References

Zimmer: Ergodic th & semisimple groups

Shalom: Growth dichotomy for lin groups

Bekka de la Harpe – Valette: Kazhdan prop T.

Vina-Sageev – Vogtmann : GGT

Outline:

① amenability

② Introduce (T) & relative (T) ③ prop (T)/rel (T)

④ Haagerup group gp. GNS construction

⑤ Median spaces & characters

Amenability

Locally compact, secondly countable gp

Ex. Γ discrete and countable gp

$G = \bigcup_n K_n \qquad K_n \subset K_{n+1}$ compact

locally compact $\Rightarrow$ Haar measure is left invariant

Def. Γ $p \in [1, \infty)$

$$\ell^p(\Gamma) = \{ f : \Gamma \to \mathbb{C} \mid \Sigma |f(x)|^p < \infty \}$$

$p = \infty$

$$\ell^\infty(\Gamma) = \{ f : \Gamma \to \mathbb{C} \mid \sup_{x \in \Gamma} |f(x)| < \infty \}$$

$$d(f, g) = \sqrt[p]{\|f-g\|_p^p} \qquad (\gamma f)(x) = f(\gamma^{-1} x)$$ # Ejercicio #

6 definitions (equivalent)

Γ is amenable $\Leftrightarrow$ one of the following conditions hold:

① $\exists$ left invariant probability finitely aditive measure in $P(\Gamma) = 2^\Gamma$ ((σ-aditive))

② $\exists$ sequence of Følner sets

③ $\ell^2(\Gamma) \succ I_\Gamma$ ($\ell^2(\Gamma)$ has almost inv vectors)

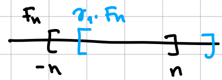
④ K compact metrizable, $\Gamma \to Homeo(\Gamma) \Rightarrow \exists\, \mu \in Prob(K)$, Γ invariant

⑤ Γ does not admit a Ponzi-scheme

⑥ Γ not paradoxical.

Def $\{F_n\}_{n\in\mathbb{N}}$ sequence of Følner sets means:

$$F_n \subset F_{n+1} \qquad \Gamma = \bigcup_n F_n \qquad \frac{|F_n \,\Delta\, \gamma F_n|}{|F_n|} < \infty \quad \forall \gamma \in \Gamma$$



$$\frac{|F_n \,\Delta\, \gamma_1 F_n|}{|F_n|} = \frac{2}{2n+1} \xrightarrow[n\to\infty]{} 0$$

③ **Def** $\mathcal{H}$ is a Hilbert space (ex. $\ell^2(\Gamma)$), $\mathcal{U}(\mathcal{H})$ is the unitary group $M: \mathcal{H} \to \mathcal{H}$ linear isomo

Def $\pi: \Gamma \to \mathcal{U}(\mathcal{H})$ is said to have almost invariant vectors if

$$\exists\, v_n \in \mathcal{H}_n \ni \frac{\| \pi(\gamma) v_n - v_n \|}{\| v_n \|} \xrightarrow[n\to\infty]{} 0 \quad \forall \gamma \in \Gamma$$

③ $I(\Gamma)$ is the trivial representation

$\ell^2(\Gamma) \succ I(\Gamma)$ means $\ell^2(\Gamma)$ almost behave as the trivial representation.

⑤ Ponzi scheme (Pyramid scheme)

$P: \Gamma \to \Gamma$ is a Ponzi-scheme

if $\# |P^{-1}(\gamma)| \geq 2, \ \forall \gamma \in \Gamma$

Bounded distance from id: $\exists\ S \subset \Gamma$ finite $\ni$ $P(\gamma) \subseteq \gamma S$

⑥ **Def** Γ is paradoxical if $\exists\ X$ (distance countable)

$\Gamma \curvearrowright X$ & partitions $X = A \sqcup B$ $\quad A = \bigsqcup_{i=1}^{n} A_i \quad B = \bigsqcup_{i=1}^{m} B_i$

