The Grigorchuk and Grigorchuk-Machi Groups of Intermediate Growth

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Vanderbilt University

July 20, 2019

References:

1. de la Harpe, *Topics in Geometric Group Theory*, Chapter VIII.
2. Grigorchuk, Machi, "A group of intermediate growth acting by homomorphisms on the real line."

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The Free Monoid

Definition

Let A be a set. Then the *free monoid* on A, denoted A^*, is the set of all words over the alphabet A. More formally, the set of all finite sequences of elements of A.

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The *word length* of an element $g \in G$ with respect to S is

$$|g|_S = \min\{|w| \mid w =_G g\}.$$

Growth of Groups

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Let G be a group generated by a finite set S. The *growth* of G with respect to S is the function $\gamma_{G,S} : \mathbb{N} \to \mathbb{N}$ given by

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Lemma (VIII.61,63)

If $\gamma_G \sim \gamma_G^2$, then there exists an $\alpha \in (0,1)$ such that $e^{n^\alpha} \preceq \gamma_G$.

The Infinite Rooted Binary Tree

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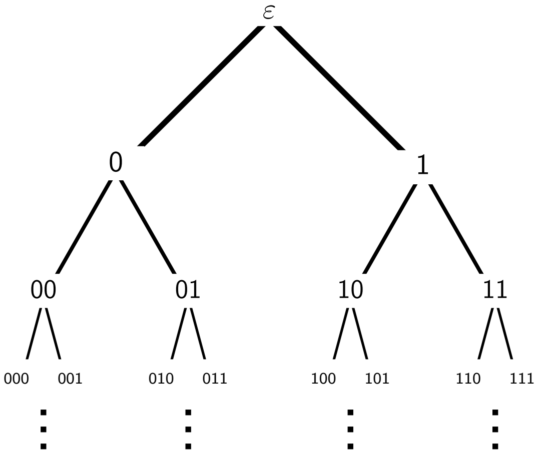
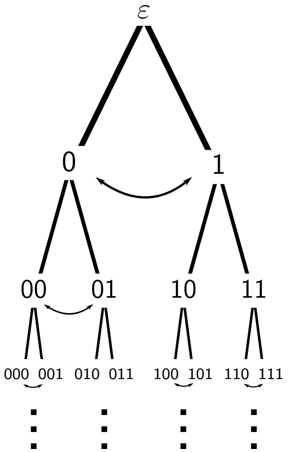


Automorphisms of T



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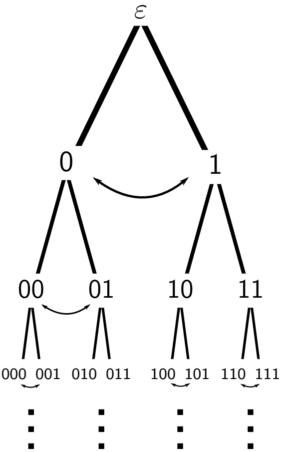

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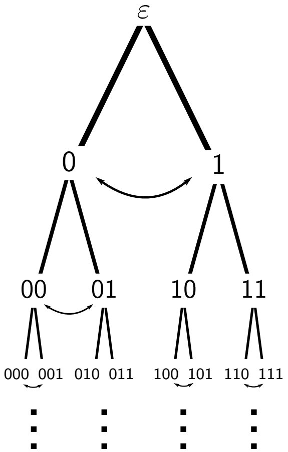

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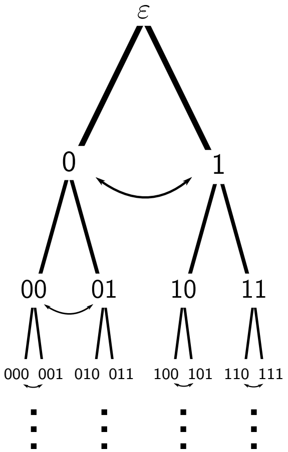

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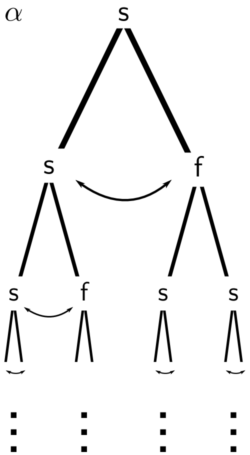

