Geometry and Topology of Automorphisms of Free Groups; Spencer Dowdell

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Lecture 1

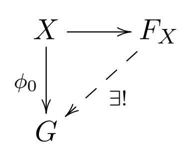
Definition 1. Let X any set, a free-group on X is the set

$$F_X = \{\text{empty word}\} \cup \{\text{freely reduced words on } X \cup X^{-1}\}$$

with the operation contatenation and freely reduced.

Exercise 1. Check that the previous satisfies group axioms.

Definition 2. For any group G and set map $\phi_0 : X \to G$ there exists a unique group homomorphism $\phi : X \to G$ such that



Up to isomorphism F_X is determined by cardinality.

We will denote by $F_n = F_{\{x_1,\cdots,x_n\}}$ the free group of rank n.

Definition 3. A graph is a 1-dimensional cell complex with oriented edges. There exists a reverse orientation map from the set of edges. We will denote by e^+ the initial and by e^- the end points.

Definition 4. A morphism of graphs is a cellular map that sends each open edge homeomorphically onto an open edge. Formally speaking, a graph morphism from $(V, E, -, \iota) \to (V', E', -, \iota)$ is a pair of functions $V \to V'$, $E \to E'$ conmuting with $-$ and ι.

Remark 5. Adjusting by a homeomorphism that is isotopic to identity at vertices, we will regard as the same graph morphism.

Definition 6. A graph morphism is an immersion if its locally injective (just need to check at vertices).

Exercise 2. Suppose that $f : X \to Y$ is an immersion of finite graphs. Show that is possible to attach finitely many 1- and 0- cells to obtain a new graph $\tilde{X}$ to which f extends ($\tilde{f} : \tilde{X} \to Y$) which is a graph morphism that is a covering map.

Definition 7. An edge path in a graph G is a (possibly degenerated) edges sequence $e_1, \cdots, e_n$ such that $e_i^- = e_{i+1}^+$.

Remark 8. An edge path is a morphism $I \to G$ where I is a graph homeomorphic to $[0,1]$ or $\{p\}$.

An edge path is tight (reduced) if $I \to G$ is an immersion to $[0,1]$ ($\{p\}$).

Definition 9. An elementary homotopy of edge path is a map that inserts/deletes consecutive edges $e, \bar{e}$.

Exercise 3.

- Edges paths are related by elementary homotopy if and only if homotopic (...) endpoints.
- Two reduced edge paths are homotopic if and only if are equal.
- Elements of $\pi_(G, v)$ are in bijection with reduced edge paths that start and stop at v.
- An immersion b/w graphs is π_1-injective.

0.1 Folding

Definition 10. If e_1, e_2 are edges such that $e_1 \neq e_2, \bar{e}_1$ and $e_1^+ = e_2^+$. Can forma a new graph G' with (quotient) morphism $G \to G'$ by identifying e_1 and e_2, and e_1^- with e_2^-.

Example 11.

Put images of foldings

Theorem 12 (Stallings)**.** *Every morphism $G \to G'$ of finite graphs factors as*

$$G = G_0 \to G_1 \to \cdots \to G_h \to G'$$

where each $G_i \to G_{i+1}$ is a fold and $G_h \to G'$ is an immersion.

Definition 13. A tree is a graph with unique reduced edge path between any two vertices. A spanning tree in a graph G is a subtree $T \subset G$ that contains all verticies of G.

Remark 14.

- For any graph G, $\pi_1(G, v)$ is free.
- For $T \subset G$ a spanning tree, choose orientation on each edge of $G \setminus T$. Get isomorphism $\pi_1(G, v) \to F_X$ that send closed reduced loop at v to a word in $X \sqcup X^{-1}$.

Theorem 15 (Nielsen-Schreier)**.** *Every subgroup of a free group is free.*

Given $g_1, \cdots, g_h \in F_n$ find free basis of $H = \langle g_1, \cdots g_h \rangle \subset F_n$.

Example 16. Let $H = \langle a^3 b, \overline{a}bab, a^2\overline{b}a \rangle$. Claim is subgroup of $F_{a,b}$.

Picture 1: Graph with two loops labelled a and b.
Picture 2: Graph with 3 loops and each loop labeled as generators of H
Picture 3: Graph each step to foldings where the graph morphism is not injective.

