorem 0.37 (Hölder)**.** *Every free action by homeomorphisms on $\mathbb{R}$ is semiconjugate to an action by translations.*

Proof. Consider $(G, \leq)$ an Archimedian left-ordered group and fix $f > id$. Define the function

$$\phi(g) : \lim_{q\to\infty} \frac{p(q)}{q}$$

such that $f^q < g^{p(q)} < f^{q+1}$. It is clear that $\phi(G) \simeq \mathbb{Z}$ and it is dense. Define

$$\psi(x) := \sup\{\phi(g) : g(0) \leq x\}.$$

We claim that ψ is a non-decreasing function, $\psi(h(x)) = \psi(x) + \phi(h)$ for all $h \in G$ and ψ is continuous.

We have that $\psi(h(x)) = \sup\{\phi(g) : g(x) \leq h(x)\} = \sup\{\phi(hg') : g'(0) \leq x\} = \phi(h) + \sup\{\phi(g') : g'(0) \leq x\}$. □

The mysterious Conrad's property

The following are equivalent definitions of Conrad's property

1. For all $f, g > 1$ there exists n such that $fg^n > g$.
2. For all $f, g > 1$ one has $fg^2 > g$.
3. The following cannot happen $h_1 < fh_1 < fh_2 < gh_1 < gh_2 < h_2$

A visualization of the action of $(\mathbb{Z}^2, \leq_{lex})$ by homeomorphisms on $\mathbb{R}$.

Image of the action: a acts translation, b acts as homeo on the intervals with boundary fixed

Lecture 4

This lecture is about to solve the most natural question about left-orderable group theory.

Consider the group $\Gamma = \langle a, b : a^2ba^2 = b, b^2ab^2 = a\rangle$. This group is a crystallographic group, i.e., Γ acts freely and co-compactly on $\mathbb{R}^3$. The action of Γ is given by:

$$\begin{aligned} a(x,y,z) &= (x+1, 1-y, -z) \\ b(x,y,z) &= (-x, y+1, 1-z) \\ c(x,y,z) &= (1-x, -y, z+1) \end{aligned}$$

The group Γ is a torsion-free group. One have to notice that the group $\langle a^2, b^2, c^2\rangle$ is isomorphic to $\mathbb{Z}^3$. One can prove that $\mathbb{Z}^3$ in Γ has finite index and the quotient $\Gamma/\mathbb{Z}^3$ is the Klein group.

If we take an element $w \in \Gamma$, we have that this element have the form $w = a^{2i}b^{2j}c^{2k}a$ where i, j, k are integers. Notice that if we take the squared power of a generic element w we have that is of the form $w^2 = a^{4i}a^2$ which is not the identity.

We claim that Γ is an example of a torsion-free group which is no left-orderable.

Exercise 0.38. Let $\epsilon, \delta \in \{-1, 1\}$. Then $(a^\epsilon b^\delta)^2(b^\delta a^\epsilon)^2 = 1$.

Exercise 0.39. If G is a torsion-free group with an index-3 left-orderable normal subgroup, then G is left-orderable.

Definition 0.40 (Unique product property (UPP))**.** A group G has the (UPP) if for all $S \subset G$ finite, there exists $s \in S \otimes S$ that "appears only on Ga.

It is not difficult to prove the following property, if G has the (UPP), then G is torsion-free. But the converse is NOT true, the proof was given by Rips-Sagecu and PRomislow independently.

Definition 0.41. For a group G, one can define the *group algebra*, denoted by $\mathbb{R}G$, whose elements are of the form $\sum r_i g_i$ of finite products real numbers and group elements. The product is the natural.

From this, if $f^n = 1$ implies that $f^n - 1 = 0$ in $\mathbb{R}G$. The last statement is equivalent that $(f-1)$ and $(f^{n-1} + \cdots + 1)$ are zero divisors. From Kaplansky, we have the conjecture that if the group algebra of the group has non-trivial zero divisors, then there exists $f^n = 1$.

Example 0.42 (Thurston/Bergman)**.** A left-ordered finitely generated group G such that there is non-trivial $\phi : G \to \mathbb{R}$.

Consider the group $\langle a, b, c : a^2 = b^3 = c^7 = 1\rangle \subset \mathrm{PSL}(2, \mathbb{R})$, which is a triangular group with triangular domain angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7})$.

If we look at the lift to $\mathbb{R}$ of the previous group. We have an example of this.

There is another characterization of the Conrad's property as if the group has an action on the line with no resilient pairs (Solodov, Plante).

Theorem 0.43. *A left-order $\leq$ on G is Conradian if and only if this cannot happen: $h_1 < fh_1 < fh_2 < gh_1 < gh_2 < h_2$.*

Proof. If $\leq$ is Conradian, the claim follows from the dynamical realization.