& $g_1, \dots, g_n, h_1, \dots, h_m \in \Gamma$ (Paradoxical decomposition)

$$\bigsqcup_{i=1}^{n} g_i A_i = X \qquad \bigsqcup_{j=1}^{m} h_j B_j = X$$

Def Γ is linear if $\exists\ \rho: \Gamma \hookrightarrow GL_n\, \mathbb{R}$

Rmk: $\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{R}^2 \quad \Rightarrow \quad GL_n\, \mathbb{C} \hookrightarrow GL_{2n}\, \mathbb{R}$

Field embeddings $\quad \overline{\mathbb{Q}_p} \to \mathbb{C} \qquad GL_n\, \overline{\mathbb{Q}_p} \to GL_n\, \mathbb{C}$ $\quad$ = Galois theory =

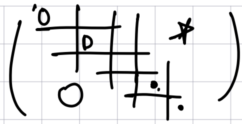
Question: Which linear groups are amenable?

Locally Compact linear groups that are amenable are essentially O_n = orthogonal gp. ← compact

Soluble groups ~ conj to Upp. triang gp

Products ‖ semidirect products ‖

Using this + homework : another kind of gp

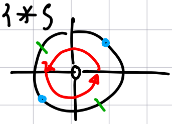
$\boxed{\text{block upper triangular}}$ 

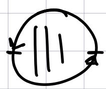
Q: Which linear groups preserves the measure on $\mathbb{R}P^n$

$\mathbb{R}P^n = \mathbb{P}(\mathbb{R}^{n+1})$

$\mathbb{P}(\mathbb{R}^n)$ = equivalence classes of lines through the origin.

$\mathbb{P}(\mathbb{R}^1) = \{*\}$

$\mathbb{P}(\mathbb{R}^2) =$  $= S^1/_{x \sim -x} \cong S^1$

$\mathbb{P}(\mathbb{R}^3) =$  $= S^2/_{x\sim -x} =$  if we take out a disk we obtain a Möbius band.

<u>Facts:</u> $\mathbb{P}(\mathbb{R}^n)$ is a compact metric space.

$Gr(k, \mathbb{R}^n)$ $k \in \mathbb{N}$, $1 \leq k \leq n$ Grassmanian variety

equivalence classes of k-dimensional spaces

$k=1$ $Gr(1, \mathbb{R}^n) = P(\mathbb{R}^n)$ Q: What kind of groups acts with an invariant measure on these spaces?

<u>Def</u> G amenable if $\forall\, G \to \mathrm{Homeo}(K)$ strongly continuous. $G \times K \to K$, $(g, k) \mapsto g \cdot k$ is continuous.

K compact metrizable space, $\exists$ $\mu \in \mathrm{Prob}(K)$ (σ-ad probmeas. on K)

$\gamma_* \mu E := \mu(\gamma^{-1} E) = \mu E$ $\forall \gamma \in \Gamma$

<u>Lemma:</u> (Furstenberg Lemma) μ is a probability measure <u>on</u> $\mathbb{P}(\mathbb{R}^n)$

$\Gamma \leq GL_N \mathbb{R}$. If μ is Γ invariant.

Then either Γ is precompact ($\bar{\Gamma}$ compact)

or $\exists\, 0 \subsetneq V_0 \subsetneq \mathbb{R}^n$ subspace $\ni$ $\mu(V_0) > 0$

zero & $\exists\, \gamma_0 < \Gamma$ finite index $\Gamma_0[V_0] = V_0$

<u>Remark</u>: $\forall\, V < \mathbb{R}^n$ we can construct $\mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathbb{R}^n)$

<u>Proof</u> Assume Γ not precompact

$\exists$ $\gamma_n \xrightarrow[n\to\infty]{} \infty$ $\Rightarrow$ escapes compact set ; chose lift of γ_n in $GL_n \mathbb{R}$

in $PGL_n \mathbb{R} = GL_n(\mathbb{R})/_{Z = \text{Diagonal}}$ $\ni \|\hat{\gamma}_n\| = \max(a_{ij} \mid \text{entries}) = 1$ in $M_n \mathbb{R}$.