From the pictures we can assure that H is free on $\langle a^3 b, a^2\overline{b}a \rangle$.

Definition 17. The core of a bassed graph (Y, v) is the smallest subgraph that contains v and to which Y deformation retracts.

Proposition 18. *For G a finitely generate subgroup of $F_n = \pi_1(G, v)$ and let $(Y_H, \tilde{v}) \to (V, B)$ corresponding core. The following are equivalent:*

1. *The core of $(Y_H, \tilde{v})$.*
2. *Results of folding algorithm.*
3. *The largest connected finite subgraph of Y_H that contains $\tilde{v}$ and has no valence one vertices except at $\tilde{v}$.*
4. *Union of all reduced edge paths in Y_H that start and stop at $\tilde{v}$.*

Q 1. Given $w \in F_n$, can we algorithmically decide if $w \in H$?

Definition 19. The group $\mathrm{Aut}(F_n)$ is the group of automorphisms of F.

Theorem 20 (Nielsen)**.** *We have that $\mathrm{Aut}(F_n)$ is finitely generated. In fact by*

1. *Permutations: permute basis of elements.*

2. *Sign change: send* $a_i \mapsto a_i^{\pm}$.

3. *Change of maximal tree: for some* i *send* $a_i \mapsto a_i^{\pm}$ *and each* $i \neq j$ *send* $\{a_j, a_i^{\pm} a_j, a_j a_i^{\pm}, a_i^{\pm} a_j a_i^{\pm}\}$.

The topological interpretation of the previous theorem:

We will identify $F_n \simeq \pi_1(G)$ by fixing: a spanning tree, orientation on each $e \in G \setminus T$, bijection between edges on $G \setminus F$. If we change bijection we obtain a type 1 automorphism. If we change orientations we obtain a type 2. Changing tree via $T \to T'$ via edge swap move (add one edge $G \setminus T'$ remove one edge of T from T') we obtain a type 3.

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Let $e \in G \setminus T$, then there exists $T_v e$ has a unique cycle. Let $f \in T$ be a edge in the cycle and let $T' = T \cup e \setminus f$ spanning tree.

Remark 21. Observe that under this identification we have that:

- $e \leftrightarrow f$
- For e_i on the side both trees, via $T-$rule for e_i as a Y. If $f \notin Y$, $e_i \to e_i$. If $f \in Y$, assume f is oriented away from base point: if $f \in \{$ base, left, right$\}$ of Y, then $e_i \mapsto \{f e_i f^{-1}, f e_i, e_i f\}$.

finish the image with two spanning trees changing in the square

Exercise 4. If T and T' are spanning tree of G there is a sequence $T = T_0 \to T_1 \to \cdots \to T_n = T'$, where each $T_i \to T_{i+1}$ is a single edge swap.

Proof of Theorem 20. Let $\alpha : F_n \to F_n$ be any automorphism. Let R be a rose corresponding to the basis $a_1, \cdots, a_n$. Let X be subdivided $n-$pedal rose with pedals labeled by words $\alpha(a_1), \cdots, \alpha(a_n)$. Get an automorphism $\varrho : X \to R$. Notice that the π_1-image under ϱ is the group generated by $\langle \alpha(a_i) \rangle = F_n$.

Factor as a sequence of folds $X = X_1 \to X_2 \to \cdots \to X_h = R$ with the last identification an homeomorphism.

Identify $\pi_1(X) \simeq F_n$ via maximal tree T (implying that $\varrho_* = \alpha$). Analyse folds $X_i \to X_{i+1}$:

Case 1 Fold 2 embedded edges, changing the max tree $T_i \to T_i'$ such that both edges in T_i'. The T_i' induces spanning tree T_{i+1} of X_{i+1} such that fold $X_i \to X_{i+1}$ is $id : F_n \to F_n$.

Case 2 Fold embedded edge over loop, changing $T_i \to T_i'$ such that embedded edge belong to T_i', then we fold $X_i \to X_{i+1}$ such that T_{i+1} of X_{i+1} induces a type 3 automorphism of $\pi_1(X_i) \to \pi_1(X_{i+1})$.

Last map $X_h \to R$ homeomorphism so correspond to a signed permutation. □

0.2 Out(F_n) and Outer space

Definition 22. The group of outer automorphism is defined as

$$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n),$$

where $F_n \in \mathrm{Inn}(F_n)$ is via conjugation.