... □

Theorem 0.44 (Plante-Solodov)**.** *If $G \subseteq \mathrm{Homeo}_+(\mathbb{R})$ is finitely generated and has no resilient pair. Then there exists a $\sigma-$finite measure μ on $\mathbb{R}$.*

Put draw of resilient pair: pairs of points that stay close together by group left action

Before we state the proof, we will define the *translation number homomorphism*, as the map $\tau : G \to (\mathbb{R}, +)$ given by

$$\tau(g) = \begin{cases} \mu([x, gx)) & \text{if } gx > x \\ -\mu([gx, x)) & \text{if } gx < x \end{cases}$$

The previous function is independent of the choice of x, and it is not difficult to prove that this map is an homomorphism.

Remark 0.45. If G is left-orderable and has sub-exponential growth. Then every order on G is Conradian.

Exercise 0.46. Assume that $f, g > 1$ such that $fg^n > g$ for all $n \in \mathbb{N}$. Then letting $u = 1, v = f^{-1}g, w = g^2$ the following hold:

1. $u < w < v$.
2. $g^n(u) < v$ and $f^n(v) > u$ for all $n \in \mathbb{N}$.
3. There exists M, N such that $f^N v < w < g^M w$.

Lecture 5

Exercise 0.47. Let G a finitely generated left-orderable group with $|LO(G)| = 2$ if and only if $G \simeq \mathbb{Z}$.

Remember, if G is finitely generated and $\leq$ is a Conrad on G then there exists $\phi : G \to \mathbb{R}$ order preserving homeomorphism.

Example 0.48. Consider $\mathbb{Z}^2$ and $\leq$ order on $\mathbb{Z}^2$, from the fact that $\mathbb{Z}^2$ is abelian we can conclude that $\leq$ is Conradian. Therefore there exists $\phi_1 : \mathbb{Z}^2 \to \mathbb{R}$.

For ϕ_1 can happen:

- $\ker(\phi_1) = \{0\}$ which is ok.
- $\ker(\phi_1) \neq \{0\}$, then for $\leq |_{\ker(\phi)}$ there exists $\phi_2 : \ker(\phi_1) \to \mathbb{R}$ and $\ker(\phi_2) = \{0\}$.

In general, the homeomorphism ϕ_1 is of the form $(m, n) \mapsto am + bn$. From this we can claim that $\mathcal{O}(\mathbb{Z}^2)$ is a Cantor set.

Figure of a hemi-plane denoting positivity based on (a,b) and the orthogonal space.

Theorem 0.49 (Rivas)**.** *The space of Conradian orders $\mathcal{CO}(G)$ is either finite or a Cantor set.*

Example 0.50. The Braid group $BS(1,2) = \langle a, b : aba^{-1} = b^2 \rangle$ has 4 Conrad orders.

Theorem 0.51 (N, Rivas-Tessera)**.** *If G is solvable left-orderable and $|LO(G)| = \infty$, then $LO(G)$ is a Cantor set.*

Q 2. When we have an analogue result to above theorem in the case of G amenable?

Theorem 0.52 (Dove-Witle-Morris)**.** *If G is amenable and left-orderable, then G is Conrad orderable.*

Proof. Let us consider that G is finitely generated and the natural action (induced by conjugations inequalities) of G in $LO(G)$ which is a compact metric space. Then, there exists μ a probability measure on $LO(G)$ which is $G-$invariant.

For the sake of completeness, we will recall Poincaré's recurrence theorem.

Theorem 0.53. *Let $T \curvearrowright (X, \mu)$ an action and μ is a probability $T-$invariant measure. Let $A \subset X$ such that $\mu(A) > 0$, then for almost every $x \in A$ there exists $m \in \mathbb{N}$ such that $T^n(x) \in A$.*

Claim $\mu(\text{Conrad orders}) = 1$.

Consider the following notation $g \neq 1$, then $V_g = \{\leq : g > 1\}$.

For $f \in G$ denote by $B_g(f) := V_g \setminus \bigcup_{n>0} f^{-n}(V_g)$. By Poincaré's recurrence theorem, we have that $\mu(B_g(f)) = 0$.

Let's take $B := \bigcup_{g \neq 1} B_g$.

We claim that $B^c \subset$ Conrad orders. We have that $\mu(B^c) = 1$. Suppose that $\leq$ that belongs to B^c and let $f, g > 1$, such that $\leq \in V_g$ and $\leq \notin B_g(f)$. Then there exists $n > 0$ such that $\leq \in f^{-n}(V_g)$. By this, $\leq_{f^n}$ belongs to V_g and for this we have that $g >_{f^n} 1$ and from this it follows the Conrad property. □

Exercise 0.54. Let $G = \mathbb{F}_2 \ltimes \mathbb{Z}^2$. Prove that

1. G has a Conrad order.
2. For all left-order $\leq$ on G there exists $f, g > 1$ such that $gf^n < f^n$ for all $n \in \mathbb{N}$.

Q 3 (Open Problem)*.* Let G be a left-orderable group, if G has no free subgroup in two generators. Is it true that G is Conrad orderable?

Q 4*.* Can be $G \curvearrowright LO(G)$ minimal?

Theorem 0.55 (McCleary,Rivas)**.** *In $LO(\mathbb{F}_2)$ there is $\leq$ with a dense orbit.*