It follows there exist a subsequence $\tilde{\gamma}_n \longrightarrow g \in M_n\mathbb{R}$ that converges.

$g \notin GL_n\mathbb{R}$ because $\gamma_n \to \infty$ in $PGL_n\mathbb{R}$

$\Rightarrow$ $\operatorname{Ker} g \neq 0$ (not invertible) but $\|g\| = 1$ $\Rightarrow$ $\operatorname{Im}(g) \neq 0$

<u>show</u>: $\mu(\operatorname{Im}(g) \cup [V]) = 1$

$[V]$ = limit point of $[\gamma_n \operatorname{Ker} g]$ ⑩

Read Furstenberg proof on the infinity of primes

Segunda sesión

Furstenberg Lemma, $\mu \in \operatorname{Prob}\mathcal{M}(\mathbb{P}\mathbb{R}^n)$ $\Gamma < PGL_n\mathbb{R}$ & μ is Γ-invariant

$\Rightarrow$ either ① $\overline{\Gamma}$ compact

② $\exists$ $V_0 \subsetneq \mathbb{R}^n$ & $V_0 \neq \{0\}$ $\ni$ $\mu(V_0) > 0$ and has a finite Γ-orbit

<u>Proof</u> Assume Γ is <u>NOT</u> precompact

$\exists$ $\gamma_n \longrightarrow \infty$ in $PGL_n\mathbb{R}$; find a lift to $GL_n\mathbb{R}$ $\ni$ $\|\tilde{\gamma}_n\| := \max\{\text{entries}\} = 1$

As $M_n(\mathbb{R})$ is locally compact $\Rightarrow$ $\exists$ convergent subsequence $\tilde{\gamma}_n \longrightarrow g \in M_n\mathbb{R}$ with $\|g\| = 1$ with $g \neq 0$ and $\operatorname{Ker} g \neq \{0\}$

Now we use the facts:

① $P(\mathbb{R}^n)$ & $Gr(?, \mathbb{R}^n)$ are compact metrizable

Considering the lift $\tilde{g}$ $\Rightarrow$ $\tilde{\gamma}_n[\operatorname{Ker}\tilde{g}] \longrightarrow [V]$ [up to passing through subsequence again]

<u>claim</u>. $\mu(\underbrace{[V] \cup [\operatorname{Im}\tilde{g}]}_{=: A_0}) = 1$ Notice that this is closed in $P(\mathbb{R}^n)$.

Fix a metric on $P(\mathbb{R}^n)$ look at $D_{A_0} : P(\mathbb{R}^n) \longrightarrow \mathbb{R}$

$x \longmapsto \operatorname{dist}(x, A_0)$

<u>Claim</u>: $D_{A_0}(\gamma_n x) \longrightarrow 0$ $\forall x \in P(\mathbb{R}^n)$

D_A is bounded and continuous

Why this claim is true?

$x \in \operatorname{Ker}[\tilde{g}]$ $D_{A_0}(\gamma_n x) \longrightarrow [V]$

$x \notin \operatorname{Ker}[\tilde{g}] \Rightarrow \gamma_n x \longrightarrow gx \in \operatorname{Im} g$

$$P(\mathbb{R}^n) \setminus A_0 = \bigcup_{m \in \mathbb{N}} \underbrace{\{x \mid D_{A_0}(x) > \tfrac{1}{m}\}}_{A_m}$$

Let's verify that $\mu(A_m) = 0$

$$\frac{1}{m}\mu(A_m) \leq \int_{A_m} D_{A_0}(x)\, d\mu(x)$$

$$\leq \int_{P\mathbb{R}^n} D_A(x)\, d\mu(x)$$

$$= \int_{P\mathbb{R}^n} D_{x_0}(x)\, d\mu(\gamma_n^{-1}x)$$

$$= \int_{P\mathbb{R}^n} D_{x_0}(\gamma_n x)\, d\mu x \longrightarrow 0$$

Dominated Convergence Thm ⊠

So far $\mu([V] \cup [W]) = 1$

Let $[V_0]$ be a minimal dimensional subspace $\ni \mu[V_0] = 0$

$[\gamma V_0]$ vs $[V_0]$? We know that μ is Γ-invariant $\Rightarrow \mu[\gamma V_0] > 0$

