UNIVERSITÉ PARIS VI — PIERRE et MARIE CURIE

U.F.R de Mathématiques

THÈSE DE DOCTORAT

Spécialité Mathématiques

Présentée par **ANDREA PULITA**
Pour obtenir le grade de docteur de l'Université Paris VI

1.– Équations différentielles $p-$adiques d'ordre un et applications.

2.– Confluence des équations aux $q-$différences $p-$adiques.

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La thèse a été preparée à l'Université de Paris sous la direction de Gilles Christol et a été co-dirigée par Bruno Chiarellotto de l'Université de Padoue dans le cadre d'une co-tutelle.

La thèse est constituée de deux articles plus des chapitres en appendice.
Nous avons ajouté un resumé detaillé de la thèse en français
pour que le lecteur puisse mieux s'orienter.

Remerciements

Je remercie avant tout les membres du Jury de cette thèse pour avoir accepté d'en faire partie. Je remercie particulièrement les rapporteurs S.Matsuda et M.Reversat pour le temps qu'il ont consacré à la correction de cette thèse. Leur corrections ont été très instructives.

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Premier parmi tous Gilles Christol, mon directeur de thèse français, qui a incarné pour moi plusieurs figures d'homme. Il a été avant tout un *maître*, soit pour les mathématiques que pour la vie. En effet, tout au long de ces années, il a essayé continûment de m'expliquer les mécanismes qui conduisent à la vérité mathématique. En ce sens il a été aussi *philosophe*. C'est à lui que je dois le concept

“ Ce sont les petits lemmes qui font les théorèmes, ”

ou encore

“ Il y a *toujours* une vérité cachée derrière. ”

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Pendant ces années de thèse on s'est vu assez régulièrement, et alors que Christol me guidait dans le monde des équations différentielles $p-$adiques, j'allais chez lui pour qu'il me parle des relations entre les équations différentielles $p-$adiques et d'autres domaines mathématiques. Ses explications sont toujours très éclairantes, a tel point qu'au début cela me semblait trop facile.

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Sa disponibilité et ses compétences sont sans égales et m'ont fait comprendre l'importance de la communication dans une équipe, car je suis convaincu

[1] Voir : http ://www.institut.math.jussieu.fr/%7Ezapponi/gdt-diff.html

que les mathématiques se transmettent bien plus en vitesse oralement que par voie écrite.

Je voudrais aussi remercier Pierre Colmez. Je n'ai pas communiqué beaucoup avec lui, mais les rencontres qu'on a eu ont été cruciales pour ma thèse, car c'est lui qui m'a signalé que les objets avec lesquels je travaillais s'appelaient "Groupes de Lubin-Tate" !

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Dans la dernière année de thèse j'ai commencé à discuter avec Lucia Di Vizio qui m'a permis de comprendre ses papier sur les équations aux $q-$différences. Elle m'a toujours encouragé dans la directions des équations aux $q-$différences, et elle m'a toujours mis à l'aise avec sa grande sympathie et son bonheur. C'est à elle à qui je dois la seconde partie de cette thèse, car c'est son bonheur, sa joie, et son amitié qui m'ont accroché à ce domaine.

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Pour comprendre jusqu'a quel point va son amour il suffit de savoir que elle est (et elle sera) peut-être l'unique personne, en dehors de moi, qui ait lu au moins trois fois cette thèse !

Andrea Pulita

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Première partie

Resumé detaillé de la Thèse

Chapitre 1

Premier Article

Le premier article se propose d'expliciter, pour les caractères, la correspondence donnée par le théorème de monodromie locale $p-$adique récemment demontré (cf. [And02], [Ked04], [Meb02]).

Définition 1.0.1. *Soit K un corps ultramétrique de caractéristique* 0. *Nous dénotons par $\mathcal{R}_K$ l'anneau "de Robba"*

$$\mathcal{R}_K := \{f(T) := \textstyle\sum_{i\in\mathbb{Z}} a_i T^i \mid a_i \in K, \exists\, \rho < 1 \text{ tel que } f(T) \text{ converge pour } \rho < |T| < 1\} \tag{1.0.1.1}$$

et par $\mathcal{E}_K^\dagger$ l'anneau de "Robba borné"

$$\mathcal{E}_K^\dagger := \{f(T) \in \mathcal{R}_K \mid \sup |a_i| < +\infty\} \,. \tag{1.0.1.2}$$

Dans cet article nous obtenons les résultats suivants :

- Une *classification complète des équations différentielles de rang un, solubles sur l'anneau de Robba* $\mathcal{R}_K$ (resp. sur $\mathcal{E}_K^\dagger$), et une étude detaillée de chaque équation, à isomorphisme prés ;
- Une correspondence explicite entre *caractères de Artin-Schreier-Witt* du groupe de Galois absolu de $k((t))$, et *équations différentielles de rang un sur* $\mathcal{R}_K$;
- *Le calcul explicite, pour ces caractères, du foncteur de Monodromie $p-$adique* qui associe a une représentation V, du groupe de Galois absolù $\mathrm{G}_{k((t))}$, un $\varphi-$module $\mathbf{D}^\dagger(\mathrm{V})$ sur $\mathcal{E}_K^\dagger$ et par conséquence une équation différentielle $p-$adique $\mathbf{M}^\dagger(\mathrm{V})$ sur $\mathcal{R}_K$;
- Pour tout corps $p-$adique L de corps residuel k_L de caracteristique $p > 0$, on donne une *description des extensions non ramifiées cycliques de L, qui proviennent, par henselianité, d'une extension finie séparable du corps k_L.*

Ces résultats passent par l'étude detaillée de la convergence/surconvergence des solutions des équations solubles sur l'anneau de Robba $\mathcal{R}_K$.

Notre travail commence par l'introduction d'une nouvelle classe d'exponentielles de type Artin-Hasse, nommées $\boldsymbol{\pi}-$*exponentielles* qui généralisent la bien connue *exponentielle de Dwork*

$$\exp(\pi_0 T) \tag{1.0.1.3}$$

où π_0 est une racine du polynôme $X^{p-1} = -p$. Ces exponentielles sont solutions d'équations de rang un solubles, et reciproquement toute solution d'une équation de rang un soluble est de ce type (après éventuel changement de base dans le module différentiel).

Les $\boldsymbol{\pi}-$*exponentielles* sont l'outil technique central de l'article et leur étude permet de clarifier et de décrire très explicitement toute la théorie en rang un.

Remarque 1.0.2. *Afin d'être simple, nous fesons dans cette introduction une suite d'hypothèses, qui en réalité ne sont pas nécessaires. Les énoncés ne sont donc pas dans leur forme la plus générale.*

Les différents définitions et les principals objets qui apparaissent dans cette introduction sont rapélée dans les premieres chapitres de la thèse.

1.1 $\pi-$exponentielles

Nous fixons une série de Lubin-Tate $P(X) \in \mathbb{Z}_p[[X]]$ et son groupe de Lubin-Tate $\mathfrak{G}_P$. Par définition la série $P(X)$ vérifie

$$P(X) \equiv \mathrm{w}X \mod X^2\mathbb{Z}_p[[X]] \quad , \quad P(X) \equiv X^p \mod p \cdot \mathbb{Z}_p[[X]] \tag{1.1.0.1}$$

où $\mathrm{w} \in \mathbb{Z}_p$ est une uniformisante. Le groupe $\mathfrak{G}_P(X,Y) \in \mathrm{Z}_p[X,Y]$ est l'unique groupe formel pour lequel $P(X)$ est un endomorphisme. De plus $\mathfrak{G}_P$ a une structure canonique de $\mathbb{Z}_p-$module pour laquelle la multiplication par w est donnée par la série $P(X)$. Les points de w^n-torsion de $\mathfrak{G}_P$ sont alors les zéros, de valuation inférieure ou égale à 1, dans une clôture algébrique $\mathbb{Q}_p^{\mathrm{alg}}$ de $\mathbb{Q}_p$, de la série $P^{(n)}(X) := \underbrace{P \circ P \circ P \circ \cdots \circ P}_{n \text{ fois}}$.

$$\mathrm{Ker}(\mathrm{w}^n) := \{x \in \mathbb{Q}_p^{\mathrm{alg}} \mid |x| \leq 1 \ , \ P^{(n)}(x) = 0\} \ . \tag{1.1.0.2}$$

Le groupe de Tate associé a $\mathfrak{G}_P$ est par définition

$$\mathrm{T}(\mathfrak{G}_P) := \varprojlim \left(\mathrm{Ker}(\mathrm{w}^{n+1}) \xrightarrow[x \mapsto P(x)]{} \mathrm{Ker}(\mathrm{w}^n) \right) \ . \tag{1.1.0.3}$$

On trouve que $\mathrm{T}(\mathfrak{G}_P)$ est un $\mathbb{Z}_p-$module libre de rang un. Un générateur de $\mathrm{T}(\mathfrak{G}_P)$ est une suite $(\pi_j)_{j \geq 0}$ compatible (i.e. $P(\pi_0) = 0$ et $P(\pi_{j+1}) = \pi_j$, pour tout $j \geq 0$) tel que $\pi_0 \neq 0$ et $|\pi_0| < 1$.

