Orders on groups — Andrés Navas

G : group (countable)
$(G, \preceq)$ total order
$\preceq$ is left invariant if $f \preceq g \Rightarrow hf \preceq hg$
bi-invariant & $fh \preceq gh$.

Examples
① $\mathbb{Z}$, with two orders : natural and reversal
$\mathbb{Z}^2$ lexicographic order
$\mathbb{Z}^{\aleph_0}$, $\mathbb{Z}^{\aleph_1}$, ...

② Exercise ① F_2 is left orderable. (actually biorderable)
② $(G, \leq)$ LO $f > g \not\Rightarrow f^{-1} < g^{-1}$
(give an example of this)
$f > g \Leftrightarrow f^{-1}f > f^{-1}g \Leftrightarrow 1 > f^{-1}g \quad \Leftrightarrow \quad g^{-1} > f^{-1}$

③ Examples : π_1(Surface) BO S oriented.
π_1(Surface) LO S not oriented $S \neq \mathbb{R}P^2$
$MCG(S, \partial)$ LO What if S is NOT orientable ?
Braid groups LO
Thompson's groups LO

Remark G is LO $\Rightarrow$ G has no torsion.
either $f < id \Rightarrow id = f^n < f < id$ # RIDICULOUS
$f > id \Rightarrow id = f^n > f > id$

Exercise. Give an example of a group, torsion free which is NOT LO.

Exercise Give a LO group $(G, \leq)$ and $f > id \ni gfg^{-1} < id$

Dedekind — Hölder Thm (Hölder) G admits an archimedean order
analytical combinatorial if and only if $G \underset{\text{order emb.}}{\hookrightarrow} (\mathbb{R}, +)$

Holder thm was for BO and proved for LO by Conrad (50's)
Def A LO is CONRAD — conradian — if
$\forall f > id, \forall g > id \ \exists m \ni fg^n > g$ # if $\exists n \ni$ ⊛ $\Rightarrow \exists m > 0 \ni$ ⊛
Exercise If this happens $\forall f > id$; $g > id$; $fg^2 > g$
crucial remark noticed 10 years ago very important

Theorem. (Conrad–Brodsky) G is conrad orderable $\iff$ G is locally indicable

l.i.: $\forall\, G_0 < G$ f.g $\exists\, \phi_0 : G_0 \to (\mathbb{R},+)$ group homomorphism, non trivial

Theorem (Dave, Morris Witte)

Let G be an amenable group. G is LO $\iff$ G is Locally Indicable.

#Exercise# Give an example of a LO which is not locally indicable.

=open question till Thurston=

Theorem 2018. (Hyde–Lodha; Matte Bon–Triestino)

$\exists\, G$ finitely generated LO and simple.

The dynamical approach

Andres Navas = The dynamical approach =

G is LO $\iff$ $G \curvearrowright (\Omega, \leq_\Omega)$ faithfully in a totally ordered space by order preserving action.

($\Rightarrow$) Cayley ($\Leftarrow$) $\Omega = \{w_1, w_2, w_3, \dots\}$

$f > g$ if $fw_1 >_\Omega g w_1$

or $fw_1 = gw_1 \wedge fw_2 >_\Omega gw_2$.

or $fw_1 = gw_1 \wedge fw_2 = gw_2 \wedge fw_3 > gw_3$

...

this order preserves the product thus it is LO.

Thm (Folklore: no body knows who proved it first)

If G is countable and LO then

$$G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$$

Conversely $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R}) \Rightarrow G$ LO.

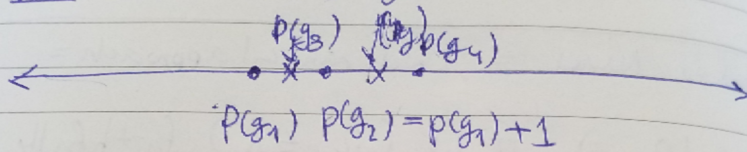
Q Assume that $\leq$ is bi-invariant

$\mathbb{R} \to \mathbb{R}$

Exercise $\quad x \mapsto x+1$, $\quad x \mapsto x^{3}$ } generate a free group

maybe it works for p any prime #.

proof $\quad G = \{g_1, g_2, g_3, \dots$



$\Rightarrow \quad g_1 < g_3 < g_2 < g_4$

G acts on $p(G)$ $\quad (g(p\, g_i)) = p(g g_i)$

and this extends continuously. Commutator

Question 1 $\quad$ Assume that $\leq$ is bi-invariant

what kind of action you get?