Exercise 5. The topological interpretation of outer automorphism to think of is that if G is a finite graph, $\pi_1(G, v) \simeq F_n$. Prove that the homotopy equivalences of $G \to G$ quotient homotopy is isomorphic to $\mathrm{Out}(\pi_1(G, v))$.

An anology from the other course, if Σ is a surface, we have that the mapping class group can be represented into $\mathrm{Out}(\pi_1(\Sigma))$. In the case that Σ is closed this representation is an isomorphims. For the case that Σ is not closed, $\pi_1(\Sigma)$ is free and $\mathrm{MCG}^{\pm}(\Sigma) \to \mathrm{Out}(F_n)$ is not an isomorphism but is injective.

We have to mention that $\mathrm{Out}(F_n)$ acts on the sets:

- Conjugacy classes in F_n.
- Conjugacy classes of free factors.

Basic init(...) of $\phi \in \mathrm{Out}(F_n)$ is stretch factor $\lambda(\phi)$ is defined as

$$\log \lambda(\phi) = \sup_{\alpha \in F_n} \limsup_{m \to \infty} \frac{\log \|\phi^m(x)\|}{m}$$

where $\|\cdot\|$ is conjugated to length in F_n with respect to a basis.

An issue is to How calculate λ? A good representative of ϕ is via the core graph Γ (a graph of valence ≥ 2 at each vertex). Fix $n-$pedal rose R_n and $\pi_1(R_n) \simeq F_n$. Marking Γ is a homotopy equivalence $f : R_n \to \Gamma$, a self map $\sigma : \Gamma \to \Gamma$ of marked core graph represents $\phi \in \mathrm{Out}(F_n)$ is $\sigma_* = \phi$.

Definition 23. Let ϕ reducible if some representative leaves a homotopically non-trivial proper subgraph invariant up to homotopy (or equivalently ϕ fixes conjugacy classes of some proper free factor). Otherwise we say that ϕ is irreducible.

We can give a metric to Γ, if we choice a length $\ell(e) > 0$ for each edge e. The volume is equal to sum of edge length. The $\ell(c)$ (where c is a path or a loop in Γ) is equal to the length of geodesic/reduced representative up to homotopy.

A direction at $x \in \Gamma$ is a germ of isometric embeddings $[0, \varepsilon) \to \Gamma$ with $0 \mapsto x$.

A turn at x is a unordered pair (d, d') of distinct directions.

Definition 24 (Illegal Turn Structure)**.** The ITS is an equivalence relation on the set of direction at each point, such that (d, d') is illegal is $d \sim d'$ and otherwise is legal. We will call the equivalence classes under this equivalence as gates.

A path is legal if only takes legal turns.

Definition 25 (Train Track Structure)**.** A TTS is an ITS with ≥ 2 gates at each point.

Definition 26. A train track representative of $\phi \in \mathrm{Out}(F_n)$ is a rep $\sigma : \Gamma \to \Gamma$ with a TTS such that:

1. Edges map to immersed legal paths.

2. legal turns map to legal turns.

Notice that legality is preserved under iterating, in particular if α is a legal loop then $\sigma^h(\alpha)$ is legal and immersed for all h. The previous imply that we can calculate $\lambda(\phi)$. We can use the transition matrix $M(\sigma)$ whit entries

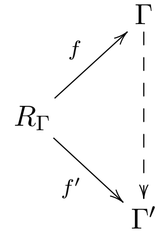
$$m_{ij} = \#\text{ times } \sigma(e_i) \text{ crosses } e_j \text{ in either direction.}$$

We can deduce that λ is the largest egenvalue of transition matrix.

Q 2. How can we find a good representative?

Definition 27. Caller-Vogtmann outer space is defined as $X_n = \{(\Gamma, f, \ell)\} / \sim$ where Γ is a core graph, f is a marking, ℓ is a metric of volume one.

We say that $(\Gamma, f, \ell) \sim (\Gamma', f', \sim')$ if there exists an isometro $\sigma : \Gamma \to \Gamma$ such that the following diagram conmutes via homotopy



X_n decomposes into open simplices.