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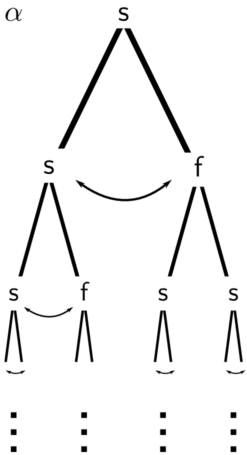

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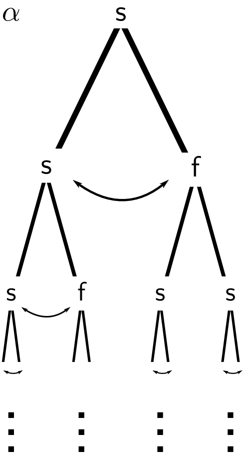

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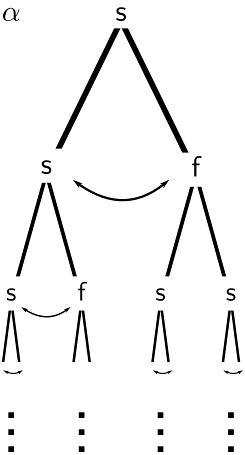

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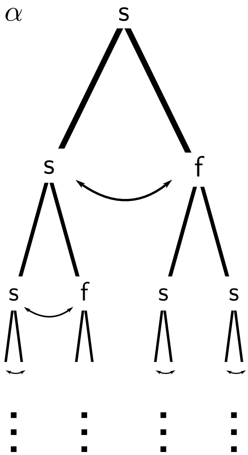

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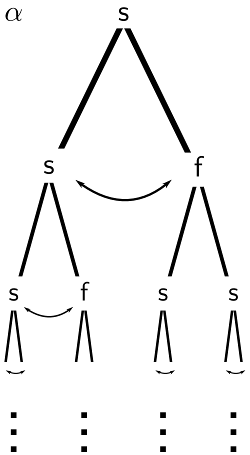

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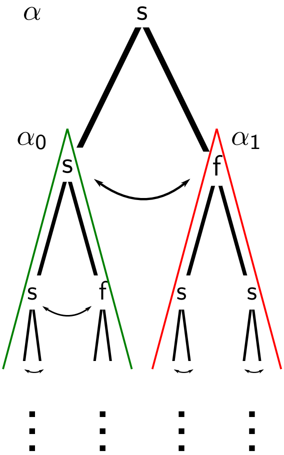
Therefore $\mathrm{Aut}(T)$ is residually finite.

Self-similarity of Aut(T)



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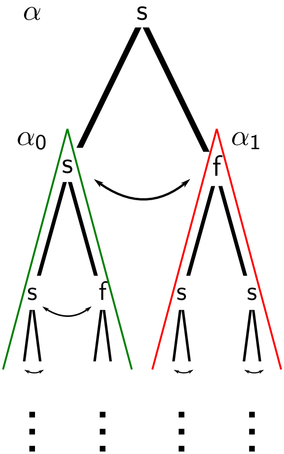

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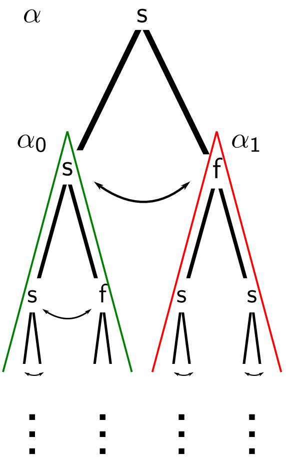

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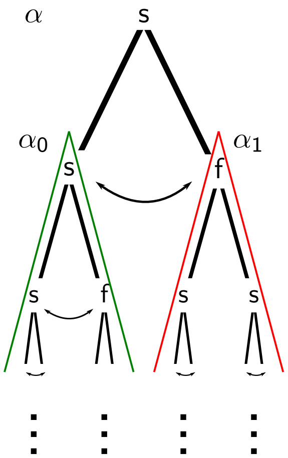

