Group Actions on Hilbert spaces: property T and Haagerup; Talia Fernós

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Lecture 1

Locally compact second countable groups, for example:

- Γ be a discrete countable group
- $G = \bigcup_n K_n$, where $K_n \subset K_{n+1}$, compact.

Definition 0.1. Let Γ be a group and $p \in [0, \infty]$. We will define

$$\ell^p(\Gamma) = \{f : \Gamma \to \mathbb{C} \|f\|_p \sum_{x \in \Gamma} : |f(x)|^p < \infty\}.$$

The set of $p-$sumable groups. For the case of $p = \infty$ we will take supremum norm.

We can make this spaces into a metric space.

Definition 0.2. For $f, g \in \ell^p(\Gamma)$, we will define the distance function by

$$d(f, g) = \sqrt[p]{\|f - g\|_p} \quad (1)$$

There are several definitions for amenable groups, we will list few of them just to give examples. We encourage the lector to proof some equivalences.

Definition 0.3. A group Γ is *amenable* if and only if one of the following equivalent conditions hold:

1. There exists a left-invariant probability finitely add measure on 2^Γ.
2. There exists a sequence of Folner sets.

3. $\ell^2(\Gamma) \succeq I_\Gamma$, i.e., has almost invariant vectors

4. K compact metrizable $\Gamma \to \mathrm{Homeo}(K)$, there exists a μ probability measure such that $\gamma_\star\mu(E) = \mu(E)$.

5. Γ does not admit a Ponzi scheme.

6. Γ is not paradoxical.

Definition 0.4. A sequence $\{F_n\}_n$ of *Folner sets* means $F_n \subset F_{n+1}, \Gamma = \bigcup_n F_n$ and

$$\frac{|F_n \Delta \gamma F_n|}{|F_n|} \to \infty$$

as n tends to infinity.

Definition 0.5. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{U}(\mathcal{H})$ is the unitary group. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a group homomorphism. We will say that π have almost invariant vectors if there exists $v_n \in \mathcal{H}$ such that

$$\frac{||\pi(\gamma)v_n - v_n||}{||v_n||} \to 0$$

as n tends to infinity.

Definition 0.6. Let $P : \Gamma \to \Gamma$ is a Ponzi scheme if $\#|P^{-1}(\gamma)| \geq 2$.

Definition 0.7. A group Γ is paradoxical if there exists X a discrete countable set such that $\Gamma \curvearrowright X$ and partitions $X = A \sqcup B$, where $A = \bigsqcup_{i=1}^n A_i$ and $B = \bigsqcup_{j=1}^n B_j$ and some elements $g_1, \cdots g_n, h_1, \cdots h_n \in \Gamma$ such that

$$X = \bigsqcup_{i=1}^n g_i A_i = \bigsqcup_{j=1}^n h_j B_j.$$

Definition 0.8. We say that Γ is linear is there exists $\varrho : \Gamma \to \mathrm{GL}_n(\mathbb{R})$

Locally compact linear groups that are amenable are:

- O_n the orthogonal group.
- Solvable groups (conjugated to upper-triangle group).
- Block upper triangular matrices, where the diagonal blocks could be orthogonal groups or diagonal groups, and the ranks of the diagonals have to sum the rank of the whole group.

Q 1. Which linear groups preserve a measure on $\mathbb{R}\mathbf{P}^2$?

We recall that $\mathbb{R}\mathbf{P}^n$ is a compact metric space.

There is a similar space related to projective spaces. The $\mathrm{Gr}(k,n)$ is the set of equivalence classes of $k-$subspaces of $\mathbb{R}^n$. In particular, for $k=1$ the set $\mathrm{Gr}(k,n) = \mathbb{R}\mathbf{P}^n$.