Example 28. X_2 and take the rose R_2, we can make cells blowing up vertices and changin orientation of loops.

put image of this simplex

Definition 29. Let $\Gamma \in X_n$ and let $\alpha \in F_n$ a conjugacy class. Then $\alpha \in F_n \simeq \pi_1(R_n) \to \pi_1(\Gamma)$. We define $\ell_\Gamma(\alpha)$ as the geodesic length in homotopy class.

Remark 30. We can define a topology on X_n as the one where all $\ell(\alpha) : X_n \to \mathbb{R}_n$ are continuous.

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Definition 31. A difference of markins from Γ to Γ' is a map σ such that:

1. $\sigma f \simeq f'$
2. σ is linear on edges.

Definition 32. Set $L(\sigma)$ is defined as the Lipschitz costant of σ and is equal to maximal edge slope.

Definition 33. The tension graph $\Delta(\sigma) \subset \Gamma$ is the union of edges of maximal slope such that:

1. $\sigma \sim D_x\sigma : \{\text{directions at } x\} \to \{\text{directions at } \sigma(x)\}$.
2. induces a ITS, if we declare $d \sim d'$ is $D\sigma(d) = D\sigma(d')$.

Proposition 34. *We have that*

$$\inf\{L(\sigma) : \sigma : \Gamma \to \Gamma' \ \ d.o.m^1\} = \sup_{\alpha \in F_n} \frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)}$$

and both are realized.

Proof. For any σ continuous map, let $\ell_{\Gamma'}(\alpha) \leq L(\sigma)\ell_\Gamma(\alpha)$ then by Arzela-Ascoli theorem the infimum is realized and let say by $\sigma : \Gamma \to \Gamma'$.

If ITS on $\Delta(\sigma)$ induced by $D\sigma$, this is not a TTS, then some vertex $v \in \Delta(\sigma)$ has only one gate. We can homotope σ in a way to decrease $\Delta(\sigma)$.

So we can assume that $D\sigma$ induces a TTS on $\Delta(\sigma)$ (a core graph). Then for any loop α immersed in $\Delta(\sigma)$. If α is legal then $\sigma(\alpha)$ is immersed and $\ell(\sigma(\alpha)) = L(\sigma)\ell_\Gamma$. □

Remark 35. To achive supremum may assume that α crosses each edge of Γ cut at most twice.

Definition 36. A difference of markings σ is optimal if:

1. Realizes $L(\Gamma, \Gamma') = \inf\{L(\sigma) : \sigma \text{ is a d.o.m.}\}$.

2. $\Delta(\sigma)$ is a core graph with induced TTS.

Definition 37. Lipschitz distance on X_n is $d(\Gamma, \Gamma') = \log(L(\Gamma, \Gamma'))$.

Proposition 38. *The Lipschitz distance is a symmetric metric on X_n.*

He have that $\mathrm{Out}(F_n)$ acts on X_n by changing the marking on

$$[\Gamma, f, \ell]\phi = [\Gamma, f\Phi, \ell].$$

One can convince itself that this action is by isometry with respecto to d.

Definition 39. Let $\phi \in \mathrm{Out}(F_n)$. The translation length $\tau(\phi)$ is defined as $\inf_\Gamma d(\Gamma, \Gamma\phi)$. Have three possibilities:

1. ϕ is elliptic if $\tau(\phi) = 0$ and realized.

2. ϕ is hyperbolic if $\tau(\phi) > 0$ and realized.

3. ϕ is parabolic if $\tau(\phi)$ is not realized.

Proposition 40. *After an arbitrary small perturbation of* Γ, σ, *maintaining* $d(\Gamma, \Gamma\phi) = \log(\lambda)$. *We may assume that*

1. $\sigma(\Delta) = \Delta$.

2. σ *sends edges of* Δ *to legal paths.*

3. σ *sends turns to legal turns.*

Example 41. Consider F_2 and $\phi(a) = ab$ and $\phi(b) = bab$. Take $\Gamma = R_2$. We put a length such that $\ell(a) + \ell(b) = 1$, by the map ϕ we need that $\lambda\ell(a) = \ell(a) + \ell(b)$ and $\lambda\ell(b) = \ell(a) + 2\ell(b)$. We can show that $\lambda = \frac{3+\sqrt{5}}{2}$. And there is only one illegal turn (turn from negative direction of a to positive direction of b).