Définition 1.1.1. *Nous fixons un générateur* $\boldsymbol{\pi} := (\pi_j)_{j\geq 0}$ *de* $\mathrm{T}(\mathfrak{G}_P)$.

Exemple 1.1.2. *Si* $P(X) = X^p + pX$*, alors* π_0 *est le bien connu "*π *de Dwork" (cf. 1.0.1.3).*

Proposition 1.1.3. Soit L un corps valué complet de caractéristique 0, qui contient les racines $p^{m+1}-$èmes de l'unité. Soit $d = np^m \geq 1$, avec $(n,p) = 1$, et $m \geq 0$. Soit $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_L)$ un vecteur de Witt. Soit $\phi = (\phi_0, \ldots, \phi_m) \in (\mathcal{O}_L)^{m+1}$ son vecteur fantôme, i.e.

$$\phi_j := \lambda_0^{p^j} + p\lambda_1^{p^{j-1}} + \cdots + p^j\lambda_j \ .$$

Alors, la serie formelle (nommée $\boldsymbol{\pi}-$*exponentielle*)

$$\mathrm{e}_d(\boldsymbol{\lambda}, T) := \exp\left(\pi_m\phi_0 T^n + \pi_{m-1}\phi_1\frac{T^{np}}{p} + \cdots + \pi_0\phi_m\frac{T^{np^m}}{p^m}\right) \quad (1.1.3.1)$$

converge pour $|T| < 1$. De plus, elle est surconvergente (i.e. converge pour $|T| < 1+\varepsilon$, avec $\varepsilon > 0$) si et seulement si $|\lambda_i| < 1$, pour tout $i = 0, \cdots, m$.

En particulier, la série

$$\mathrm{e}_d(\boldsymbol{\lambda}, T)^{p^{m+1}} = \mathrm{e}_d(p^{m+1}\cdot\boldsymbol{\lambda}, T) \ , \quad (1.1.3.2)$$

est toujours surconvergente.

Exemple 1.1.4. *Pour* $m = 0$ *et* $n = 1$ *(i.e.* $d = 1$*), on trouve*

$$\begin{aligned} \mathrm{e}_1(\lambda_0, T) &= \exp(\pi_0\lambda_0 T) \ , & (1.1.4.1)\\ \mathrm{e}_1(\lambda_0, T)^p &= \exp(\pi_0\lambda_0 T)^p = \exp(\pi_0 p\lambda_0 T) \ . & (1.1.4.2) \end{aligned}$$

Dans l'article on trouve une étude détaillée de l'équation différéntielle satisfaite par une $\boldsymbol{\pi}-$exponentielle.

Proposition 1.1.5. *L'équation différentielle satisfaite par* $\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})$ *est*

$$\mathrm{L}_d(\boldsymbol{\lambda}) := \partial_T - \frac{\partial_T(\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1}))}{\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})} = \partial_T + n\cdot\big(\sum_{j=0}^{m}\pi_{m-j}\cdot\phi_j\cdot T^{-np^j}\big) \ . \quad (1.1.5.1)$$

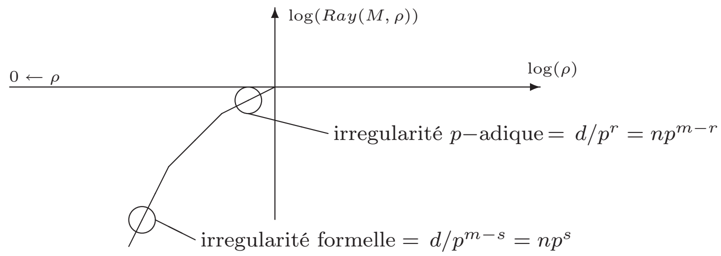
Ou $\partial_T = T\frac{d}{dT}$*. On définit* $s, r \leq m$ *respectivement par* $\phi = \langle\phi_0, \ldots, \phi_s, 0, \ldots, 0\rangle$*, avec* $\phi_s \neq 0$*, et* $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m)$*, avec* $|\lambda_0|, \ldots, |\lambda_{r-1}| < 1$ *et* $|\lambda_r| = 1$*. Par convention, si* $|\lambda_i| < 1$ *pour tout* i*, on pose* $r = \infty$*. Alors :*

- *L'opérateur* $\mathrm{L}_d(\boldsymbol{\lambda})$ *est trivial sur* $\mathcal{R}_K$ *(i.e.* $\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1}) \in \mathcal{R}_K$*) si et seulement si* $r = \infty$*, (i.e.* $|\lambda_0|, \ldots, |\lambda_m| < 1$*).*
- *Plus précisement l'irrégularité* formelle *de* $\mathrm{L}_d(\boldsymbol{\lambda})$ *est égale a* $d/p^{m-s} = np^s$ *et l'irregularité* $p-$adique *est égale à* $d/p^r = np^{m-r}$.

Voici le graphe logarithmique de la fonction

$$\rho \mapsto Ray(\mathrm{L}_d(\boldsymbol{\lambda}), \rho)/\rho \tag{1.1.5.2}$$

où $Ray(\mathrm{L}_d(\boldsymbol{\lambda}), \rho)$ est la rayon de convergence générique en ρ de $\mathrm{L}_d(\boldsymbol{\lambda})$:



• Dans les notations précédentes, soit k_L le corps residuel de L. Soit $\varphi : \mathcal{O}_L \to \mathcal{O}_L$ un Frobenius qui réleve la puissance $p-$ème de k_L. Nous dénotons encore par $\varphi : \mathbf{W}(\mathcal{O}_L) \to \mathbf{W}(\mathcal{O}_L)$ le morphisme d'anneau déduit par fonctorialité.

La proposition suivante dépend fortement de la théorie de Lubin-Tate :

Proposition 1.1.6. Pour tout $\boldsymbol{\lambda} \in \mathbf{W}(\mathcal{O}_L)$, la série formelle

$$\theta_d(\boldsymbol{\lambda}, T) := \frac{\mathrm{e}_d(\varphi(\boldsymbol{\lambda}), T^p)}{\mathrm{e}_d(\boldsymbol{\lambda}, T)} \tag{1.1.6.1}$$

est surconvergente si et seulement si le groupe $\mathfrak{G}_P$ est isomorphe au groupe multiplicatif formel $\widehat{\mathbb{G}}_m$. Dans ce cas, on peut considérer sa valeur en 1.

Exemple 1.1.7. *Pour $d = 1$ on trouve*

$$\theta_1(\lambda_0, T) = \exp(\pi_0(\varphi(\lambda_0)T^p - \lambda_0 T)) \ . \tag{1.1.7.1}$$

En particulier, si $P(X) = X^p + pX$ (i.e. π_0 est le π de Dwork), alors on retrouve la bien connue "splitting function" de Dwork (cf. [Dwo62, §4,a)])

$$\theta_1(1, T) = \exp(\pi_0(T^p - T)) \ . \tag{1.1.7.2}$$

1.2 Une déformation du complexe de Artin-Schreier-Witt dans le complexe de Kummer

Dorénavant nous supposerons que $\mathfrak{G}_P$ est isomorphe à $\widehat{\mathbb{G}}_m$. Ceci révient a demander que $\mathrm{w} = p$.

Les théories de Artin-Schreier-Witt et de Kummer consistent dans la donnée de deux complexes qui calculent la cohomologie galoisienne.

$$\begin{tikzcd}
\boxed{\text{char } 0} & 0 \arrow[r] & \mathbb{G}_m(L) \arrow[r, "x \to x^{p^{m+1}}"] & \mathbb{G}_m(L) \arrow[r] & 0 & \text{Kummer} \\
\text{char } p \arrow[u, dotted] & 0 \arrow[r] & \mathbf{W}_m(k_L) \arrow[r, "\mathrm{F}-1"'] \arrow[u, dotted, "\theta"] & \mathbf{W}_m(k_L) \arrow[r] \arrow[u, dotted, "e"'] & 0 & \text{Artin-Schreier}
\end{tikzcd}$$

Le théorème suivant "*déforme*" le complexe de Artin-Schreier-Witt dans celui de Kummer. Les applications qui déforment un complexe dans l'autre sont *la valeur en* $T = 1$ *des* $\boldsymbol{\pi}-$*exponentielles surconvergents* $\theta_{p^m}(-,T)$ *et* $\mathrm{e}_{p^m}(-,T)^{p^{m+1}}$.