Lemma 0.9 (Furstenberg Lemma)**.** *Let μ be a probability measure on $\mathbb{R}\mathbf{P}^n$ and Γ a subgroup of $P\mathrm{GL}_n(\mathbb{R})$. If μ is $\Gamma-$invariant then either:*

- *Γ is pre-compact,*
- *Exists $0 \subsetneq V_0 \subsetneq \mathbb{R}^n$ subspace such that $\mu([V_0]) > 0$ and Γ_0 a finite index subgroup such that $\Gamma_0[V_0] = [V_0]$.*

Proof. Assume Γ is not precompact, i.e., it means that there exists $\gamma_n \to \infty$ in $P\mathrm{GL}_n(\mathbb{R})$. Choose lifts of γ_n in $\mathrm{GL}_n(\mathbb{R})$ such that $\|\tilde{\gamma}\| = 1$ for the max-norm. From the fact that the set of squared real matrices space is locally compact, it follows that there exists a sequence of $\tilde{\gamma}_n \to g$ and $g \notin \mathrm{GL}_n(\mathbb{R})$. From this fact, we can assure that $\ker(\tilde{g}) \neq 0$ and $\mathrm{im}(\tilde{g}) \neq 0$.

Lecture 2

We have to mention that $\gamma_n([\ker tildag]) \to [V]$ up to subsequence. We claim that $\mu([V] \cup [\mathrm{im}(\tilde{g})]) = 1$. It is clear that $[V] \cup [\mathrm{im}(\tilde{g})]$ is closed.

Let us denote by $A_0 = [V] \cup [\mathrm{im}(\tilde{g})]$. Fix a metric on $\mathbb{R}\mathbf{P}^n$ and define the function $D_{A_0} : \mathbb{R}\mathbf{P}^n \to \mathbb{R}$ defined as $D_{A_0} = dist(x, A_0)$ based in the fixed metric. It is easy to prove that $D_{A_0}(\gamma_n x) \to 0$ for all point $x \in \mathbb{R}\mathbf{P}^n$.

We have that $\mathbb{R}\mathbf{P}^n \setminus A_0 = \bigcup_{m \in \mathbb{N}} \{x : D_{A_0}(x) > \frac{1}{m}\}$. Let A_m denote the elements of the previous partition.

We claim that $\mu(A_m) = 0$ for each $m \in \mathbb{N}$. We have the following inequalities

$$\frac{1}{m}\mu(A_m) \leq \int_{A_m} D_{A_m}(x) d\mu \leq \int_{\mathbb{R}\mathbf{P}^n} D_{A_0}(x) d\mu$$

Since μ is a $\Gamma-$invariant, we obtain that

$$\frac{1}{m}\mu(A_m) \leq \int_{\mathbb{R}\mathbf{P}^n} D_{A_0}(\gamma_n(x)) d\mu.$$

The claim follows from the dominated convergence theorem.

The rest follows from the fact that $\mu(A_0) = 1$. □

With the Furstenberg lemma we can answer question: If there is a $P\mathrm{GL}_n(\mathbb{R})$ amenable subgroup, how it looks like? Since we have that for Γ there exists Γ_0 that stabilizes some V_0. If we take the quotient representation, in which Γ_0 is amenable and taking this process a finite number of times, we can ensure that Γ contains a finite index subgroup Γ' which is conjugated to a block upper triangular with diagonal blocks compact.

0.1 Relative Property (T) ans semi-direct products.

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ a strongly continuous unitary representation.

Definition 0.10. We will say that π has almost invariant vectors ($\pi \succcurlyeq I_G$) if there exists v_n non-zero element of $\mathcal{H}$ such that

$$\sup_{\gamma \in K} \frac{\|\pi(\gamma)v_n - v_n\|}{\|v_n\|} \to 0$$

as $n \to \infty$ for all $K \subset G$ compact.

Definition 0.11. We will say that π has invariant vectors if there exists a non-zero vector v such that $\pi(\gamma)v = v$ for all element of G.

Definition 0.12. Let G a locally compact second countable group, and $H < G$ be a closed subgroup. We will say that (G, H) has relative property (T) if $\pi : G \to \mathcal{U}(\mathcal{H})$ such that $\pi \succcurlyeq I_G$ then $\pi|_H \geq I_H$.