Since: intersection of subspaces is subspace

$\Rightarrow \mu([\gamma V_0] \cap [V_0]) = 0$

or $[\gamma V_0] = [V_0]$

$\Rightarrow$ Γ-orbit is finite; ie $\exists \Gamma_0 < \Gamma$ finite index $\ni \Gamma_0 \cdot [V_0] = [V_0]$ ☒

We got to this Lemma via the question $\Gamma < PGL_n\mathbb{R}$ is amenable?

By Furstenberg's Lemma $\exists \underset{\text{finite index}}{\Gamma_0 < \Gamma}$ stabilizing V_0, then we can look to the quotient representation. = We can chose basis of V_0 and extend to $\mathbb{R}^n$; then

$$\underset{\substack{\text{conj}\\ \text{into}\\ \text{new basis}}}{\Gamma_0 \sim} \left(\begin{array}{c|c} V_0 & * \\ \hline 0 & * \end{array}\right) \quad \text{ie} \quad \Gamma_0 \longrightarrow PGL(\mathbb{R}^n/V_0)$$

$\Downarrow$ $\Gamma_0 \to PGL_n(\mathbb{R}^n/V_0)$ has an inv. meas. on the corresponding proj. space.

* The quotient of amenable is amenable *

Answer: Γ contains a finite index subgroup $\Gamma' \underset{\text{conj}}{\sim} \begin{pmatrix} \boxed{\text{comp}} & * \\ 0 & \ddots \quad \boxed{\text{comp}} \end{pmatrix}$
& this is amenable, locally compact and second countable.

Tit's alternative: $\Gamma < GL_n\mathbb{R}$ either
① $\Gamma \geq F_2$ ② Γ is virtually soluable

⚠ $F_2 \hookrightarrow SO(3)$ as a dense subgroup.

✗ Relate property (T) & semidirect products

A group; $Aut(A) = \{\alpha : A \to A \,|\, \alpha$ is an iso$\}$
Γ is a group $\varphi : \Gamma \longrightarrow Aut(A)$

$\Gamma \ltimes_\varphi A$ is a group.

$$\gamma a \eta b = \gamma \eta \eta^{-1} a \eta b = \underset{\Gamma}{\gamma\eta}\,(\underset{A}{\underline{\gamma\eta^{-1} a}})\, b$$

observe: $N \triangleleft G$ then G acts on N by conjugation $G \longrightarrow \mathrm{Aut}\, N$

* Being a semidirect product exactly corresponds to finding an homomorphism σ

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & Q \arrow[r] \arrow[l, bend left, "\sigma"] & 1 \end{tikzcd}$$

$\exists\, \sigma$ hom $\iff G = Q \ltimes N$ ∎

LCSC

Recall: $\pi: G \longrightarrow \mathcal{U}(H)$ strongly continuous

Say π has almost invariant vectors ($\pi \succsim I_G$) means
$\exists\ v_n \in H \setminus \{0\} \ni$
$$\sup_{\gamma \in K} \|\pi(\gamma) v_n - v_n\| \xrightarrow{n\to\infty} 0 \quad \forall\, K \subset G \text{ compact}$$

Recall. π has invariant vectors if $\exists\ v \in H \setminus \{0\} \ni \pi(\gamma)(v) = v \quad \forall\, \gamma \in G.$

Def G LCSC $H < G$ closed; Say (G, H) has relative property (T)
if $\pi: G \longrightarrow \mathcal{U}(H)$ then $\pi \succsim I_G$
$\pi|_H \geq I_H$

Ex. $(SL_2\mathbb{Z} \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has rel (T).