Lecture 4

0.3 Hyperbolic spaces

Definition 42. A geodesic γ in a metric space X is $D-$strongly contracting if $d(y,y') \leq d(y,\gamma)$ then $\mathrm{Diam}(\pi_\gamma(y),\pi_\gamma(y')) \leq D$.

Definition 43. A geodesic γ is Morse if for all $k \geq 1, C \geq 0$ there exists $N = N(k,c)$ such that for any $(k,C)-$quasi-geodesic $\rho : [a,b] \to X4$ with $\rho(a), \rho(b) \in \gamma$ and

$$d_{\text{Haus}}(\gamma|_{[a,b]}, \rho) \leq N.$$

Call $N : [1,\infty) \times [1,\infty) \to \mathbb{R}$ a Morse gauge

Remark 44. In a $\delta-$hyperbolic space there exists a Morse gauge N such that every geodesic is $N-$Morse.

Exercise 6. For all D, there exists a Morse gauge N such that every $D-$strongly contracting geodesic is $N-$Morse.

Theorem 45. *For X a geodesic metric space. The following are equivalent:*

1. *There exists δ such that X is $\delta-$hyperbolic.*
2. *There exists D such that every geodesic in X is $D-$strongly contracting.*
3. *There exists a Morse gauge N such that every geodesic is $N-$Morse.*

Ideas of the Proof:

1. (1) implies (2): Notice that if X is $\delta-$hyperbolic, we have that all quadrilaterals are $2\delta-$thin. Let say that γ is a geodesic such that $\mathrm{Diam}(\pi_\gamma(y),\pi_\gamma(y')) \geq 10\delta$. Then we can assure that $d(y,y') \geq d(y,\gamma) - 2\delta$ and by contrapositive we have the claim.
2. (2) implies (3):
3. (3) implies (1):

   Consider the following lemma

   Lemma 46. *Let X be any geodesic space, γ a geodesic, $y \in X$ and $z \in \pi_\gamma(y)$. The concatenation path $[y,z] \cup \gamma$ is a $(3,0)-$quasi-geodesic.*

   This lemma implies that the triangles are thin.

□

Remark 47. Strongly contracting geodesic behave like geodesics in hyperbolic spaces. Even if the space is not $\delta-$hyperbolic it may have some strongly contracting geodesics.

Example 48. Let $G = \langle a, b, c : ab = bc \rangle$. We have that G is not hyperbolic.

Definition 49. A map $\gamma : I \to X$ is a $L-$local geodesic if the restriction to each length $\leq L$ subinterval of I is geodesic.

Lemma 50. *In a hyperbolic space, every local geodesic is a quasi-geodesic.*

Proof. Given δ, let $N = N(2, 0)$ be a Morse constant for $(2, 0)-$quasi-geodesics and take $\frac{L}{4} > 2N + 2\delta$. We claim that every $L-$local geodesic $\gamma : I \to X$ is a $(2, 0)-$quasi-geodesic.

Nts (?)

$$(t - s) \geq d(\gamma(s), \gamma(t)) \geq \frac{(t - s)}{2}$$

fot $s < t$.

Induct on $(t - s)$:

Suppose it holds for all $t - s \leq R$. Show it holds for $R < t - s \leq R + L$. We can find some w with $d(w, z) \leq \delta$ for some $z \in [x, y]$ and such that:

$$d(x, y) \geq d(x, w) + d(w, y) - 2\delta$$

□

Remark 51. Let $G = \langle S \rangle$ is an hyperbolic group such that there exists L such that $s_1 \cdots s_n = 1$ with $s_i \in S$ then some subword of length $\leq L$ is NOT a geodesic.

Definition 52 (Dehn's algorithm for word problem)**.** Look for subwords of length $\leq L$ that are not geodesic, and decide: if you find one, reduce and repeat, if none exists the word is not the identity.

Remark 53. The previous paragraph imply that the Hyperbolic groups are finitely generated.

Remark 54. Hyperbolic groups have linear isoperimetric inequality area$(w) \leq$ lenght(w).

Theorem 55 (Gromov). *If a finitely presented group has a subquadratic isoperimetric inequality then the group is hyperbolic.*

0.4 Extensions

Q 3. How can we build new hyperbolic spaces out of old ones?