Example 0.13. Consider the groups $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and $\mathbb{Z}^2$ has rel(T).

Exercise 0.14. Consider Γ be an amenable group and $H < \Gamma$ such that $|H| = \infty$. (Γ, H) does not has rel(T).

Definition 0.15. We will say that G has property (T), if (G, G) has rel(T).

Lemma 0.16. *Let G be a compact group, then G has property (T).*

Using that the group has a finite Haar measure you can prove that there are invariant vector.

Exercise 0.17. Assume that H has property (T) and $H < G$ is closed. Then (G, H) has rel(T).

Lemma 0.18. *Consider the following short exact sequence*

$$1 \to A_0 \to A \to A_1 \to 1.$$

Let $\Gamma \to \mathrm{Aut}(A, A_0)$, *where the later one is the subgroup of* $\mathrm{Aut}(A)$ *that stabilizes* A_0. $(\Gamma \ltimes A_0, A_0)$ *and* $(\Gamma \rtimes A_1, A_1)$ *has rel(T) if and only if* $(\Gamma \ltimes A, A)$ *has rel(T).*

The idea of the proof is to look at the properties of the short exact sequence

Lecture 3

Exercise 0.19. Show that if $H < G$ is a closed subgroup such that H has (T). Then (G, H) has (T).

Example 0.20. Consider $\Gamma \to \mathrm{SL}_n\mathbb{Z}$, $A = \mathbb{Z}^n, V = \mathbb{R}^n$ and $T = V/A$. Satisfies the previous lemma.

Example 0.21. Consider $\Gamma \to \mathrm{SL}_n\mathbb{Z}[\frac{1}{p}], A = (\mathbb{Z}[\frac{1}{p}])^n$ and $V = \mathbb{R}^n \times \mathbb{Q}_p^n$, with $T = V/A$ satisfies the lemma.

Theorem 0.22. *Let* $\Gamma \to \mathrm{SL}_n\mathbb{Z}$ *is a group homomorphism. Then* $(\Gamma \ltimes \mathbb{Z}^n, \mathbb{Z}^n)$ *has Rel(T) if and only if* $(\Gamma \ltimes \mathbb{R}^n, \mathbb{R}^n)$ *has Rel(T).*

The previous theorem prove has relation with Burger's criterion. We recall that Burger's criterion says that for a $\psi : \Gamma \to \mathrm{GL}_n\mathbb{R}$ such that there are no $\Gamma-$invariant probability measures on $\mathbf{P}(\hat{\mathbb{R}}^n)$, then $(\Gamma \ltimes_\psi \mathbb{R}^n, \mathbb{R}^n)$ has Rel(T).

Recall that $\mathbb{R}$ is a local field (locally compact with respect the norm and is a countable union of compact sets). There is an isomorphism between $\hat{\mathbb{R}}^n := \{\pi : \mathbb{R}^n \to \mathcal{U}(\mathcal{H}) : \pi \text{ is irred.}\}$ and $\mathrm{Hom}(\mathbb{R}^n, \mathrm{S}^1)$.

Remark 0.23. In the previous paragraph that in definition of dual unitary space of $\mathbb{R}^n$ we look for representation in any Hilbert space, but the irreducibility of the representation will assure that this Hilbert space has to be one dimensional.

If there is a unitary representation $\pi : \Gamma \ltimes \mathbb{R}^n \to \mathcal{U}(\mathcal{H})$, there exists a map $P : \mathcal{B}(\hat{\mathbb{R}}^n) \to \mathrm{Proj}(\mathcal{H})$, where $\mathcal{B}(\hat{\mathbb{R}}^n)$ is the set of Borel sets and $\mathrm{Proj}(\mathcal{H})$ is the set of orthogonal projections of $\mathcal{H}$. We will call P a projection valued measure, it is clear that depends on π and have the following properties:

- $P(\hat{\mathbb{R}}^n) = id$
- For every $v \in \mathcal{H}$, $B \mapsto \langle P(B)v, v\rangle$ is a positive Borel measure.