Ce théorème constitue l'analogue d'une partie de la théorie de Sekigichi-Suwa (cf. [SS94]).

Théorème 1.2.1. Posons $\mathrm{G}_L := \mathrm{Gal}(L^{\mathrm{alg}}/L)$, $\mathrm{G}_{k_L} := \mathrm{Gal}(k_L^{\mathrm{sep}}/k_L)$ et $\mathcal{O}_L^{\varphi=1} := \{a \in \mathcal{O}_L \mid a^{\varphi} = a\}$. On a un diagramme commutatif

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_{p^{m+1}} \arrow[r] & L^{\times} \arrow[r, "x \mapsto x^{p^{m+1}}"'] & L^{\times} \arrow[r, two heads, "\delta_{\mathrm{Kum}}"] & \mathrm{H}^1(\mathrm{G}_L, \boldsymbol{\mu}_{p^{m+1}}) \\
 & \mathbf{W}_m(\mathcal{O}_L^{\varphi=1}) \arrow[u] \arrow[d, two heads] \arrow[r, hook] & \mathbf{W}_m(\mathcal{O}_L) \arrow[u, "\theta_{p^m}(-,1)"] \arrow[d, two heads] \arrow[r, "\varphi-1"'] & \mathbf{W}_m(\mathcal{O}_L) \arrow[u, "\mathrm{e}_{p^m}(-,1)^{p^{m+1}}"'] \arrow[d, two heads] & \\
0 \arrow[r] & \mathbb{Z}/p^{m+1}\mathbb{Z} \arrow[uu, dotted, bend left, "\wr"] \arrow[r] & \mathbf{W}_m(k_L) \arrow[r, "\bar{\mathrm{F}}-1"'] & \mathbf{W}_m(k_L) \arrow[r, two heads, "\delta"'] & \mathrm{H}^1(\mathrm{G}_{k_L}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \arrow[uu, dotted, hook, "\overline{e}"]
\end{tikzcd}$$

où $\bar{\mathrm{F}}$ est le Frobenius de $\mathbf{W}_m(k_L)$. De plus, $\theta_{p^m}(-,1)$ induit une identification

$$1 \mapsto \xi_m^{-1} \ : \ \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \mu_{p^{m+1}} \ , \tag{1.2.1.1}$$

où ξ_m est l'unique racine $p^{m+1}-$ème de 1 tel que $|(\xi_m - 1) - \pi_m| < |\pi_m|$.

Exemple 1.2.2. *La dernière partie de ce théorème généralise le fait bien connu que la fonction surconvergente* $\theta_1(1,T) = \exp(\pi_0(T^p - T))$ *a pour valeur en* $T = 1$ *l'unique racine* $p-$*ème de l'unité* ξ_0^{-1} *qui vérifie* $|\xi_0 - 1 - \pi_0| < |\pi_0|$ *(cf. ex.1.1.7).*

Ceci permet de donner la description suivante des extensions cycliques *non ramifiées* de L dont l'ordre est une puissance de p :

Corollaire 1.2.3. Soit $\overline{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_L)$. Soit k'/k_L l'extension définie par $\overline{\boldsymbol{\lambda}}$.[1] Alors l'extension non ramifiée L' de L correspondante à k' est donnée par

$$L' = L(\theta_{p^m}(\boldsymbol{\nu}, 1)) \ , \tag{1.2.3.1}$$

où $\boldsymbol{\nu} \in \mathbf{W}_m(\mathcal{O}_{L'})$ est une relèvement arbitraire d'une solution $\bar{\boldsymbol{\nu}}$, dans $\mathbf{W}_m(k_L^{\mathrm{sep}})$, de l'équation

$$\bar{\mathrm{F}}(\bar{\boldsymbol{\nu}}) - \bar{\boldsymbol{\nu}} = \overline{\boldsymbol{\lambda}} \ . \tag{1.2.3.2}$$

Exemple 1.2.4. *Si* $m = 0$*, alors la suite de Artin-Schreier est*

$$0 \to \mathbb{Z}/p\mathbb{Z} \to k_L \xrightarrow[\nu \mapsto \nu^p - \nu]{} k_L \xrightarrow{\delta} \mathrm{H}^1(\mathrm{G}_{k_L}, \mathbb{Z}/p\mathbb{Z}) \to 0 \ , \tag{1.2.4.1}$$

---

[1] Rappelons que l'extension k' est par définition le corps fixé par le noyau du caractère $\delta(\boldsymbol{\lambda}) : G_{k_L} \to \mathbb{Z}/p^{m+1}\mathbb{Z}$, où δ est le morphisme cohomologique de Artin-Schreier-Witt défini dans le diagramme du théorème 1.2.1.

Si $\bar{\lambda} \in k_L$, alors l'extension k'/k_L définié par $\bar{\lambda}$ est, par définition, le sous-corps de k^{sep} stabilisé par le noyau de $\delta(\lambda)$ et est égale à $k' = k_L(\bar{\nu})$, où $\bar{\nu} \in k^{\mathrm{sep}}$ est une racine de l'équation $\bar{\nu}^p - \bar{\nu} = \bar{\lambda}$. L'extension de L'/L est alors donnée par

$$L' = L(\theta_1(\nu, T)_{|T=1}) = L(\,\exp(\pi_0(\varphi(\nu)T^p - \nu T))_{|T=1}\,)\ . \tag{1.2.4.2}$$

Remarque 1.2.5. *Le corollaire 1.2.3 n'est pas entièrement explicite car il necessite le calcul de $\bar{\boldsymbol{\nu}}$. Nous allons voir que, bien que $\mathcal{E}_K^{\dagger}$ ne soit pas complèt, ce corollaire s'applique aussi au cas $L = \mathcal{E}_K^{\dagger}$. Dans ce cas nous aurons une description meilleure qui dépend uniquement de $\boldsymbol{\lambda}$, et ne nécessite pas le calcul de $\boldsymbol{\nu}$ (cf. 1.4.2). En effet on remarque qu'on a l'identité*

$$\theta_{p^m}(\boldsymbol{\nu}, 1)^{p^{m+1}} = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}\ . \tag{1.2.5.1}$$

Au contraire, l'écriture $\theta_{p^m}(\boldsymbol{\nu}, 1) = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)$ n'a pas de sens car la série $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, T)$ n'est pas surconvergente et on ne peut pas prendre sa valeur en $T = 1$. Dans le cas du corps $\mathcal{E}_K^{\dagger}$ nous allons donner un sens au symbole $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)$ pour une classe de vecteurs $\boldsymbol{\lambda}$ qui est suffisamment grande pour décrire toutes les extensions non ramifiée de $\mathcal{E}_K^{\dagger}$ qui proviennent par henselianité d'une extension séparable de son corps residuel $k(\!(t)\!)$ (voir théorème 1.4.2).

1.3 Equations différentielles solubles de rang un

Posons $\partial_T := T\frac{d}{dT}$. Les modules différentiels solubles de rang un sur $\mathcal{R}_K$ sont, dans une base convenable, définis par un équation $\partial_T - g(T)$, où $g(T) \in K[T^{-1}]$. La solution de Taylor à l'infini d'une telle équation s'exprime comme produit de $\boldsymbol{\pi}-$exponentielles. Cette solution est l'analogue de l'élément $\theta_{p^m}(\boldsymbol{\nu}, 1)$ du corollaire 1.2.3. Remarquons que le diagramme du théorème 1.2.1 donne l'équation

$$\theta_{p^m}(\boldsymbol{\nu}, 1)^{p^{m+1}} = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}\ . \tag{1.3.0.2}$$

Bien que la série entière $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, T)$ ne soit pas en général surconvergente, cette remarque justifie la définition suivante

Définition 1.3.1. *Soit $m \geq 0$ et soit $\boldsymbol{f}^-(T) := (f_0^-(T), \ldots, f_m^-(T))$ un vecteur de Witt dans $\mathbf{W}_m(T^{-1}\mathcal{O}_K[T^{-1}])$. Soit $(\phi_0(T), \ldots, \phi_m(T))$ son vecteur fantôme. Nous posons*

$$\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1) := \exp\left(\pi_m\phi_0(T) + \pi_{m-1}\frac{\phi_1(T)}{p} + \cdots + \pi_0\frac{\phi_m(T)}{p^m}\right)\ . \tag{1.3.1.1}$$

• La série formelle $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)$ appartient à $1 + \pi_m T^{-1}\mathcal{O}_K[[T^{-1}]]$, et est donc convergente pour $|T| > 1$.