Exercise: Γ amenable & $H < \Gamma$ $|H| = \infty \Rightarrow (\Gamma, H)$ does not have rel (T)

Direct proof Shalom "Bounded generation"

$\boxed{\text{THM}}$ Route to proof:
show $(SL_2\mathbb{Z} \times \mathbb{Z}^2, \mathbb{Z}^2)$ has rel (T) $\iff (SL_2\mathbb{Z} \times \mathbb{R}^2, \mathbb{R}^2)$ has rel (T).

Def. G has property T, if (G, G) has rel (T)
ie. $\pi: G \longrightarrow \mathcal{U}(H)$; $\underbrace{\pi \succsim I_G}_{\text{almost inv. vectors}} \Rightarrow \underbrace{\pi \geq I_G}_{\text{inv. vectors}}$.

Lema G compact $\Rightarrow G$ has property (T)

Idea: G locally compact second countable $\Rightarrow G$ has a L-invariant Haar measure.
G compact $\iff \mathrm{Haar}(G) < \infty$

Using finite Haar measure you can average vectors & get an invariant vector
Assume $|G| < \infty$; counting is a G-inv. measure on G
$\pi: G \longrightarrow \mathcal{U}(H)$ assume $\pi \succsim I_G$; pick $v \neq 0$ is "almost invariant"
$\frac{1}{|G|} \sum_{\gamma \in G} \pi(\gamma) \cdot v \overset{?}{\neq} 0$ yes: we are done
No: we have to work choosing elements #.

Exercise. Assume H has property (T) & $H < G$ closed
$\Rightarrow (G, H)$ has rel (T)

Lemma Suppose that $1 \to A_0 \to A \to A_1 \to 1$ S.E.S. and
$\Gamma \to \mathrm{Aut}(A, A_0)$ the subgroup that stabilizes A_0.

$(\Gamma \ltimes A_0, A_0)$ & $(\Gamma \ltimes A_1, A_1)$ have rel $(T) \Leftrightarrow (\Gamma \ltimes A, A)$ has rel (T).

proof $\Rightarrow$ Let's see that $(\Gamma \ltimes A, A)$ has rel (T).

$\pi : \Gamma \ltimes A \longrightarrow \mathcal{U}(A)$ with almost inv. vectors

$\Gamma \ltimes A \supseteq \Gamma \ltimes A_0$ $\Rightarrow \exists\, v \in H \setminus \{0\}$ that is A_0-invariant

Take the orbit & the quotient will act on that

$\Gamma \ltimes A / A_0 \cong \Gamma \ltimes A_1$ # the lot invariant vectors survive the quotient #
Exercise

property $(T) \Leftrightarrow$ Trivial rep is isolated

Mercredi:

Ex. Show that if $H < G$ closed, H has prop $(T) \Rightarrow (G, H)$ has prop (T).

Corollary: Assume $1 \to A \to V \twoheadrightarrow T \to 1$ SES. & $\Gamma \to \mathrm{Aut}(V, A)$
T has prop (T) $(\Gamma \ltimes A, A)$ has rel $(T) \Leftrightarrow (\Gamma \ltimes V, V)$ has rel (T)

Example: $\Gamma \longrightarrow SL_n \mathbb{Z}$, $A = \mathbb{Z}^n$ $V = \mathbb{R}^n$ $T = V/A$

Example $\Gamma \to SL_n \mathbb{Z}[\frac{1}{p}]$ $A = (\mathbb{Z}[\frac{1}{p}])^n$
$V = \mathbb{R}^n \times \mathbb{Q}_p^n$ $T = V/A$ $n =$ dimensión
$p \sim$ solenoid

Exercise: Show that $\mathbb{Z}[1/p] \hookrightarrow \mathbb{R} \times \mathbb{Q}_p$ discrete compact.

Exercise: $A = \mathcal{O}^n$ ring of integers of finite extension of $\mathbb{Q}$.
$V^n = (\mathbb{R}^? \times \mathbb{C}^?)^n$ $T = V/A =$ Torus
$\Gamma \longrightarrow SL_n \mathcal{O}$.