- For all $\gamma \in \Gamma$, we have $\pi(\gamma^{-1})P(B)\pi(\Gamma) = P(\gamma^* B)$

Example 0.24. Consider $\pi : \mathbb{R} \to \mathbb{C}\times\mathbb{C}$ given by $\pi(x)(v_1, v_2) = (e^{ix}v_1, e^{2ix}v_2)$. In this case, we have that

$$P(\psi) = \begin{cases} Proj(v_j) & \text{if} \quad \psi = j \\ 0 & \text{otherwise} \end{cases}$$

Proof of Theorem 0.22. Let $\pi : \Gamma \ltimes \mathbb{R}^n \to \mathcal{U}(\mathcal{H})$ an unitary representation with almost invariant unit vectors $\{v_n\}$. Then there exists a projection valued measure $P : \mathcal{B}(\hat{\mathbb{R}}^n) \to \mathrm{Proj}(\mathcal{H})$ such that $P(\{0\})$ is the projection onto the $\mathbb{R}^n-$invariant vectos. Assume that $P(\{0\}) = \{\mathbf{0}\}$ the zero-subspace of $\mathcal{H}$.

Define $\mu_n(B) = \langle P(B)v_n, v_n\rangle$ a Borel measure. We have that $\mu_n(\{0\}) = 0$. If we look at the push-forward measure from $\hat{\mathbb{R}}^n \setminus \{0\} \to \mathbf{P}(\hat{\mathbb{R}}^n)$ given by $\mu_n \mapsto \overline{\mu}_n$ From this it follows that

$$\|\gamma_*\mu_n - \mu_n\| := 2 \sup_{B\in\mathcal{B}(\hat{\mathbb{R}}^n)} |\gamma_*\mu_n(B) - \mu_n(B) \leq 2\|\pi(\gamma)v_n - v_n\| \to 0$$

From the previous, we obtain a sequence $(\overline{\mu}_n)$ of almost invariant measures on $\mathbf{P}(\hat{\mathbb{R}}^n)$. From the Banach-Alaoglu theorem, we have that $(\overline{\mu}_n)$ has a weak-* limit point up to passing to some subsequence. Denote by $\overline{\mu}_\infty$ which is a $\Gamma-$invariant and by Burguers criterion, $(\Gamma \ltimes \mathbb{R}^n, \mathbb{R}^n)$ has Rel(T). □

Example 0.25. Let $\Gamma \leq \mathrm{SL}_2\mathbb{Z}$ and $\Gamma \ltimes \mathbb{R}^2$ an amenable group. Then $(\Gamma \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has Rel(T) and also $(\Gamma \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has Rel(T).

Lecture 4

The previous example able us to claim that $\mathrm{SL}_n\mathbb{Z}$ has property (T) for $n \geq 3$.

Notice that $\mathrm{SL}_2\mathbb{Z} \ltimes \mathbb{Z}^2 \hookrightarrow \mathrm{SL}_3\mathbb{Z}$ in several ways, but in particular of the form

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

where $A \in \mathrm{SL}_2\mathbb{Z}$ and $b \in \mathbb{Z}^2$. Also, we recall that $\mathrm{SL}_3\mathbb{Z} = \langle E_{ij}(1) : i \neq j\rangle$.

Furthermore, inside $\mathrm{SL}_3\mathbb{Z}$, they are all conjugate, what implies $(\mathrm{SL}_3\mathbb{Z}, \langle E_{ij}(1)\rangle)$ has Rel(T) for all $i \neq j$.

Theorem 0.26 (Carter-Keller)**.** *Let* $\mathrm{SL}_n\mathbb{Z}$ *with* $n \geq 3$*. We have that* $\mathrm{SL}_n\mathbb{Z}$ *is boundedly generated by elemental matrices.*

Definition 0.27. We understand that $\mathrm{SL}_n\mathbb{Z}$ is boundedly generated if there exists $B \in \mathbb{N}$ and $\varphi : \{1, \cdots, B\} \to \{(i,j) : i \neq j, i, j = 1, \cdots n\}$ such that $\mathrm{SL}_n\mathbb{Z} = \langle E_{\varphi(1)}\rangle \cdots \langle E_{\varphi(B)}\rangle$.