• Sa différentielle logarithmique $g(T) := \partial_T(\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1))/\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)$ est un polynôme en T^{-1} sans terme constant. On pose

$$\mathrm{L}(\boldsymbol{f}^-(T),0) := \partial_T - g(T)\ , \tag{1.3.1.2}$$

cette équation est triviale sur $\mathcal{R}_K$ si et seulement si sa solution $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)$ appartient a $\mathcal{R}_K$.

• On étudie la série $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)$ en la décomposant en produit fini de $\boldsymbol{\pi}-$exponentielles élémentaires de la forme 1.1.3.1. On développe le langage qui permet de passer d'une écriture à l'autre.

• La fonction $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)$ est algébrique sur $\mathcal{R}_K$ (resp. $\mathcal{E}_K^\dagger$) et est aussi le générateur d'une extension de Kummer de $\mathcal{E}_K^\dagger$, car on a

$$\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)^{p^{m+1}} = \mathrm{e}_{p^m}(p^{m+1}\cdot\boldsymbol{f}^-(T),1) \in \mathcal{E}_K^\dagger\ . \tag{1.3.1.3}$$

Plus précisement, on a les résultats suivants qui font le pont entre la théorie de Artin-Schreier-Witt pour le corps $k(\!(t)\!)$ et les équations différentielles sur $\mathcal{R}_K$.

Théorème 1.3.2. Tout module différentiel soluble sur $\mathcal{R}_K$ a une base dans laquelle l'opérateur associé est défini à l'infini. La solution de Taylor à l'infini de cet opérateur est alors de la forme

$$T^{a_0}\cdot\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1) \tag{1.3.2.1}$$

pour un $m\geq 0$ convenable, un $a_0\in\mathbb{Z}_p$, et un vecteur de Witt $\boldsymbol{f}^-(T)\in\mathbf{W}_m(T^{-1}\mathcal{O}_{K_m}[T^{-1}])$, où $K_m := K(\boldsymbol{\mu}_{p^{m+1}})$.

Théorème 1.3.3. La série formelle $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)$ est surconvergente si et seulement si l'équation d'Artin-Schreier-Witt

$$\bar{\mathrm{F}}(\bar{\boldsymbol{g}}) - \bar{\boldsymbol{g}} = \overline{\boldsymbol{f}^-(T)} \in \mathbf{W}_m(k(\!(t)\!)) \tag{1.3.3.1}$$

a une solution $\bar{\boldsymbol{g}}\in\mathbf{W}_m(k(\!(t)\!)^{\mathrm{sep}})$. En particulier si $\varphi:\mathcal{E}_K^\dagger\to\mathcal{E}_K^\dagger$ est un morphisme d'anneaux qui relève la puissance $p-$ème de $k(\!(t)\!)$, alors la série

$$\theta_{p^m}(\boldsymbol{f}^-(T),1) := \mathrm{e}_{p^m}(\varphi(\boldsymbol{f}^-(T)) - \boldsymbol{f}^-(T),1) \tag{1.3.3.2}$$

est surconvergent pour tout $\boldsymbol{f}^-(T)\in\mathbf{W}_m(T^{-1}\mathcal{O}_K[T^{-1}])$.

Exemple 1.3.4. *Si $P(X) = X^p + pX$, et si $\boldsymbol{f}^-(T) = T^{-1}$, $m=1$, alors on retrouve*

$$\mathrm{e}_1(T^{-1},1) = \exp(\pi_0 T^{-1})\ , \tag{1.3.4.1}$$

dans ce cas $\exp(\pi_0T^{-1})^p = \exp(\pi_0 pT^{-1})\in\mathcal{E}_K^\dagger$. *D'autre part on trouve*

$$\theta_1(T^{-1},1) = \exp(\pi_0(T^{-p} - T^{-1}))\ . \tag{1.3.4.2}$$

Définition 1.3.5. *Soit* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)$ *le groupe, pour le produit tensoriel, des classes d'isomorphisme des modules différentiels de rang un sur* $\mathcal{R}_K$*. Soit* $K_\infty := \cup_{m\geq 0} K_m$*, où* $K_m := K(\pi_m)$*. Nous posons*

$$\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}) := \bigcup_{m\geq 0} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_m}) \ . \tag{1.3.5.1}$$

Corollaire 1.3.6. Soit k_∞ le corps résiduel de K_∞. On a

$$\begin{aligned} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}) &= \mathbb{Z}_p/\mathbb{Z} \times \mathrm{Hom}(\mathcal{I}_{k_\infty((t))}, \mathbb{Q}_p/\mathbb{Z}_p) & (1.3.6.1)\\ &= \mathbb{Z}_p/\mathbb{Z} \times \frac{\mathbf{CW}(k_\infty((t)))}{(\bar{\mathrm{F}}-\mathrm{Id})\mathbf{CW}(k_\infty((t)))} \ . & (1.3.6.2)\end{aligned}$$

• Le groupe $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})$ est le groupe des caractères du groupe de Galois tannakien de la catégorie des modules différentiels sur l'anneau de $\mathcal{R}_{K_\infty}$.[2]

• Nous donnons aussi une description du sous-groupe de $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})$ formé par les équations qui sont trivialisées par une extension spéciale $\mathcal{R}'$ de $\mathcal{R}_K$ (i.e. une extension étale finie de $\mathcal{R}_K$ qui "provient" d'une extension finie séparable de $k((t))$ cf. [Mat02, 5.1]).

1.3.1 Un critère de solubilité

Nous obtenons un critère de solubilité pour les équations différentielles de rang un à coefficients dans $\mathcal{R}_K$, sans avoir besoin de passer à K_∞.

Théorème 1.3.7. Soit $g(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K$. Alors l'équation différentielle $\partial_T - g(T)$ est soluble si et seulement si, pour tout entier n premier à p, les vecteurs

$$\Big(-\frac{a_{-n}}{n}, -\frac{a_{-np}}{n}, -\frac{a_{-np^2}}{n}, \ldots\Big) \quad , \quad \Big(\frac{a_n}{n}, \frac{a_{np}}{n}, \frac{a_{np^2}}{n}, \ldots\Big)$$

sont les vecteurs fantômes de vecteurs de Witt $\boldsymbol{\lambda}_{-n} = (\lambda_{-n,0}, \lambda_{-n,1}, \ldots)$ et $\boldsymbol{\lambda}_n = (\lambda_{n,0}, \lambda_{n,1}, \ldots)$ qui appartiennent à $\mathbf{W}(\mathcal{O}_K)$. Très explicitement $\partial_T - g(T)$ est soluble si et seulement si pour tout couple $(n,m) \in \mathbb{Z}\times\mathbb{N}$, avec n premier a p, ils existent $\lambda_{n,m} \in \mathcal{O}_K$, de telle sorte que

$$\begin{aligned} a_{-np^m} &= -n\cdot\phi_{-n,m} &:=& -n\cdot(\lambda_{-n,0}^{p^m} + p\lambda_{-n,1}^{p^{m-1}} + \cdots + p^m\lambda_{-n,m}) \ ,\\ a_{np^m} &= n\cdot\phi_{n,m} &:=& n\cdot(\lambda_{n,0}^{p^m} + p\lambda_{n,1}^{p^{m-1}} + \cdots + p^m\lambda_{n,m}) \ .\end{aligned}$$

• On remarque que si l'on considère une famille arbitraire de vecteurs $\{\boldsymbol{\lambda}_{-n}, \boldsymbol{\lambda}_n\}_{(n,p)=1}$ dans $\mathbf{W}(\mathcal{O}_K)$, et si l'on forme la série formelle

$$g(T) := \sum_{n\in\mathbb{Z}}\sum_{m\geq 0} n\phi_{n,m} T^{np^m},$$

[2] Remarquons que le corps K_∞ n'est pas complet. L'anneau $\mathcal{R}_{K_\infty}$ est alors *défini* par $\mathcal{R}_{K_\infty} := \mathcal{R}_K \otimes_K K_\infty$. De cette façon tout module différentielle sur $\mathcal{R}_{K_\infty}$ provient, per extension des scalaires, d'un module sur $\mathcal{R}_{K_m}$, pour un K_m convenable.

il n'est pas vrai en général que cette série soit dans $\mathcal{R}_K$, ni qu'elle converge sur une couronne. Nous donnons une caractérisation des familles $\{\boldsymbol{\lambda}_{-n}, \boldsymbol{\lambda}_n\}_{(n,p)=1}$ pour lesquelles cette série est bien dans $\mathcal{R}_K$.