Thm: $\Gamma \longrightarrow SL_n\mathbb{Z} < Aut(\mathbb{Z}^n) \leq Aut(\mathbb{R}^n) = GL_n\mathbb{R}$

Corollary: $(\Gamma \ltimes \mathbb{Z}^n, \mathbb{Z}^n)$ has rel (T) $\iff$ $(\Gamma \ltimes \mathbb{R}^n, \mathbb{R}^n)$ has rel (T).

Motivation for the next section

Burger's Criterion:

Assume that $\varphi : \Gamma \longrightarrow GL_n\mathbb{R}$ & there are no Γ-inv prob. measures on $P(\hat{\mathbb{R}}^n)$ then $(\Gamma \ltimes_\varphi \mathbb{R}^n)$ has rel (T)

Fact: $\mathbb{R}^n \cong \hat{\mathbb{R}}^n$ vector spaces

~~XXX~~

$\mathbb{R}$ local field (locally compact w/r/t $\|\cdot\|$ & countable union of compact sets)

$$\hat{\mathbb{R}}^n := \{\pi : \mathbb{R}^n \longrightarrow \mathcal{U}(H) \mid \pi \text{ is irreducible}\}$$ unitary dual

!

$$\cong Hom(\mathbb{R}^n, S^1) \cong \mathbb{R}^n$$

Schur's lemma

⊕ $\pi : \Gamma \longrightarrow \mathcal{U}(H)$ is irred if $\nexists$ $H' \subsetneq H$ $\quad H' =$ clos Hilbert sp.

⊕ $\hat{\mathbb{R}}^n = \{\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \mid \mathbb{R}\text{-linear}\} = \text{lin}\langle \varphi_1, \dots, \varphi_n \rangle$ where $\varphi_j(v) = \langle v_j, v\rangle$

$(\varphi . f) = f(\gamma^{-1}.x)$

Define: $\pi_j : \mathbb{R}^n \longrightarrow S^1$ $\quad$ check: All characters are of this form.

$\pi_j(v) = e^{i\varphi_j(v)} = e^{i\langle w, v\rangle}$

Observe: $GL_n\mathbb{R} \longrightarrow GL_n\hat{\mathbb{R}}$ $\quad$ this is the only automorphism (up to conj) that is not a conjugation in $GL_n\mathbb{R}$

$M \longmapsto (M^t)^{-1}$

$$(M.\varphi_j)(v) = \varphi_j(M^{-1}v) = \langle v_j, M^{-1}v\rangle = \langle (M^{-1})^t v_j, v\rangle$$

Recall: $\Gamma \leq PGL_n\mathbb{R}$ amenable?

$$\Gamma \leq \begin{pmatrix} K_1 & * & * \\ 0 & K_2 & * \\ & & \ddots \end{pmatrix} \xrightarrow[\text{transpo}]{\text{inverse}} \begin{pmatrix} K_1' & & 0 \\ * & K_2' & \\ * & * & K' \\ & & & \ddots \end{pmatrix}$$

* $\mathbb{R}^n$ abelian

$\pi : \Gamma \ltimes \mathbb{R}^n \longrightarrow \mathcal{U}(H) \implies \exists\ \underset{\pi}{P} : \underbrace{\mathcal{B}(\widehat{\mathbb{R}^n})}_{\text{Borel } \sigma \text{ algebra}} \longrightarrow \underbrace{\mathrm{proj}(H)}_{\text{orthogonal projection in } \mathcal{H}}$ $(p^2 = p)$

<u>π will be fixed</u> denote $P \equiv P_\pi$ — P is a "projection valued measure : instead of taking values on $\mathbb{R}$ it takes value on $\mathcal{U}(H)$

- $P(\widehat{\mathbb{R}^n}) = \mathrm{Id}_H$
- $\forall\, v \in H,\ B \longmapsto \langle P(B)v, v\rangle$ is a positive Borel measure.
- $P(\{0\})$ = projection onto $\mathbb{R}^n$ invariant vectors
- $\forall\, \gamma \in \Gamma\quad \pi(\gamma^{-1})\, P(B)\, \pi(\gamma) = P(\gamma^* B)$