Theorem 56 (Bestvina-Feighn Combination Theorem)**.** *Suppose X is a finite graph of spaces such that:*

1. *Universal covr $\tilde{X}_e$ and $\tilde{X}_v$ of edge and vertex spaces are $\delta-$hyperbolic.*

2. *Gluings $\tilde{X}_e \to \tilde{X}_v$ are quasi-isometric embeddings.*

3. *The "flaring condition"[2] is satisfied.*

Then the universal cover $\tilde{X}$ is a Gromov hyperbolic.

Remark 57. This theorem can be apply to graphs of hyperbolic groups.

The proof uses flaring to conclude that it is satisfied the subquadratic isoperimetric inequality and by Gromov it follows.

Definition 58. For K, Q groups, an extension of K by Q is any group fitting into a short exact sequence $1 \to K \to G \to Q \to 1$.

Q 4. If K and Q are hyperbolic, when G is hyperbolic?

It depends! For example is we take the stupid extension $G = K \times Q$ is never hyperbolic if $|K|, |Q| = \infty$.

For example, the "monodromy" $\rho : Q \to \mathrm{Out}(K)$. For G to be hyperbolic, ρ must be "complicated", i.e., if there exists $q \in Q, k \in K$ of infinite orders such that $\rho(q)$ fixes conjugacy classes of k we get $\mathbb{Z} \oplus \mathbb{Z} \leq G$.

Lecture 5

0.5 Hyperbolic Extensions

Let S be a finite generating set of G, that contains the generating set of K. We have the word length $|\cdot|$ on G and the conjugacy length $\|\cdot\|$ on K.

There is a "bungle map" of Cayley graphs as

$$\begin{array}{ccc} Cay(K) & \longrightarrow & Cay(G) \\ & & \downarrow \\ & & Cay(Q) \end{array}$$

[2]The lecturer doesn't provide a definition but in the later will talk about it.

We have $Cay(G)$ have vertical and horizontal edges.

The monodromy $Q \to \mathrm{Out}(K)$. Also, we have a Q action on the set of conjugacy classes of K.

Theorem 59 (Mj-Sardar Combination Theorem)**.** *The group Q is hyperbolic provided flaring condition holds, if there exists $\lambda > 1$, $M \in \mathbb{N}$ such that for any non-trivial $\alpha \in K$ and geodesics $g_{-n}, \cdots, g_0, \cdots, g_n$ in G with $n \geq M$*

$$\lambda \|g_0^{-1}\alpha\| \leq \max\{\|g_{-n}^{-1}\alpha\|, \|g_n^{-1}\alpha\|\}.$$

Remark 60. The JSJ theory of hyperbolic groups implies that if K is torsion free, G is hyperbolic, must have K is a free product of surface or free groups.

Example 61. Historically the first example of extension is the following. Let Σ a closed surface and $f : \Sigma \to \Sigma$ homeomorphism, consider the mapping torus of (Σ, f) then we have

$$1 \to \pi_1(\Sigma) \to \pi_1(M_f) \to \mathbb{Z} \to 1.$$

Remember by Thurston's theorem, if f is a pseudo-Anosov then M_f admits a hyperbolic Riemannian metric. Therefore $\pi_1(M_f)$ is an hyperbolic group. Even more, from the previous we have that if $\pi_1(M_f)$ is hyperbolic then f is a pseudo-Anosov.

Example 62. Let $\phi \in \mathrm{Out}(F_n)$ and consider $F_n \rtimes_\phi \mathbb{Z}$ defined as

$$\langle F_n, t : t^{-1}wt = \phi(w), w \in F_n \rangle.$$

We have the short exact sequence

$$1 \to F_n \to F_n \rtimes_\phi \mathbb{Z} \to \mathbb{Z} \to 1$$

and by Brinkmann say that $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if ϕ is atoroidal.[3]

Remark 63. Some work of Farb-Masher, Kent-Leninger, Hamerstadt about theory of hyperbolic extensions of surfaces group is related to properties of convex-cocompact subgroups of $\mathrm{MCG}(\Sigma)$.

[3]No power ϕ^k fixes conjugacy classes.