$\boxed{\text{Space of orders}}$

$LO(G)$ = left order on G

$BO(G)$ = bi-order on G

$CO(G)$ = Circular order on G

Topology on a space of orders:

$(G, \leq)$ LO assume $f_1 > g_1, \dots, f_n > g_n$

$\Rightarrow N_{\leq}((f_1, g_1), (f_2, g_2), \dots (f_n, g_n))$

$= \{ \leq' \mid f_1 >' g_1, f_2 >' g_2, \dots f_n >' g_n \}$

todos los ordenes que preservan las desigualdades planteadas

Exercise $LO(G)$ is totally disconnected and COMPACT.

(use Tychonov's theorem)

If G is finitely generated $G = \langle g_1, \dots g_n \rangle$

$B(n) = \{ g : g = g_{i_1}^{\pm 1} \dots g_{i_m}^{\pm 1} ; m \leq n \}$

$\text{dist}(\leq, \leq') = \frac{1}{n}$ if

~~dist~~ n is the smallest radius s.t. $\leq$ and $\leq'$ do not coincide on $B(n)$

Theorem (D. Witte-Morris)

$$\begin{tikzcd}
G \text{ is LO and amenable} \arrow[r, Rightarrow] \arrow[dr, dashed] & G \text{ is locally indicable.} \arrow[d, leftrightarrow] \\
& G \text{ is Conrad orderable.}
\end{tikzcd}$$

$G \curvearrowright LO(G)$ by conjugacy

given $\leq$ and g, $(\leq, g) \mapsto \leq_g$

$$f_1 \leq_g f_2 \iff g f g^{-1} \leq g f_2 g^{-1}$$

We will see a property that is ONLY local

Let G f.g. + amenable + LO

$LO(G)$ metric compact space.

G amenable $\Rightarrow \exists\, \mu \in \mathrm{Prob}(LO(G))$ invariant

Key Lemma (consequence of Poincaré recurrence Th)

$$\mu\left(\begin{matrix}\text{Conrad} \\ \text{orders}\end{matrix}\right) = 1$$

Segunda sesión = positive & negative cones.

Prop G is LO $\iff G = P \sqcup P^{-1} \sqcup \{id\}$
where P semigroup

($\Rightarrow$) $\leq$: LO on G $\quad P_{\leq} = \{g \in G \mid g > 1\}$ positive cone of $\leq$

$g,h \in P_{\leq} \Rightarrow \quad h > id \Rightarrow gh > g > id \quad \Rightarrow gh \in P_{\leq}$

($\Leftarrow$) $G = \underline{P} \sqcup \underline{P}^{-1} \sqcup \{id\}$

$\leq$ on G : $\quad g > h \underset{\text{def}}{\iff} h^{-1}g \in P$

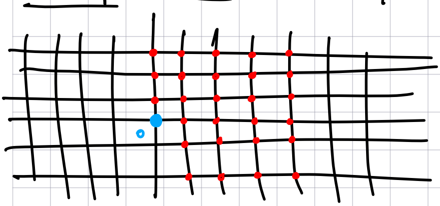
Total: $g \neq h \quad h^{-1}g \in P \Rightarrow g > h$
$\qquad\qquad g^{-1}h \in P \Rightarrow h > g$

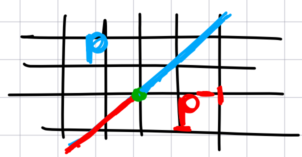
Transitivity: $f > g \wedge g > h$
$g^{-1}f \in P \quad h^{-1}g \in P$
$(h^{-1}g)(g^{-1}f) \in P$
$h^{-1}f \in P \Rightarrow f > h$

Sometimes it is easier to describe the cones than the orders

Example: $\mathbb{Z}^2$: $P = \{(a,b) \mid a > 0 \vee (a = 0 \wedge b > 0)\}$

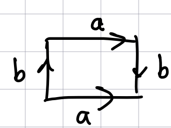
$\mathbb{Z}^2 = \langle 1, \alpha \rangle \subseteq \mathbb{R}$