Lemma 0.28. *Let* $\Gamma = \mathrm{SL}_n\mathbb{Z}$ *acting by isometries on* X *and let* $A_1, \cdots A_B$ *be subgroups of* Γ *such that:*

1. $\mathrm{SL}_n\mathbb{Z} = A_1 \cdot A_2 \cdots A_B$
2. *Every* A_i-*orbit is bounded for every* $i = 1, \cdots, B$*.*

Then $\mathrm{SL}_n\mathbb{Z}$ *has a bounded orbit.*

Theorem 0.29 (Delone-Guichardet)**.** Γ *has (T) if and only if every affine isometric action on Hilbert has a fixed point.*

Remark 0.30. In the previous theorem we can change fixed point by bounded orbit.

Theorem 0.31 (Relative Version of previous theorem)**.** (Γ, A) *has Rel(T) if and only if every affine* $\Gamma-$*action on a Hilbert space has an* $A-$*fixed point.*

Let us fix $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ if $\mathcal{H}$ is a $\mathbb{C}-$vector space (for the real vector space we change by $\mathcal{O}(\mathcal{H})$). Fix $b : \Gamma \to \mathcal{H}$ set theoretically such that $\alpha(\gamma)(v) = \pi(\gamma)v + b(\gamma)$.

Q 2. Is α an action of Γ on $\mathcal{H}$?

The answer is yes if and only if $b(\gamma_1, \gamma_2) = b(\gamma_1) + \pi(\gamma_1)b(\gamma_2)$ $(\star)$ is a 1-cocylcle relation.

Definition 0.32. Consider the following sets

$$\mathcal{Z}^1(\Gamma, \pi) = \{b : \Gamma \to \mathcal{H}, b \text{ satisfies } (\star)\} \quad (2)$$
$$\mathcal{B}^1(\Gamma, \pi) = \{b : \Gamma \to \mathcal{H}, b(\gamma) = \pi(\gamma)v - v\} \quad (3)$$

Exercise 0.33. Let $\Gamma \upharpoonright X$ by permutations and $f : X \to \mathbb{C}$. Define $b(\gamma) = \gamma f - f$. Prove that b satisfies $(\star)$

It is easy to prove that $\mathcal{B}^1(\Gamma, \pi) \subset \mathcal{Z}^1(\Gamma, \pi)$.

Definition 0.34. The first cohomology group of Γ with $\pi-$coefficients is the group defined as

$$H^1(\Gamma,\pi)=\frac{\mathcal{Z}^1(\Gamma,\pi)}{\mathcal{B}^1(\Gamma,\pi)}.$$

Remark 0.35. The first cohomology group of Γ with $\pi-$coefficients is "parametrizing affine actions of Γ, with linear part π, modulo those with fixed points".

Exercise 0.36. The following are equivalent:

Lemma 0.37. *If $X \subset \mathcal{H}$ is a bounded set. Then there exists a unique closed ball of minimal radius containing X.*

With the previous paragraphs we can obtain the has (T) is equivalent that all affine actions on $\mathcal{H}$ have fixed point and that the first cohomology group of Γ with $\pi-$coefficients is trivial for all π.

Exercise 0.38. Let T denote a tree with V its vertex set and $E \subset V \times V$ its oriented edges set. Notice that under this conditions any edge is uniquely determined by its initial and end point $e=(e^-,e^+)$.

Fix an edge $e \in E$ and consider the set

$$h_e=\{v\in V : d(v,e^+)<d(v,e^-)\}.$$

For each $e \in E$ denote that $V = h_e \sqcup h_{\overline{e}}$, where $\overline{e}$ denotes the inverse orientation on e.

Fix $v \in V$ and define $\mathbf{1}_v(e)=1$ if $v \in h_e$ and 0 otherwise, which is the characteristic function of the set $\{e : v \in h_e\}$.