Le théorème précédent donne alors le corollaire suivant

Corollaire 1.3.8. Si K est non ramifié sur $\mathbb{Q}_p$, alors toute équation différentielle soluble de rang un sur $\mathcal{R}_K$ est modérée (i.e. isomorphe à une equation du type $\partial_T - a_0$, avec $a_0 \in \mathbb{Z}_p$).

1.4 Calcul explicite du foncteur de monodromie pour les caractères

Supposons que k soit un corps parfait de caractéristique $p > 0$. Soit $\Lambda/\mathbb{Q}_p$ une extension finie de corps résiduel $\mathbb{F}_q$, avec $q = p^r$, qui contient les racines $p^{m+1}-$èmes de 1. Nous supposons de plus que Λ est munie d'un Frobenius $\sigma_0 : \Lambda \to \Lambda$ qui relève la puissance $p-$ième de $\mathbb{F}_q$, et tel que $\sigma_0^r = \mathrm{Id}_\Lambda$ et $\sigma_0(\pi_m) = \pi_m$. Nous notons encore par σ_0 (resp. σ) le Frobenius $\sigma_0 \otimes \mathrm{F}$ (resp. $\mathrm{Id} \otimes \mathrm{F}^r$) sur l'anneau

$$K := \Lambda \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathbf{W}(k) \ .$$

Notons par $\mathcal{O}_K^\dagger$ l'anneau des entiers du corps $\mathcal{E}_K^\dagger$. Nous fixons donc un Frobenius $\varphi_0 : \mathcal{O}_K^\dagger \to \mathcal{O}_K^\dagger$ qui prolonge $\sigma_0 : K \to K$. Alors φ_0 et $\varphi := \varphi_0^r$ s'étendent de manière unique à toute extension finie non ramifié de $\mathcal{E}_K^\dagger$.

• Soit $\mathrm{G}_{k(\!(t)\!)}$ le groupe de Galois absolu du corps $k(\!(t)\!)$. Pour tout caractère

$$\alpha : \mathrm{G}_{k(\!(t)\!)} \to \Lambda^\times \ ,$$

notons V_α la représentation de $\mathrm{G}_{k(\!(t)\!)}$ donnée dans une base $\mathbf{e} \in \mathrm{V}_\alpha$ par

$$\gamma(\mathbf{e}) := \alpha(\gamma) \cdot \mathbf{e} \ , \quad \text{pour tout } \ \gamma \in \mathrm{G}_{k(\!(t)\!)}.$$

Si maintenant

$$\alpha : \mathrm{G}_{k(\!(t)\!)} \to \mathbb{Z}/p^{m+1}\mathbb{Z}$$

est un caractère, et si ξ_m est la racine définie dans le théorème 1.2.1, nous notons encore par V_α la représentation obtenue par composition $\imath \circ \alpha : \mathrm{G}_{k(\!(t)\!)} \to \boldsymbol{\mu}_{p^{m+1}}$ ou

$$\imath \ : \ 1 \mapsto \xi_m \quad : \quad \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\ \sim\ } \boldsymbol{\mu}_{p^{m+1}} \ . \qquad (1.4.0.1)$$

• Notons par $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ le $\varphi-$module sur $\mathcal{E}_K^\dagger$ défini par

$$(\mathbf{D}^\dagger(\mathrm{V}_\alpha) \ , \ \varphi_{\mathbf{D}^\dagger(\mathrm{V}_\alpha)}) := ((\mathrm{V}_\alpha \otimes_\Lambda \mathcal{E}_K^{\dagger,\mathrm{unr}})^{\mathrm{G}_{k(\!(t)\!)}} \ , \ 1 \otimes \varphi \ ) \ .$$

L'application $1 \otimes \partial_T$ est alors une connection sur $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$. Nous notons par $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ l'équation différentielle de rang un sur $\mathcal{R}_K$ ainsi définie.

• L'abélianisé $\mathrm{G}_{k(\!(t)\!)}^{\mathrm{ab}}$ se décompose comme

$$\mathrm{G}_{k(\!(t)\!)}^{\mathrm{ab}} = \mathcal{I}_{k(\!(t)\!)}^{\mathrm{ab}} \oplus \mathrm{G}_k^{\mathrm{ab}} \;,$$

où G_k est le groupe de Galois absolu de k. Ceci implique que tout caractère α de $\mathrm{G}_{k(\!(t)\!)}$ est produit d'un caractère α_- de $\mathcal{I}_{k(\!(t)\!)}$ avec un caractère α_0 de G_k. En termes de représentations ceci s'exprime en disant que

$$\mathrm{V}_\alpha = \mathrm{V}_{\alpha_-} \otimes \mathrm{V}_{\alpha_0} \;. \tag{1.4.0.2}$$

On trouve que $\mathbf{M}^\dagger(\mathrm{V}_{\alpha_0}) = 0$.

• Soit maintenant $\alpha : \mathcal{I}_{k(\!(t)\!)} \to \Lambda$ un caractère avec *image finie*. Alors le groupe abélien $\alpha(\mathcal{I}_{k(\!(t)\!)}^{\mathrm{ab}}) \subseteq \boldsymbol{\mu}_n$ se décompose en un produit de groupes cycliques. Pour calculer $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ et $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$, on peut alors supposer que α est un caractère cyclique, c'est à dire que :

$$\alpha : \mathcal{I}_{k(\!(t)\!)} \to \boldsymbol{\mu}_{\ell^m} \tag{1.4.0.3}$$

où ℓ est un nombre premier.

• Pour ℓ premier a p on trouve que l'extension définie par le noyau de α est $k(\!(t^{1/\ell^m})\!)$, et alors on a $\alpha(\gamma) = \gamma(t^{1/\ell^m})/t^{1/\ell^m}$ pour tout $\gamma \in \mathcal{I}_{k(\!(t)\!)}$:

$$\gamma(\mathbf{e}) := \gamma(t^{1/\ell^m})/t^{1/\ell^m} \cdot \mathbf{e} \;.$$

On relève, par hensélianité, $k(\!(t^{1/\ell^m})\!)$ en une extension kummérienne de $\mathcal{E}_K^\dagger$, dont un générateur de kummer est donné visiblement par T^{1/ℓ^m}. Donc une base de $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ est donnée par

$$\mathbf{e} \otimes T^{-1/\ell^m} \in (\mathrm{V}_\alpha \otimes_\Lambda \mathcal{E}_K^{\dagger,\mathrm{unr}})^{\mathrm{G}_{k(\!(t)\!)}} \;. \tag{1.4.0.4}$$

On calcule alors facilement la matrice du Frobenius et de la connexion :

$$(1 \otimes \varphi)(\mathbf{e} \otimes T^{-1/\ell^m}) \;=\; T^{1/\ell^m}\varphi(T^{-1/\ell^m}) \cdot (\mathbf{e} \otimes T^{-1/\ell^m}) \;, \tag{1.4.0.5}$$

$$(1 \otimes \partial_T)(\mathbf{e} \otimes T^{-1/\ell^m}) \;=\; -1/\ell^m \cdot (\mathbf{e} \otimes T^{-1/\ell^m}) \;. \tag{1.4.0.6}$$

En particulier l'équation différentielle est $\mathrm{L}(0, a_0)$, avec $a_0 = 1/\ell^m$ (cf. 1.3.1.2).

Le cas $\ell = p^m$ est traité dans le paragraphe suivant.

1.4.1 Cas de la $p-$torsion

Pour $\ell = p$ on rencontre le problème technique de trouver un générateur de Kummer explicite de l'extension non ramifiée de $\mathcal{E}_K^\dagger$ attachée a une extension finie séparable de $k(\!(t)\!)$ cyclique de degrée p^m. Ce problème est résolu par le corollaire 1.2.3. De plus l'équation 1.3.0.2 montre comment décrire l'élement $\theta_{p^m}(\boldsymbol{\nu}, 1)$. On obtient donc les théorèmes 1.4.2 et 1.4.4.

