AMENABILITY

Let Γ be a discrete countable group.

Exercise 1. *Show that* $(\gamma f)(x) := f(\gamma^{-1}x)$ *actually defines an action of* Γ *on* $\ell^p(\Gamma)$ *and that it preserves the norm* $\|f\|_p^p = \sum_{x\in\Gamma} |f(x)|^p$ *and hence is an isometric action for the associated metric.*

Exercise 2. *Let* X *be a discrete countable set with an action of* Γ *by permutations. Show that for a suitably chosen* S *and suitably chosen subgroups* H_s, $s \in S$ *we have that*

$$\ell^2(X) \cong \bigoplus_{s\in S} \ell^2(\Gamma/H_s)$$

Exercise 3. *Continuing with* X *as above, prove that* $\ell^2(X)$ *has a nonzero* Γ*-invariant vector if and only if the index* $[\Gamma : H_s] < \infty$ *for some* $s \in S$.

Exercise 4. *Assume that* Γ *is amenable, and* $H \leqslant \Gamma$. *Prove that* H *is amenable. Warning!! This is only true for* Γ *a discrete countable group. If we pass to the world of locally compact second countable groups, we must assume* H *is closed and then the result holds again. We will see how this fails when we study the Banach-Tarski Paradox.*

Exercise 5. *Let* $1 \to N \to \Gamma \to Q \to 1$ *be a short exact sequence. Prove that* Γ *is amenable if and only if* N *and* Q *are both amenable.*

Exercise 6. *Let* $\{\Gamma_i\}_{i\in I}$ *be a directed family of groups, and* $\Gamma = \bigcup_{i\in I} \Gamma_i$ *be the direct limit. Prove that* Γ *is amenable if and only if* Γ_i *is amenable for every* $i \in I$.

Exercise 7. *Prove that the following classes of groups are amenable:*

- *Finite groups*
- $\mathbb{Z}$
- *finitely generated abelian groups*
- *abelian groups*
- *solvable groups*

Exercise 8. *Prove that groups of subexponential grown are amenable.*

Exercise 9. *Let* Γ *act on the discrete countable set* X. *If there is a finitely additive measure* μ *on* X *with* $\mu(X) = 1$ *and* μ *is* Γ*-invariant then* X *is not* Γ*-paradoxical.*

Exercise 10. *Prove that* F_2 *is not amenable. You get to pick which definition you use.*

Exercise 11. *When we defined amenability, many supposedly equivalent conditions were given. How many directions can you prove? One place to start is to prove the Følner condition implies the other ones.*

Exercise 12. *Let* $\{a_n : n \in \mathbb{N}\}$ *be a sequence of real numbers that is subadditive, i.e. for every* $n, m \in \mathbb{N}$ *we have that* $a_{n+m} \leqslant a_n + a_m$. *Prove that*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}.$$

Exercise 13. *Let* $\{A_n : n \in \mathbb{N}\}$ *be a sequence of positive real numbers that is submultiplicative, i.e. for every* $n, m \in \mathbb{N}$ *we have that* $A_{n+m} \leqslant A_n \cdot A_m$. *Prove that*

$$\lim_{n\to\infty} A_n^{1/n} = \inf_{n\in\mathbb{N}} A_n 1/n.$$

Relative Property (T) and Semidirect Products

Exercise 14. *Assume that $N \trianglelefteqslant G$ and let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G. Prove that the subspace $\mathcal{H}_N := \{v \in H : \pi(n)v = v \text{ for all } n \in N\}$ is G-invariant and hence so is*

$$\mathcal{H}_N^{\perp} := \{v \in \mathcal{H} :< v, v' >= 0 \text{ for all } v' \in \mathcal{H}_N\}.$$

Set $\pi^{\perp} : G \to \mathcal{U}(\mathcal{H}_N^{\perp})$ to be the restricted representation. Conclude that if (G, N) has relative property (T) then, $\pi^{\perp}$ does not have G-almost invariant vectors.

Exercise 15. *Prove that if Γ is discrete countable amenable group and $H \leqslant \Gamma$ is an infinite subgroup then (Γ, H) does not have property (T).*

Exercise 16. *(Requires knowledge of Haar measure and induced representations.) Assume G is a locally compact second countable amenable group and $H \leqslant \Gamma$. Prove that if (G, H) has relative property (T) then H is compact.* Warning! H may have measure 0 in G.

Exercise 17. *Let $1 \to A_0 \to A \to A_1 \to 1$ be a short exact sequence. Let $\Gamma \to Aut(A, A_0)$ be an action by automorphism of A preserving A_0. Prove that $(\Gamma \ltimes A_0, A_0)$ and $(\Gamma \ltimes A_1, A_1)$ both have relative property (T) if and only if $(\Gamma \ltimes A, A)$ has relative property (T).*

Exercise 18. *Assume that $H \leqslant G$ is a closed subgroup with property (T). Prove that (G, H) has relative property (T).*

Exercise 19. *Let p be a prime. Recall that the p-adics $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ in the p-norm. If $x \in \mathbb{Q}_p$ then there is a $v \in \mathbb{Z}$ and integers $a_k \in \{0, \dots, p-1\}$ such that:*

$$x = \sum_{k=-v}^{\infty} a_k p^k.$$

It follows that $\mathbb{Z}[\frac{1}{p}] := \left\{\frac{a}{p^k} : a, k \in \mathbb{Z}\right\}$ diagonally embeds in $\mathbb{R} \times \mathbb{Q}_p$. Prove that this makes $\mathbb{Z}[\frac{1}{p}]$ a discrete subgroup with pre-compact fundamental domain.