Assume $\Gamma \curvearrowright T$ by isometries define $b(\gamma)=\mathbf{1}_{\gamma v}-\mathbf{1}_v$ is a 1-cocycle. Prove that

$$\|b(\gamma)\|^2=\sum_{e\in E}|b(\gamma)(e)|^2=2d(\gamma v,v)<\infty.$$

From this exercise we can conclude that if Γ has (T) and it acts on a simplicial tree then:

- Γ have bounded orbits.
- Γ has a fixed vertex or an invariant edge, called the property (FA).

Q 3. $\|b(\gamma)\|^2 \to \infty$ if $\gamma \to \infty$?

There are few lines mistaken

Lecture 5

Remember that a group has (T) if and only if for all $\pi \succcurlyeq I_\Gamma$, which the later one implies that $\pi \geq I_\Gamma$. Another equivalent definition is that the representation has (FH) which means that every affine isometric action on $\mathcal{H}$ has a fixed point.

Also remember that a group has the Haagerup property is and only if there exists $\pi \succcurlyeq I_\Gamma$ and π is C_0, i.e.,

$$\langle \pi(\gamma), w \rangle \to 0$$

as $\gamma \to \infty$ for all $v, w \in \mathcal{H}_\pi$. Another equivalent definition is that $b \in H^1(\Gamma, \pi)$ is proper.

In the case of *median spaces* (think in trees) that (T) and Haagerup property looks as:

- (T) if and only if every isometric action on median space has a bounded orbit.
- Haagerup if and only if there exists a proper action on a median space.

Consider $\Gamma \curvearrowright (X, \mu)$ with $\mu(X) = 1$ measure preserving then $\Gamma \to \mathcal{U}(L^2(X, \mu))$ ergodic. In this scenario, Γ has (T) if and only if every measure preserving ergodic action on (X, μ) is strongly ergodic, which means that for all $E_n \subset X$ means $\mu(\gamma E_n \triangle E_n) \to 0$ as n tends to infinity for all γ, then $\mu(E_n)(1 - \mu(E_n)) \to 0$

Also, Γ has Haagerup if and only if there exist $\Gamma \curvearrowright (X, \mu)$ a measure preserving ergodic action such that there exists a sequence $E_n \subset X$ measurable sets such that

$$\frac{\mu(\gamma E_n \triangle E_n)}{\mu(E_n)} \to 0$$

for all $\gamma \in \Gamma$ and strongly mixing ($\mu(\gamma A \cap A) \to \mu(A)\mu(B)$ as γ tends to infinity).

Definition 0.39. Let X be a discrete countable set. We say that $\Phi : X \times X \to \mathbb{C}$ is a Kernel of *positive type* if

1. $\Phi(x, x) \geq 0$
2. $\Phi(x, y) = \overline{\Phi(y, x)}$

3. for all $x_1, \cdots, x_n \in X$ and $c_1, \cdots, c_n \in \mathbb{C}$ it is hold

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \bar{c}_j \Phi(x_i, x_j) \geq 0$$

Theorem 0.40. Φ *if and only if* $f : X \to \mathcal{H}$ *such that* $\Phi(x, y) = \langle f(x), f(y)\rangle$.

Definition 0.41. Let X be a discrete countable set. We say that $\Psi : X \times X \to \mathbb{R}$ is a Kernel of *conditionally negative type* if

1. $\Psi(x, x) = 0$
2. $\Psi(x, y) = \Psi(y, x)$
3. for all $x_1, \cdots, x_n \in X$ and $r_1, \cdots, r_n \in \mathbb{R}$ it is hold

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_i r_j \Psi(x_i, x_j) \leq 0$$

when $\sum_{i=1}^{n} r_i = 0$.

Theorem 0.42. *There exist* $f : X \to \mathcal{H}$, $\Psi(x, y) = \|f(x) - f(y)\|^2$.