Définition 1.4.1. *Soit* $\overline{\boldsymbol{f}(t)} = (\overline{f_0(t)}, \dots, \overline{f_m(t)}) \in \mathbf{W}_m(k(\!(t)\!))$ *un vecteur de Witt. Si* $\overline{f_i(t)} = \sum_{n \geq n_i} \bar{a}_i t^i$*, nous posons* $\overline{f_i^-(t)} := \sum_{n_i \leq n \leq -1} \bar{a}_i t^i$ *et*

$$\overline{\boldsymbol{f}^-(t)} := (\overline{f_0^-(t)}, \dots, \overline{f_m^-(t)}) \; . \tag{1.4.1.1}$$

Le théorème suivant donne une description des extensions Kummeriennes non ramifiées de $\mathcal{E}_K^\dagger$ dont le corps résiduel est une extension d'Artin-Schreier-Witt donnée de $k(\!(t)\!)$ (i.e. cyclique d'ordre une puissance de p). Rappelons la suite d'Artin-Schreier-Witt :

$$0 \to \mathbb{Z}/p^{m+1}\mathbb{Z} \to \mathbf{W}_m(k(\!(t)\!)) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{W}_m(k(\!(t)\!)) \xrightarrow{\delta} \mathrm{H}^1(\mathrm{G}_{k(\!(t)\!)}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to 0 \tag{1.4.1.2}$$

Théorème 1.4.2. Soit $\overline{\boldsymbol{f}(t)} \in \mathbf{W}_m(k(\!(t)\!))$ et soit $\alpha = \delta(\overline{\boldsymbol{f}(t)})$ le caractère de Artin-Schreier-Witt défini par $\overline{\boldsymbol{f}(t)}$. Soit $k'(\!(t')\!)$ l'extension finie séparable de $k(\!(t)\!)$ fixée par le noyau de α. Soit $(\mathcal{E}^\dagger)'$ l'extension non ramifiée de $\mathcal{E}_K^\dagger$ obtenue par hensélianité de $k'(\!(t')\!)$. Alors on a

$$(\mathcal{E}^\dagger)' = \mathcal{E}_{K'}^\dagger(\; \mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1) \;), \tag{1.4.2.1}$$

ou K' est l'extension non ramifiée de K attachée par henselianité a k'/k, et $\boldsymbol{f}^-(T) \in \mathbf{W}_m(T^{-1}\mathcal{O}_K[T^{-1}])$ est un relèvement arbitraire de $\overline{\boldsymbol{f}^-(t)}$.

Exemple 1.4.3. *1.– Si* $m = 0$*, et* $\boldsymbol{f}^-(T) = T^{-1}$*, alors, comme* $\boldsymbol{f}^-(T)$ *n'a pas de terme constant, alors l'extension* $k'(\!(t')\!)/k(\!(t)\!)$ *est totalement ramifié (i.e. on a* $k' = k$*). Plus précisement l'on a*

$$\frac{1}{(t')^p} - \frac{1}{t'} = \frac{1}{t} \;, \qquad k(\!(t')\!) = k(\!(t)\!)[t'] \; . \tag{1.4.3.1}$$

L'extension non ramifiée $(\mathcal{E}^\dagger)'/\mathcal{E}_K^\dagger$ *qui correspond a* $k(\!(t')\!)$ *est donnée par*

$$\frac{1}{(T')^p} - \frac{1}{T'} = \frac{1}{T} \;, \qquad (\mathcal{E}^\dagger)' = \mathcal{E}_K^\dagger[T'] \; . \tag{1.4.3.2}$$

Mais, par le théorème 1.4.2, elle peut s'exprimer comme

$$(\mathcal{E}^\dagger)' = \mathcal{E}_K^\dagger(\; \exp(\pi_0 T^{-1}) \;) \; . \tag{1.4.3.3}$$

2.– On remarque que $(\mathcal{E}^\dagger)'$ *et* $\mathcal{E}_K^\dagger$ *sont isomorphes par l'isomorphisme (non canonique) donné par* $T \mapsto T'$*. Si l'on exprime* $\exp(\pi T^{-1})$ *dans la nouvelle variable* T' *on trouve (cf. 1.4.3.2)*

$$\exp(\pi T^{-1}) = \exp(\pi \; ((T')^{-p} - (T')^{-1}) \;) = \theta((T')^{-1}, 1) \tag{1.4.3.4}$$

et l'on a donné un sens à l'égalité

$$\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1) = \theta_{p^m}(\boldsymbol{\nu}, 1) \tag{1.4.3.5}$$

qu'on avait envisagée dans 1.2.5.1.

Théorème 1.4.4. Soit $\overline{\boldsymbol{f}(t)} \in \mathbf{W}_m(k(\!(t)\!))$ et soit $\alpha = \delta(\overline{\boldsymbol{f}(t)})$ le caractère de Artin-Schreier-Witt défini par $\overline{\boldsymbol{f}(t)}$. Alors

1. Une base de $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ est donnée par
$$\mathbf{e} \otimes \theta_{p^m}(\boldsymbol{\nu}, 1) \;, \tag{1.4.4.1}$$
où $\boldsymbol{\nu} \in \mathbf{W}_m(\widehat{\mathcal{E}}_K^{\mathrm{unr}})$ est une solution de l'équation
$$\varphi_0(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{f}(T) \;, \tag{1.4.4.2}$$
où $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathcal{O}_K[[T]][T^{-1}])$ est un relèvement arbitraire de $\overline{\boldsymbol{f}}(t)$;
2. Le Frobenius φ_0 agit sur V_α, et l'on a
$$\varphi_0\big(\mathbf{e} \otimes \theta_{p^m}(\boldsymbol{\nu}, 1)\big) = \theta_{p^m}(\boldsymbol{f}(T), 1) \cdot \big(\mathbf{e} \otimes \theta_{p^m}(\boldsymbol{\nu}, 1)\big) \;. \tag{1.4.4.3}$$
Donc, si
$$\mathrm{Tr}(\boldsymbol{f}(T)) := \boldsymbol{f}(T) + \varphi_0(\boldsymbol{f}(T)) + \cdots + \varphi_0^{r-1}(\boldsymbol{f}(T)),$$
alors
$$\varphi\big(\mathbf{e} \otimes \theta_{p^m}(\boldsymbol{\nu}, 1)\big) = \theta_{p^m}(\mathrm{Tr}(\boldsymbol{f}(T)), 1) \cdot \big(\mathbf{e} \otimes \theta_{p^m}(\boldsymbol{\nu}, 1)\big) \;; \tag{1.4.4.4}$$
3. La classe d'isomorphisme de $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ dépend seulement du caractère α_- et le module différentiel $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ est isomorphe au module différentiel défini par l'opérateur $\mathrm{L}(\boldsymbol{f}^-(T), 0)$ (cf. 1.3.1.2) où $\boldsymbol{f}^-(T) \in \mathbf{W}_m(T^{-1}\mathcal{O}_K[T^{-1}])$ est un vecteur de Witt arbitraire qui relève $\overline{\boldsymbol{f}^-(t)}$.
4. L'irrégularité de $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ est égale au conducteur de Swan de la représentation V_α.

Exemple 1.4.5. *Soit* $\alpha : \mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!)) \longrightarrow \mathbb{Z}/p\mathbb{Z}$ *le caractère de Artin-Schreier défini par* $t^{-1} \in k(\!(t)\!)$*. Si* y *est une solution de l'équation* $y^p - y = t^{-1}$*, alors on a*
$$\alpha(\gamma) = \gamma(y) - y \in \mathbb{F}_p \;, \qquad \textit{pour tout} \quad \gamma \in \mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!)) \;. \tag{1.4.5.1}$$

Notons encore par $\alpha : \mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!)) \longrightarrow \boldsymbol{\mu}_p$ *le morphisme obtenu par composition avec 1.4.0.1. Dans cet exemple* $\Lambda = \mathbb{Q}_p(\xi_0)$*, avec* $\xi_0^p = 1$*,* $\xi_0 \neq 1$*, et*
$$\sigma_0 = \sigma = \mathrm{Id}_\Lambda \otimes \mathrm{F} \tag{1.4.5.2}$$
$$\varphi_0(T) = \varphi(T) = T^p \tag{1.4.5.3}$$
$$\alpha = \alpha^- \tag{1.4.5.4}$$

Dans les notations de l'exemple 1.4.3 on obtient que une base de $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ *est donnée par (cf.1.4.4.1 et 1.4.3.4) :*

$$\mathbf{e} \otimes \exp(\pi_0 T^{-1}) \ , \tag{1.4.5.5}$$

$$\boldsymbol{\nu} = (t')^{-1} \tag{1.4.5.6}$$

$$\mathrm{Tr}(T^{-1}) = T^{-1} \tag{1.4.5.7}$$

La matrice de φ *agissant sur* V_α *est alors égale à*

$$\theta_{p^m}(T^{-1}, 1) = \exp(\pi_0(T^{-p} - T^{-1})) \ . \tag{1.4.5.8}$$

L'équation différentielle définie par $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ *est*

$$T\frac{d}{dT} + \pi_0 T^{-1} \ . \tag{1.4.5.9}$$

Le conducteur de Swan de V_α *et l'irrégularité de cette équation sont égales à* 1*.*

Remarque 1.4.6. *En general si l'on ne considère que des vecteurs de Witt de longueur* 0*, on retrouve les calculs faits par Dwork. Nous envisageons donc, dans le futur, de reprendre ses calculs sur la fontion Gamma* $p-$*adique, sur le principle de Boyarsky, et sur les fonctions zéta des variétés en caractéristique* p *pour les généraliser.*

Chapitre 2

Deuxième article

Dans le deuxième article nous étudions, dans le cadre $p-$adique, le phénomèns de la *confluence* d'une famille d'équations aux $q-$différences vers une équation différentielle et celui de la *déformation* d'une équation différentielle vers une équation aux $q-$différences.