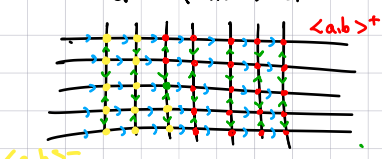


Exercise: P is NOT Finitely generated

$G = \pi_1(\text{Klein bottle}) = \langle a,b \mid bab = a \rangle$



Remark: $G = \langle a,b \rangle^+ \sqcup \langle a,b \rangle^- \sqcup \{id\}$ ①
$G = \langle a,b^{-1} \rangle^+ \sqcup \langle a,b^{-1} \rangle^- \sqcup \{id\}$ ②



Corollary $\pi_1(\text{Klein bottle})$ has exactly 4 orders:
①, ①$^-$, ②, ②$^-$

Proof $\quad a > id,\ b > id \qquad a < id,\ b < id \qquad a > id,\ b < id \qquad a < id,\ b > id$

$a > id,\ b < id$
$\Downarrow$
$a > id$
$b^{-1} > id$
$\Downarrow$
$\langle a, b^{-1} \rangle \subseteq P$
$\langle a, b^{-1} \rangle^- \subseteq P^{-1}$ ②

Ex. $BS(1,2) = \langle a,b \mid aba^{-1} = b^{-2} \rangle$
Has only 4 orders BUT
cones are not fin. generated

Theorem (Tororin) A complete classification of groups with finitely many LO.

$\langle a,b,c \mid aba^{-1}=b^{-1},\ bcb^{-1}=c^{-1},\ ac=ca\rangle$ has exactly 8 LO

= it is done by decomposing through generators =

Remark (Linnell) Let $\leq$ a LO on G $\ni$ $P_{\leq}$ is f.g.

$\Longrightarrow$ $\leq$ is an isolated pt of $LO(G)$

Proof $G = \langle a_1,\dots a_k\rangle^+ \sqcup \langle a_1^{-1},\dots,a_k^{-1}\rangle^- \sqcup \{id\}$

we can find a neighborhood on $a_1>id,\dots,a_k>id$

Example $B_3 = \langle \sigma_1,\sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle \cong \langle a,b \mid ba^2b = a\rangle$

$a=\sigma_1\sigma_2 \quad b=\sigma_2^{-1}$

and $B_3 = \langle a,b\rangle^+ \sqcup \langle a,b\rangle^- \sqcup \{id\}$

In B_4 this is not trivial # it is highly not trivial

Exercise : $\oplus$ find $x>y \ni x^2<y^2$ in $\pi_1(K) = \langle a,b \mid bab=a\rangle = \langle a,b\rangle^+ \sqcup \langle a,b\rangle^- \sqcup \{id\}$

$\oplus$ $x=a$, $y=ab^{-1}$ verify that $x>y$ + $x^{-1}>y^{-1}$

Free groups : ((Ping pong lemma by Klein))

$\langle f,g\rangle = G \leq Bij(X)$ with $\overset{\neq\phi}{A} \subset X$, $\overset{\neq\phi}{B} \subset X$ $A\cap B=\phi$

with $f^n(A) \subseteq B$ $\forall n\neq 0$
$g^n(B) \subseteq A$ $\forall n \neq 0$ then $\langle f,g\rangle \cong F_2$.

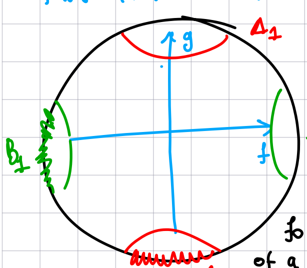
Proof (Very simple) :

W: word on f,g reduced.

$w = fgf^{-1}g^2$ we will prove that it is NOT the identity

$\bar{w}=1 \iff \overline{fwf^{-1}}=1 \qquad fwf^{-1} = gf^{-1}g^2f^{-1}$

$fwf^{-1}(A) = B \Rightarrow \overline{fwf^{-1}} \neq 1$



$A = A_1 \cup A_2$
$B = B_1 \cup B_2$

$\Rightarrow \langle f,g\rangle = F_2$

follow the action of a word on this intervals.