0.6 Free Group Extensions

Folding paths: Let $\Gamma, \Gamma' \in X_n$ with optimal map $\sigma : \Gamma \to \Gamma'$ with TTS on tension graph $\Delta \subset \Gamma$.

Assume Δ is all of Γ (If not, move Γ to Γ_0 by stretching metric on Δ and shrinking other edges to maintain volume is 1, until all edges in Δ).

For $0 \leq t < \epsilon$ small, deform Γ to Γ_t by folding all illegal turns for length t. σ descends to $\sigma_t : \Gamma_t \to \Gamma'$ with TTS.

Theorem 64. *This process yields a path $\{\Gamma_t\}_{t\in[0,1)}$ from Γ to Γ' may parametrize such that a directed geodesic in X_n*

$$d(\Gamma_s, \Gamma_t) = t - s$$

for $s < t$. This path comes with a TTS on Γ_t for all t.

Remark 65. Train track representative of $\phi \in \mathrm{Out}(F_n)$, gives a bi-infinite folding axis for ϕ acting on X_n.

Definition 66 (Free Factor Complex)**.** Simplicial simplex graph $\mathscr{F}$ with:

- Vertices: conjugacy classes of cyclic free factor.
- Edges: $[\alpha] - [\beta]$ if α and β may be jointly part of a basis.

Definition 67 (Lipschitz Projection)**.** Let $\pi : X_n \to \mathscr{F}$ given by Γ maps to shortest conjugacy class of Γ.

Theorem 68 (Bestvina-Feighn)**.** *$\mathscr{F}$ is $\delta-$hyperbolic.*

Sketch of proof: Use folding paths $\gamma : \{\Gamma_t\}_{t\in[0,L]}$. For $\alpha \in \mathscr{F}$ look at the cre of subgroup $\langle\alpha\rangle$ over Γ_t with pull-back TTS.

Fact: Any legal subpath with length ≥ 3 grows exponentially.

As you fold, illegal turns can disappear or illegal turns can collide and combine. Therefore the number of illegal turns can only decrease.

Definition 69.

$$\begin{aligned} left_\gamma(\alpha) &= \inf\{t : \alpha|\Gamma_t \text{ has legal segment of length} \geq 3\}. \\ right_\gamma(\alpha) &= \sup\{t : \alpha|\Gamma_t \text{ has a long segment without llegal subsegment of length} \geq 3\}. \end{aligned}$$

Proposition 70. *$\pi\left(\gamma([left_\gamma(\alpha), right_\gamma(\alpha)]\right) \subset \mathscr{F}$ has uniform bounded diameter.*

The family $\{\pi(\gamma) : \gamma \text{ is a folding path}\}$ of paths in $\mathscr{F}$ with projections $p : \mathscr{F} \to \pi(\gamma)$ given by $\alpha \mapsto \pi(\gamma(left_\gamma(\alpha))$. The key is that these projections are uniformly strongly contracting and therefore $\mathscr{F}$ is hyperbolic. □

For $\gamma \subset X_n$ folding path, define the map $\mathscr{L}_\gamma : X_n \to \gamma$ given by $H \mapsto left_\gamma(\pi(H))$.

Theorem 71 (Bestvina-Feighn)**.** *Given (K, C), there exists D such that if $\gamma : I \to X_n$ is a folding path such that $\pi\gamma : I \to \mathscr{F}$ is a $(K, C)-$quasi-geodesic. Then γ is $D-$strongly contracting in X_n with respect to $\mathscr{L}_\gamma$.*

Theorem 72 (D.-Taylor)**.** *Given (K, C) there exists D such that if $\gamma : I \to X_n$ is any $(K, C)-$quasi-geodesic such that $\pi\gamma : I \to \mathscr{F}$ is also a $(K, C)-$quasi-geodesic then the closest point projection $X_n \to \gamma$ is $D-$strongly contracting.*

Definition 73. Say $1 \to F_n \to E \to G \to 1$ an extension:

1. is convex cocompact if the orbit map $G \to \mathrm{Out}(F_n) \curvearrowright \mathscr{F}$ is a quasi-isometric embedding.
2. is properly atoroidal if no infinite order element of G fixes any non-trivial conjugacy class.

Theorem 74 (D-Taylor)**.** *If $1 \to F_n \to E \to G \to 1$ is convex cocompact and purely atoroidal the E is hyperbolic.*