Soit B un anneau de fonctions sur un domaine D. Soit $\mathcal{Q}$ un sous-groupe multiplicatif de $K^{\times}$ tel que D est stable par tout homothetie $x \mapsto qx$, avec $q \in \mathcal{Q}$. Pour tout $q \in \mathcal{Q}$, notons par σ_q l'automorphisme de B donné par $f(T) \mapsto f(qT)$. La $q-$dérivation est alors

$$\Delta_q(f(T)) := \frac{f(qT) - f(T)}{q-1} \ . \tag{2.0.6.1}$$

Pour q qui tend vers 1, la $q-$dérivation tend vers la dérivation $\delta_1 := T\frac{d}{dT}$ de B. Soit maintenant

$$\{\Delta_q(Y) = H(q,T) \cdot Y\}_{q \in \mathcal{Q}} \tag{2.0.6.2}$$

une famille d'équations aux $q-$différences. Dans le cadre classique, la confluence et la déformation décrivent le fait heuristique que certaines de ces familes d'équations ont la propriété suivante : lorsque q tend vers 1, l'équation $\Delta_q - H(q,T)$ "tend" vers une équation différentielle $\delta_1 - G(1,T)$, en ce sens que la solution $Y_q(T)$ de l'équation $\Delta_q - H(q,T)$ "tend" vers la solution $Y_1(T)$ de l'équation différentielle $\delta_1 - G(1,T)$:

$$\lim_{q \to 1} Y_q(T) = Y_1(T) \ . \tag{2.0.6.3}$$

Nous montrons, dans le contexte $p-$adique, que si l'on part d'une seule équation aux q_0-différences $\sigma_{q_0}(Y) = A_{q_0}(T) \cdot Y$, alors, si la solution Y_{q_0} satisfait certaines conditions, il existe un sous-groupe ouvert $U \subseteq \mathcal{Q}$ et une famille canonique d'équations aux $q-$différences

$$\{\sigma_q(Y) = A(q,T) \cdot Y\}_{q \in U} \ , \tag{2.0.6.4}$$

tels que la matrice $A(q,T)$ est localement analytique pour $(q,T) \in U \times D$, et vérifie $A_{q_0}(T) = A(q_0,T)$. En fait la solution de Taylor $Y(T,c)$ de 2.0.6.4, en tout point c du domaine D, est la même pour toutes les équations de cette famille.

En d'autres termes, dans ce contexte $p-$adique, l'équation 2.0.6.3 s'écrit

$$Y_q(T) = Y_1(T) \ , \quad \forall\, q \in U \ . \tag{2.0.6.5}$$

On remarque que, comme la solution $Y(T,c)$ est en même temps solution de chacune des équations de la famille 2.0.6.4, la famille peut être entièrement reconstituée à partir de la solution $Y(T,c)$. En effet $A(q,T) = Y(qT,c) \cdot Y(T,c)^{-1}$.

Ceci montre que le module M aux q_0-différences donné au départ est canoniquement muni d'une action de σ_q, pour tout $q \in U$. Ceci s'exprime en disant que M est un *faisceau* de modules sur le faisceau d'anneaux $\mathcal{O}_{\mathcal{Q}}$ défini par

$$\mathcal{O}_{\mathcal{Q}}(U) := \mathrm{B}[\{\sigma_q, \sigma_q^{-1}\}_{q\in U}] \ . \tag{2.0.6.6}$$

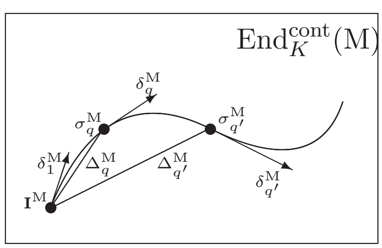
Nous dirons que M est un $\sigma-$module analytique. Nous dirons que M est *constant* sur U s'il existe une solution $Y(T,c)$ qui soit solution simultanée de toute fibre de M.

Pour $q = 1$ la fibre n'est pas définie, car Δ_q n'est pas défini. Plus généralement, si q est égal à une racine ξ de l'unité, alors l'opérateur σ_ξ est d'ordre fini et la catégorie des équations aux $\xi-$différences est de nature fondamentalement différente par rapport à la catégorie des $q-$différences pour q proche de ξ.

Par exemple, la fibre naturelle d'un $\sigma-$module analytique, pour q proche de 1, est une équation différentielle $\delta_1 - G(1,T)$ définie par

$$G(1,T) := \lim_{q\to 1} H(q,T) = \left(q\frac{d}{dq} A(q,T) \right)_{|q=1} \ . \tag{2.0.6.7}$$

Notons par σ_q^{M}, Δ_q^{M}, δ_1^{M} l'action de $\delta_1, \sigma_q, \Delta_q$ sur M, que nous visualisons dans ce dessin :



Plus généralement pour $q = \xi$ la "vrai" fibre d'un tel faisceau est la donnée du module M, de l'opérateur d'ordre fini σ_ξ^{M}, et de l'opérateur $\xi-$tangent

δ_q^{M} défini par

$$\delta_q^{\mathrm{M}} := q \lim_{q' \to q} \cdot \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q' - q} = \text{“}q \frac{d}{dq}(\sigma_q^{\mathrm{M}})\text{”} \tag{2.0.6.8}$$

Cet opérateur vérifie $\delta_q^{\mathrm{M}} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}$ et donc, pour tout $f \in \mathrm{B}$ et $m \in \mathrm{M}$:

$$\delta_q^{\mathrm{M}}(f \cdot m) = \sigma_q(f) \cdot \delta_q^{\mathrm{M}}(m) + \delta_q(f) \cdot \sigma_q^{\mathrm{M}}(m) \ . \tag{2.0.6.9}$$

Pour tenir compte de ces phénomènes nous introduisons la notion de $(\sigma, \delta)-$modules analytiques.

• Les notions de “confluence forte” et de “déformation forte” consistent alors à passer d'une fibre à l'autre du $(\sigma, \delta)-$module analytique canoniquement attaché à l'équation aux q_0-différences du départ.

Exemple 2.0.7. L'équation différentielle de rang un $\delta_1 + \pi T^{-1}$, avec $\pi^{p-1} = -p$, a pour solution de Taylor à l'infini la fonction

$$y(T) = \exp(\pi T^{-1}) \ . \tag{2.0.7.1}$$

Nous considerons cette équation sur l'anneau des fonctions analytiques sur la couronne $\mathcal{A}_K(I)$, avec $I =]1 - \varepsilon, 1[$, $0 < \varepsilon < 1$. Alors, si $q \in \mathrm{D}^-(1, \tau)$, avec $\tau = 1 - \varepsilon$, la déformation forte envoie cet opérateur différentiel dans l'équation aux $q-$différences $\sigma_q - A(q, T)$ où

$$A(q, T) = \exp(\pi(q^{-1} - 1)T^{-1}) \ . \tag{2.0.7.2}$$

En effet, comme $q \in \mathrm{D}^-(1, \tau)$, alors $A(q, T) \in \mathcal{A}_K(]1 - \varepsilon, 1[)^\times$, et de plus la solution de $\sigma_q - A(q, T)$ est toujours égale à $\exp(\pi T^{-1})$, pour tout $q \in \mathrm{D}^-(1, \tau)$.