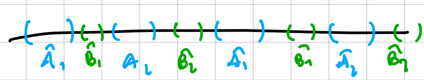
according to its beginning and ending letters

Let $\hat{f},\hat{g} \in Homeo_+(\mathbb{R})$

$\hat{f}(x+1) = \hat{f}(x)+1 \qquad \hat{g}(x+1) = \hat{g}(x)+1$

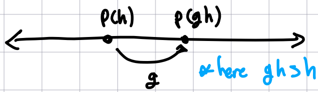
$\hat{A}$ lift to A $\qquad \hat{B}$ lift of B

then we have a kind of Cantor set



$F_2 \leq \mathrm{Homeo}_+(\mathbb{R})$ Recall: every countable LO group is in $\mathrm{Homeo}_+(\mathbb{R})$

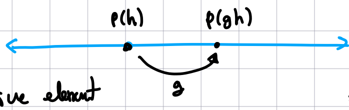
$\preceq$ on $G \xrightarrow{\text{induces}} G \curvearrowright \mathbb{R}$ ie $p: G \longrightarrow \mathbb{R}$ ordered.



where $gh > h \iff \leq$ is bi-invariant & $g > \mathrm{id}$

Understand this action $\Rightarrow$ Understand dynamics of the group order.

<u>Suppose</u> $\leq$ is bi-invariant $g > \mathrm{id} \Rightarrow$ moves every $p(h)$ to the right ie $p(h) \leq p(gh)$



any positive element will move your point to the right

$\Rightarrow g(x) \geq x \quad \forall x \in \mathbb{R}$ if $g > \mathrm{id}$

$ g(x) \leq x \quad \forall x \in \mathbb{R}$ if $g < \mathrm{id}$

⊕ F_2 is BO

⊕ $PL_+([0,1])$ is BO

Both have an action as described before.

<u>Exercise</u> Conversely, assume $G \leq \mathrm{Homeo}_+(\mathbb{R})$ is such that $\forall g \neq \mathrm{id}$

either $g(x) \geq x \quad \forall x$

or $g(x) \leq x \quad \forall x$

<u>then</u> G is biorderable

<u>Mercredi:</u>

<u>Some remarks:</u>

① G_1, G_2 ordered $\Longrightarrow G_1 \times G_2$ is ordered with lexicographic order.

This holds for LO for semidirect products.

= this is a good exercise =

= this is NOT true for BO =

<u>Example</u> $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ has Khazdan pro.

and $\underset{\cup}{F_2} \ltimes \mathbb{Z}^2$ is lo.

Open problems

① $\exists$? G LO infinite with prop (T)

② $BO(F_2)$ is a Cantor set?

<u>Thm.</u> G_1, G_2 are BO $\Rightarrow G_1 * G_2$ is BO

G_1, G_2 are LO $\Rightarrow G_1 * G_2$ is LO

= An idea of the proof for LO =

G_i is LO $\Rightarrow G_i \overset{\eta}{\curvearrowright} \mathrm{Homeo}_+ \mathbb{R}$ # no global fixed points

the trick is consider an special φ such that

$\langle \eta(G_1), \varphi\, \eta(G_2) \varphi^{-1} \rangle \cong G_1 * G_2$.

Thm (Rivas) $LO(G_1 * G_2)$ is a Cantor set.

In particular, $LO(F_2)$ is a Cantor set (Mc Cleary) !

Open problem ②: $BO(F_2)$ is a Cantor set ?

why is it important that it is a Cantor set ?

= if LO is infinite $\overset{?}{\Rightarrow}$ is a Cantor set ‖ NO: Braids are not $\mathcal{C}$ =

= when it is not $\mathcal{C}$ = what kind of groups doesn't have $LO(G) \cong \mathcal{C}$? =

= Dehornoy: generalisations of braids share this property $LO(G) \ncong \mathcal{C}$ =

Theorem (Linnell) If $LO(G)$ is infinite then it is uncountable and contains a Cantor set.

(Tomorrow will work on the proof of this)

Some remarks on compactness:

(Burns-Hale)

⊕ G is LO $\iff$ $\forall \{g_1, \dots, g_m\} \subset G \setminus \{id\}$
$\exists\ k_1, \dots, k_m \in \{-1, +1\} \ni id \notin \langle g_1^{k_1}, g_2^{k_2}, \dots, g_m^{k_m} \rangle^+$

($\Rightarrow$) $\leq$: LO in G $\quad k_i = \begin{cases} 1 & g_i > id \\ -1 & g_i < id \Rightarrow g_i^{-1} > id \end{cases}$

$\Rightarrow \langle g_1^{k_1}, \dots, g_m^{k_m} \rangle^+ \subseteq P_{\leq} \not\ni id$

($\Leftarrow$) Use compactness of LO, nested balls thm, Tychonov's thm; really nice ☺