<u>Ex = Example</u>: $\rho : \mathbb{R} \longrightarrow \mathbb{C} \times \mathbb{C} = V_1 \times V_2$

$\rho(x)(v_1, v_2) = (e^{ix} v_1, e^{2ix} v_2)$

$$P(\varphi) = \begin{cases} \mathrm{Proj}(V_j) & \text{if } \varphi = j \\ 0 & \text{otherwise} \end{cases}.$$

$\varphi \in \widehat{\mathbb{R}} \cong \mathbb{R}.$

<u>Note</u> $\Gamma \ltimes \mathbb{R}^n = G$ is a locally compact group.

<u>Burger's criterion</u>

$\varphi : \Gamma \longrightarrow GL_n\mathbb{R}$ with no Γ-inv. prob. measure on $P(\widehat{\mathbb{R}^n})$

$\implies (\Gamma \ltimes_\varphi \mathbb{R}^n, \mathbb{R}^n)$ has rel (T).

<u>proof</u> (Combination of Burger's original & Shalom's)

$\pi : \Gamma \ltimes \mathbb{R}^n \longrightarrow \mathcal{U}(H)$ with almost invariant vectors $\{v_n\}$ unit vectors.

$\implies \exists$ projective valued measure $P : \mathcal{B}(\widehat{\mathbb{R}^n}) \longrightarrow \mathrm{Proj}(H)$

& $P(\{0\})$ = projects onto the $\mathbb{R}^n$ invariant vectors

Assume by contradiction that $P(\{0\})$ = projection onto $\{0\} \subseteq H$

Define $\quad \mu_n(B) = \langle P(B)v_n, v_n\rangle$ this is a Borel Measure

$\quad \mu_n(\{0\}) = \langle \mathrm{Proj}\{0\}\, v_n, v_n\rangle = 0$

look at the push forward measure $\quad \widehat{\mathbb{R}^n} \setminus \{0\} \longrightarrow P(\widehat{\mathbb{R}^n})$

$\mu_n \longmapsto \bar{\mu}_n$ measure on $P(\widehat{\mathbb{R}^n})$.

(*) $\|\gamma_* \bar{\mu}_n - \bar{\mu}_n\| := 2 \sup_{B \in \mathcal{B}(\hat{\mathbb{R}}^n)} |\gamma_* \mu_n(B) - \mu_n(B)|$

$$\underset{\text{Ex.}}{\leq} 2 \|\pi(\gamma) v_n - v_n\|$$

$\Rightarrow$ $\bar{\mu}_n$ is a sequence of almost invariant measures on $P(\mathbb{R}^n)$

(Banach-Alaoglu Thm: prob. measure on a compact metrizable space is w-* compact)

$\{\bar{\mu}_n\}$ & find a w-* limit point up to passing to a subsequence $\bar{\mu}_n \to \bar{\mu}_\infty$ & $\bar{\mu}_\infty$ is Γ-inv.

Burger's criterion: If Γ does not have an invariant measure on $P(\mathbb{R}^n)$ then $(\Gamma \ltimes_\gamma \mathbb{R}^n, \mathbb{R}^n)$ has rel (T)

Assume $\exists$ $\pi : \Gamma \ltimes \mathbb{R}^n \to \mathcal{U}(H)$ with almost inv. vectors but not $\mathbb{R}^n$-inv. vectors ∎

Furstenberg's lemma: If $\Gamma \to PGL_n \hat{\mathbb{R}}$ & $\bar{\mu}_\infty$ is Γ-inv. prob measure on $P(\hat{\mathbb{R}}^n)$ then either Γ is precompact or $\exists$ $V_0 \neq \mathbb{R}^n$ $0 \neq V_0$ subspace which is virtually Γ inv. & $\mu[V_0] > 0$

Example $\Gamma = SL_2\mathbb{Z}$ $\Gamma \ltimes \mathbb{R}^2$

$\Rightarrow$ $(\Gamma \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has rel (T)