Theorem 0.43 (Schoenberg)**.** *Let* $\Psi : X \times X \to \mathbb{R}$ *such that* $\Psi(x, x) = 0$ *and* $\Psi(x, y) = \Psi(y, x)$, *the following are equivalent*

1. *Ψ is conditional negative type kernel*
2. *$e^{-t\Psi}$ is a kernel of positive type for all $t > 0$.*

Proposition 0.44. *Let* $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ *and* $b \in \mathcal{Z}^1(\Gamma, \pi)$ *such that* $\alpha = \pi + b$. *For* $t > 0$ *take* $(\mathcal{H}_t, \pi_t)$ *and* $\Phi_t : \mathcal{H} \to \mathcal{H}_t$ *into the unit sphere, such that*

1. $\langle \Phi_t(v), \Phi_t(w)\rangle = e^{-t\|v-w\|^2}$ *for all* v, w.
2. $\pi_t(\gamma)\Phi_t(v) = \Phi_t(\alpha(\gamma)v)$ *for all* $\gamma \in \Gamma$ *and* $v \in \mathcal{H}$.
3. *if* $\|v_n\| \to \infty$, *then* $\Phi_t(v_n) \to 0$ *weakly.*
4. *α has a fixed point if and only if π_t has a fixed unit vector for every t.*
5. *Let* $\pi := \bigoplus \pi_{1/n} : \Gamma \to \mathcal{O}(\bigoplus \mathcal{H}_{1/n})$, *then* $\pi \succcurlyeq I_\Gamma$.

Lemma 0.45. *Let* $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ *such that* $\pi \not\geq I_\Gamma$ *and* $\pi \succcurlyeq I_\Gamma$ *if and only if* $\mathcal{B}^1(\Gamma, \pi) \neq \overline{\mathcal{B}^1(\Gamma, \pi)}$.

Corollary 0.46. *If $\pi \ngeq I_\Gamma$ and $\pi \succcurlyeq I_\Gamma$, then $H^1(\Gamma, \pi) \neq 0$*

Proof of Delome Guichardet. Assume that Γ does not have (T), then there exists $\pi \succcurlyeq I_\Gamma$ and $\pi \ngeq I_\Gamma$, wich implies that $H^1(\Gamma, \pi) \neq 0$, and it follows that Γ does not have (FH).

Assume that Γ does not have (FH), which implies that $H^1(\Gamma, \pi_0) \neq 0$ for some π_0, i.e., there exists $\alpha : \Gamma \to \mathrm{Isom}(\mathcal{H})$ without fixed points. We can conclude that $\pi_{1/n} \ngeq I_\Gamma$ which implies that $\bigoplus \pi_{1/n} \succcurlyeq I_\Gamma$ and $\bigoplus \pi_{1/n} \ngeq I_\Gamma$. So we can conclude that Γ does not have (T). □

(Metric)Median Spaces

Let (M, d) be a metric space. We will say that M is a median space if for all (x, y, z) there exists a unique point $m = m(x, y, z)$ such that $m = I(x, y) \cap I(y, z) \cap I(z, x)$ where $I(x, y) = \{u \in M : d(x, u) + d(u, y) = d(x, y).\}$

Example 0.47. Consider $\mathbb{R}^2$ with the ℓ^1-metric.

Put the figure of squares with median

Example 0.48. $\mathbb{Z}$ and $M_1 \times M_2$ with $d = d_1 + d_2$, so $\mathbb{Z}^2$. CAT(0)-cube complexes.

Theorem 0.49 (Chatterji-Drutu-Haglund)**.** *Every median space (M, d) can be isometrically embed into $L^1[0, 1]$.*

Theorem 0.50 (Chatterji-Drutu-Haglund)**.** *There are Kernels of median type if and only if there exists embedding into median spaces.*

From the previous theorem, we can claim the (T) and Haagerup property are characterized by isometric action on Median spaces.

So, we can ask us: if we select our favourite Γ with the Haagerup property, what is the "simplest" median space necessary? In order to answer this question we have to consider if we want simplicity as finite dimension or simplicial vs not.