⊕ G is BO $\iff$ $\forall \{g_1, \dots g_m\} \subset G \setminus \{id\}$ $\exists\ k_1, k_2, \dots, k_m \in \{\pm 1\}$
$\ni id \notin \langle\langle g_1^{k_1}, \dots g_m^{k_m} \rangle\rangle^+$ smallest Normal subsemigroup containing $\{g_i^{k_i}\}_{i=1}^m$

Dynamical realization of the order (continue)

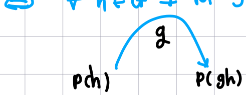
$(G, \leq)$ LO $\Longrightarrow$ $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ # the action has NO global fixed point #

$\leq$ archimedian $\Longrightarrow$? Ⓐ

$\leq$ Conrad $\Longrightarrow$? Ⓑ

⊛ Archimedean : $\forall f > id, g \in G\ \exists N \ni f^N > g$
$\iff \forall h \in G\ \exists M \ni f^M < h$ ⊛

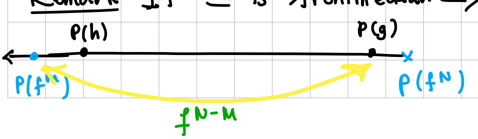
Let's study Ⓐ



$p: G \longrightarrow \mathbb{R}$ ordered map

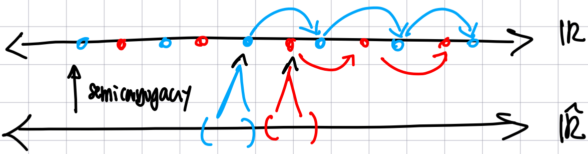
also true !

Remark If $\leq$ is Archimedean $\Longrightarrow$ the action is FREE
($\forall f \neq id$; $fx \neq x\ \forall x \in \mathbb{R}$)



Free actions by Homeo($\mathbb{R}$)

(Holder) Every such action is SEMICONJUGATE to an action by translation



$\mathbb{Z}^2 = \langle 1, \alpha \rangle \quad \alpha \notin \mathbb{Q}$

this is a semiconjugation

= construction by Poincaré =

some actions projects intervals

$(G, \leq)$ Archimidean ; fix $f > id$

$$\psi(g) = \lim_{q \to \infty} \left\{ \frac{p(q)}{q} \text{ with } f^q \leq g^{p(q)} \leq f^{q+1} \right\}$$

$$\phi : G \xhookrightarrow[\text{order embedding}]{} (\mathbb{R}, +)$$

Then either $\phi(G) \cong \mathbb{Z}$ $\rightsquigarrow$ boring (the generator becomes 1 by change of coordinates)

or $\phi(G)$ is dense.

We will focus on this kind of actions ; ie free actions by $\text{homeo}_+(\mathbb{R})$

$\varphi : \mathbb{R} \to \mathbb{R} \qquad \varphi(x) := \sup \{ \phi(g) \mid g(0) \leq x \}$

claims :

① φ is not decreasing

② $\varphi(hx) = \varphi(x) + \phi(h) \qquad \forall h \in G$

③ φ is continuous .

②

$\varphi(hx) = \sup \{ \phi g \mid g(0) \leq hx \}$

$\qquad\qquad\qquad\qquad h^{-1}g\ 0 \leq x$

$= \sup \{ \phi(hg') \mid g'(0) \leq x \}$

$= \phi h + \sup \{ \phi(g') \mid g'(0) \leq x \}$

$= \phi h + \varphi(x)$

φ sends $\mathbb{R}$ into $\mathbb{R}$. ∎

③ Continuity : Exercise.

Hint : use the density ! ! !

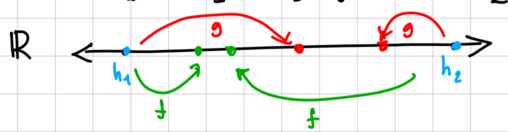
The misterious Conrad's property

① $\forall f > id \ \forall g > id \ \exists n$ such that $fg^n > g$

② $\forall f > id \ \forall g > id$ one has $fg^2 > g$ ⟺ Exercise

③ The following can not happen:

$$h_1 < fh_1 < fh_2 < gh_1 < gh_2 < h_2$$



many groups have this # property

④ $\leq$ lexicographic on $\mathbb{Z}^2 = \langle a, b \rangle$