Exercise 20. *Using the Ping-Pong Lemma prove that the following two elements generate a free group:*

$$a := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } b := \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Hint: Consider the action on $\mathbb{R}^2$ let A (or B?) be $\{(x, y) : |y| > |x|\}$ and the other set (B or A) be $\{(x, y) : |y| < |x|\}$. Now, draw a picture of what is happening on $\mathbb{P}(\mathbb{R}^2) \cong S^1$. How many fixed points does a and b have on S^1?

Bounded Generation, Affine Actions on Hilbert Space, Haagerup Property

Exercise 21. *Read the paper by Carter-Keller that $\mathrm{SL}_N\mathcal{O}$ is boundedly generated by elementary matrices provided that $N \geqslant 3$ (or $N = 2$ and $\mathcal{O}$ has infinitely many units). (You may read through this assuming that $\mathcal{O} = \mathbb{Z}$, and of course $N \geqslant 3$).*

Exercise 22. *Let Γ be a discrete countable group and $\mathcal{E}_1, \dots, \mathcal{E}_M \leqslant \Gamma$ such that $\Gamma = \mathcal{E}_1 \cdots \mathcal{E}_M = \Gamma$, i.e. for every $\gamma \in \Gamma$ there exists $\gamma_i \in \mathcal{E}_i$ such that $\gamma = \gamma_1 \cdots \gamma_M$. Let $\varphi : \Gamma \to \mathrm{Isom}(X)$ be an isometric action of Γ on the metric space X. Prove that if every $\mathcal{E}_i$ orbit is bounded (for every $i = 1, \dots, M$) then there is a bounded Γ orbit (and hence all Γ-orbits are bounded).*

Exercise 23. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation and $b : \Gamma \to \mathcal{H}$ a set theoretic map. Prove that $\alpha_b(\gamma) : v \mapsto \pi(\gamma) + b(\gamma)$ is an action if and only if $b \in Z^1(\Gamma, \pi)$, i.e. b satisfies the 1-cocycle relation with respect to π:*

$$b(\gamma_1\gamma_2) = b(\gamma_1) + \pi(\gamma_1)b(\gamma_2).$$

Exercise 24. *Let Γ act by permutations on a set X and let $f : X \to \mathbb{C}$ be a function. As usual define $(\gamma.f)(x) = f(\gamma^{-1}x)$. Prove that $b(\gamma) := \gamma.f - f$ satisfies the cocycle relation above.*

Exercise 25. *Recall that $B^1(\Gamma, \pi) := \{b : \Gamma \to \mathcal{H} |\ \exists v \in \mathcal{H}$ s.t. $b(\gamma) = \pi(\gamma)v - v\ \forall \gamma \in \Gamma\}$. Prove that $B^1(\Gamma, \pi) \subset Z^1(\Gamma, \pi)$.*

Exercise 26. *Prove that the following are equivalent for a cocycle $b \in Z^1(\Gamma, \pi)$:*

(1) *$b \in B^1(\Gamma, \pi)$;*
(2) *α_b as defined above has bounded orbits;*
(3) *b is bounded*
(4) *α_b is conjugate via a translation to π.*

*This requires the "*Lemma of the Center*" which can be stated for metric spaces but holds true for certain spaces such as Hilbert spaces: Given a nonempty bounded subset, there exists a unique closed ball of minimal radius containing the subset.*

Exercise 27. *Let T be a simplicial countable tree with vertex set V and oriented edge set $\mathcal{E} \subset V \times V$. Let $o(e), t(e) \in V$ denote the origin and terminal vertices of e respectively (so $e = (o(e), t(e)) \in V \times V$). Denote the reverse orientation on e by $\overline{e} := (t(e), o(e))$. Let $h_e = \{v \in V : d(v, t(e)) < d(v, o(e))\}$. Then $h_e \sqcup h_{\overline{e}} = V$. Let $v \in V$ and consider the function $\mathbb{1}_v : \mathcal{E} \to \{0, 1\}$ defined by $\mathbb{1}_v(e) = \begin{cases} 1 & \text{if } v \in h_e \\ 0 & \text{otherwise} \end{cases}$.*

For Γ acting simplicially (and hence by isometries on the tree T, let $b(\gamma) := \mathbb{1}_{\gamma v} - \mathbb{1}_v$. As shown above, b formally satisfies the cocycle relation. Prove that $\|b(\gamma)\|^2 = \sum_{e \in \mathcal{E}} |b(\gamma)(e)|^2 = 2d(\gamma v, v)$ and conclude that $b \in Z^1(\Gamma, \ell^2(\mathcal{E}))$. Conclude that if Γ acts properly on T then Γ has the Haagerup property. Similarly, if Γ has property (T) then it has a bounded orbit in T (and hence a fixed vertex of preserved edge.

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