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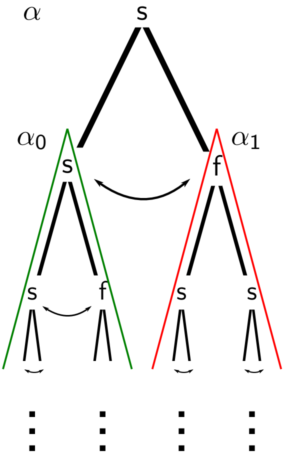

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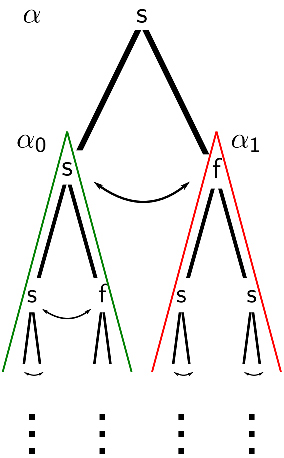

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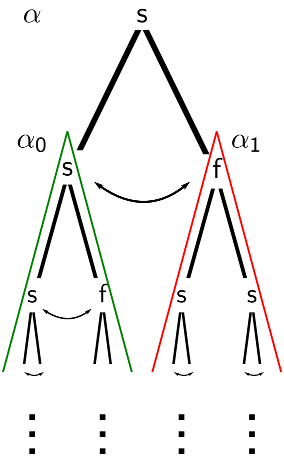

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But $\mathrm{Aut}(T)$ is uncountable!

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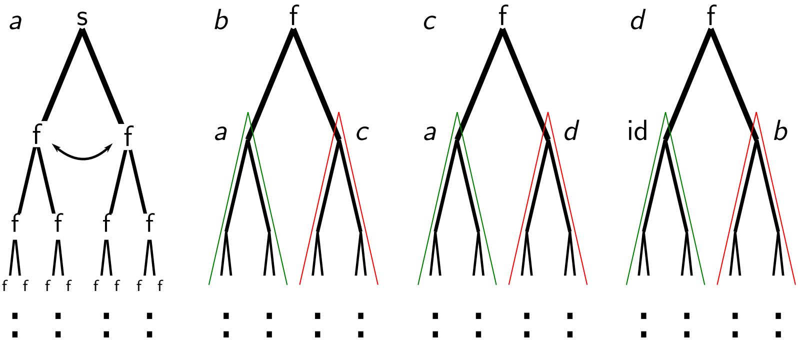

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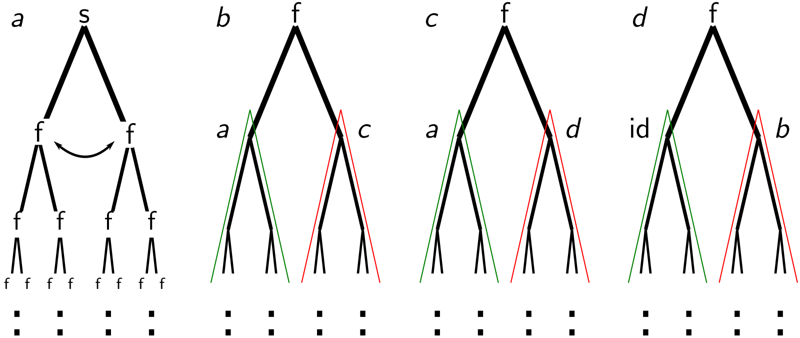
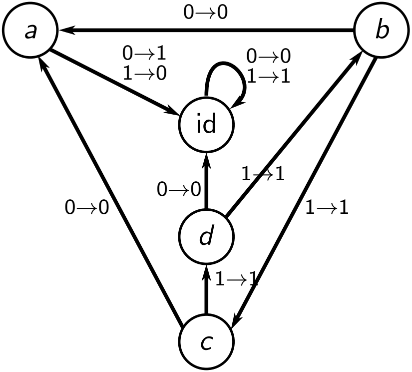


Now let $\Gamma = \langle a, b, c, d \rangle$.

Γ via a Deterministic Finite Automaton



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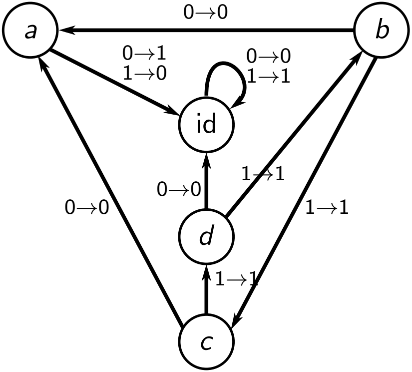

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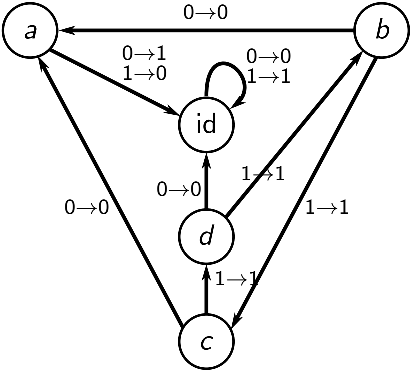

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$$\begin{aligned} d(110101) &= 1b(10101) \\ &= 11c(0101) \\ &= 110a(101) \\ &= 1100\mathrm{id}(01) \\ &= 11000\mathrm{id}(1) \\ &= 110001\mathrm{id}(\varepsilon) \\ &= 110001 \end{aligned}$$

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By recursive definition, $\psi_1 : \mathrm{St}_{1,\Gamma} \to \Gamma^2$. Let $\psi_1 = (\varphi_0, \varphi_1)$. Can check that

$$\varphi_0 = \begin{cases} b & \mapsto a \\ c & \mapsto a \\ d & \mapsto 1 \\ b^a & \mapsto c \\ c^a & \mapsto d \\ d^a & \mapsto b \end{cases} \qquad \varphi_1 = \begin{cases} b & \mapsto c \\ c & \mapsto d \\ d & \mapsto b \\ b^a & \mapsto a \\ c^a & \mapsto a \\ d^a & \mapsto 1 \end{cases}$$

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In particular, $\varphi_0 : \mathsf{St}_1 \to \Gamma$ is surjective. Therefore Γ is infinite.

Recall $\psi_1|_{\mathsf{St}_1}$ is injective.

Lemma (VIII.28)

$[\psi_1(\mathsf{St}_{1,\Gamma}) : \Gamma^2] = 8.$

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Lemma (VIII.62)

The contracting property implies that there exists a $\beta \in (0,1)$ *such that* $\gamma_\Gamma \preceq e^{n^\beta}$.

Therefore for some $0 < \alpha < \beta < 1$,

$$e^{n^\alpha} \preceq \gamma_\Gamma \preceq e^{n^\beta}$$

so Γ is a group of intermediate growth.

Other Properties of Γ

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Theorem (Bartholdi '98, $\preceq_2$| Erschler, Zheng '18, $\preceq_1$)

Let λ *be the positive root of* $x^3 - x^2 - 2x - 4$, $\lambda \approx .7674\ldots$
For all $\varepsilon > 0$,

$$e^{n^{\lambda-\varepsilon}} \preceq_1 \gamma_\Gamma \preceq_2 e^{n^\lambda}.$$

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  - Γ is a 2-group.
  - Γ is not orderable.
  - Γ is not bounded torsion ($\Leftarrow$ Zelmanov: bounded torsion + rf $\Rightarrow$ finite).
- Γ is amenable ($\Leftarrow$ subexp growth).

The Grigorchuk-Machi Group

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Theorem

If X is a countable set with a total order $\leq_X$ such that $\leq_X$ is dense and contains no first or last element, then $(X, \leq)$ is order-isomorphic to $(\mathbb{Q}, \leq)$.

Proof.

Exercise. □

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Let $\tilde{\Gamma} = \langle a, b, c, d \mid [a^2, b], [a^2, c], [a^2, d], [b, c], [b, d], [c, d] \rangle$.

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Theorem (Grigorchuk '84)

The Grigorchuk-Machi group is of intermediate growth.

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Corollary

Every left order on Γ is Conradian (and one exists).

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Then prove that the order is dense and left-invariant.

Thank You!

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If you are from the U.S., happy Independence Day!
Otherwise, happy 4th of July!