Remarque 2.0.8. *Dans un souci de concision, les énoncés de ce résumé ne sont pas dans leur forme la plus générale, et la présentation qui en est faite ne suit pas toujour l'ordre logique des démonstrations.*

2.1 Théorème principal

Soit $(K, |.|)$ un corps ultramétrique sphériquement complet. Dans la suite B sera un anneau de fonctions sur un affinoïde A, ou, plus généralement, une limite inductive ou projective de tels anneaux. Par exemple $\mathrm{B} = \mathcal{R}_K, \mathcal{E}_K^\dagger$, ou $\mathrm{B} = \mathcal{H}_K^\dagger$ (les definitions de $\mathcal{R}_K$, $\mathcal{E}_K^\dagger$, ont été données dans la première partie de ce resumé, cf. 1.0.1.1, 1.0.1.2). On pose :

$$\begin{aligned} \mathcal{H}_K^\dagger := \{f(T) = \textstyle\sum a_i T^i \mid f(T) & \text{ converge pour } \rho_1 < |T| < \rho_2, \\ & \text{pour } \rho_1 < 1 < \rho_2 \text{ non precisés}\} \ . \end{aligned} \tag{2.1.0.1}$$

Nous notons $\mathcal{Q}$ le sous groupe topologique de $K^\times$ formé des q pour lesquels σ_q est un automorphisme de B. Le groupe $\mathcal{Q}$ est ouvert dans $K^\times$.

• Pour fixer les idées nous prenons pour B l'anneau des fonctions analytiques sur une couronne $\mathcal{C}_K(I) := \{x \in K \mid |x| \in I\}$, où $I \subseteq \mathbb{R}_\geq$ est un intervalle *borné*.

$$\mathcal{A}_K(I) := \{f(T) = \sum_{i\in\mathbb{Z}} a_i T^i \mid f(T) \text{ converge pour } |T| \in I\} \tag{2.1.0.2}$$

Alors $\mathcal{Q} = \{q \in K \mid |q| = 1\}$.

Définition 2.1.1. *Soit $q \in \mathcal{Q}$ un point. On dénote par*

$$\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I)) \tag{2.1.1.1}$$

la catégorie des équations aux $q-$différences sur $\mathcal{A}_K(I)$

Un module aux $q-$différences M est un B$-$module libre de type fini muni d'une action bijective d'un opérateur σ_q-semilinéaire σ_q^{M} (i.e. $\sigma_q^{\mathrm{M}}(f \cdot m) = \sigma_q(f) \cdot \sigma_q^{\mathrm{M}}(m)$, pour tout $f \in \mathcal{A}_K(I)$ et $m \in \mathrm{M}$). Les morphismes sont ceux qui commutent à σ_q^{M}.

Si $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ est une base de M, et si $\sigma_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j a_{i,j}(q,T)\mathrm{e}_j$, alors l'opérateur attaché à M dans cette base est $\sigma_q - A(q,T)$, où $A(q,T) = (a_{i,j}(q,T))_{i,j}$. Les solutions de cet opérateur dans une B$-$algèbre C, munie d'un opérateur σ_q, sont alors en correspondance naturelle avec les morphismes $\alpha : \mathrm{M} \to \mathrm{C}$ qui commutent à σ_q.

Notation 2.1.2. 1.– Pour $n \geq 0$, nous posons $[0]_q = 0$, $[n]_q := \frac{q^n-1}{q-1}$, $[0]_q^! = 1$,

$$[n]_q^! := \prod_{i=1}^{n} [i]_q \ .$$

2.– Nous posons encore pour tout $n \geq 0$

$$(x-y)_{q,n} := (x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y) \ . \tag{2.1.2.1}$$

3.– Si $\sigma_q - A(q,T)$ est un opérateur aux $q-$différences, alors nous notons $H_{[n]}(T) \in M_n(\mathcal{A}_K(I))$ la matrice de l'opérateur $(\frac{1}{T}\Delta_q^{\mathrm{M}})^n$ (i.e. telle que $(\frac{1}{T}\Delta_q)^n(Y) = H_{[n]}Y$) on a alors $H_{[0]} = \mathrm{I}$, $H_{[1]}(T) = \frac{A(q,T)-\mathrm{I}}{(q-1)T}$, et

$$H_{[n+1]} = \frac{1}{T}\Delta_q(H_{[n]}) + \sigma_q(H_{[n]}) \cdot H_{[1]} \ .$$

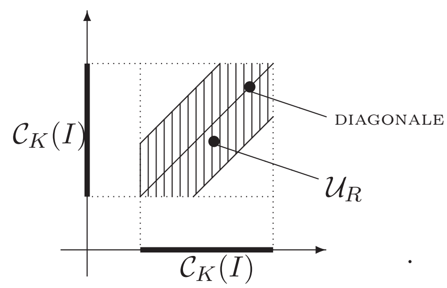
Si q n'est pas une racine de l'unité, on note

$$Y(x,y) = \sum_{n\geq 0} H_{[n]}(y) \frac{(x-y)_{q,n}}{[n]_q^!} \tag{2.1.2.2}$$

la $q-$résolvante de l'équation $\sigma_q - A(q,T)$.

Remarque 2.1.3. *Si $q = \xi$ est une à une racine $m-$ème de l'unité, alors $[m]_\xi = 0$ et donc $[n]^!_\xi = 0$, pour tout $n \geq m$. Pour cette raison il n'existe pas de $q-$résolvante si q est une racine de l'unité.*

Définition 2.1.4. *Soit q différent d'une racine de l'unité. Un opérateur aux $q-$différences $\sigma_q - A(q,T)$ est dit* admissible *si la $q-$résolvante $Y(x,y)$ converge dans un voisinage de la diagonale de $\mathcal{C}_K(I) \times \mathcal{C}_K(I)$ de la forme $\mathcal{U}_R := \{(x,y) \in \mathcal{C}_K(I) \mid |x-y| < R\}$, pour un $R > 0$.*



La sous catégorie pleine de $\sigma_q-\mathrm{Mod}(\mathcal{A}_K(I))$ dont les objets sont admissibles sera notée

$$\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} . \tag{2.1.4.1}$$

Théorème 2.1.5 (Théorème principal)**.** Soient $q_0 \in \mathcal{Q}$ et $\mathrm{M} \in \sigma_{q_0} - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}$ défini dans une base par l'opérateur

$$\sigma_{q_0} - A_{q_0}(T) , \tag{2.1.5.1}$$

avec $A_{q_0}(T) \in GL_n(\mathcal{A}_K(I))$, alors il existe un sous groupe ouvert $U \subseteq \mathcal{Q}$, et une famille unique de matrices $\{A(q,T)\}_{q\in U}$ qui vérifient :

1. $A(q_0,T) = A_{q_0}(T)$;
2. $A(q,T) \in GL_n(\mathcal{A}_K(I))$ pour tout $q \in U$;
3. Pour tout $q \in U$ il existe un disque $\mathrm{D}^-(q,\tau_q)$ tel que la matrice $A(q,T)$ est analytique sur le domaine
$$\mathrm{D}^-(q,\tau_q) \times \mathcal{C}_K(I) ; \tag{2.1.5.2}$$
4. Pour tout point $c \in \mathcal{C}_K(I)$, la solution $Y(T,c)$ de l'équation 2.1.5.1 est solution aussi de chaque opérateur $\sigma_q - A(q,T)$ pour tout $q \in U$.

de plus $Y(T,c)$ coïncide avec la solution de Taylor de l'équation différentielle $\delta_1 - G(1,T)$ définie par

$$G(1,T) := \lim_{q\to 1} \frac{A(q,T)-1}{q-1} = \left(q\frac{d}{dq}A(q,T)\right)_{\big|q=1} \in M_n(\mathcal{A}_K(I)) . \tag{2.1.5.3}$$

Preuve : On démontre ce théorème en utilisant les propriétés de la résolvante : on a $Y(x,z) = Y(x,y) \cdot Y(y,z)$. Si $c \in \mathcal{C}_K(I)$, et si $Y(T,c)$ est la solution de l'équation près de c, alors la matrice cherchée est

$$A(q,T) := Y(qT,c) \cdot Y(T,c)^{-1} = Y(qT,T)\ . \tag{2.1.5.4}$$

qui converge dans un voisinage de $q = 1$. On démontre que le lieu de convergence de cette matrice est en realité un sous groupe ouvert de $\mathcal{Q}$. $\square$

Remarque 2.1.6. *Ce théorème montre que le module* M *reçoit de façon canonique l'action de* σ_q *pour tout* $q \in U$*.*

2.2 Les $\sigma-$modules analytiques

Définition 2.2.1. *Soit* $U \subseteq \mathcal{Q}$ *un ouvert et* $\langle U \rangle$ *le sous-groupe engendré par* U*. Un* $\sigma-$*module analytique sur* U *est un* $\mathcal{A}_K(I)-$*module libre de type fini, muni d'un morphisme de groupes :*

$$\sigma^{\mathrm{M}} : \langle U \rangle \xrightarrow[q \mapsto \sigma_q^{\mathrm{M}}]{} \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M}) \tag{2.2.1.1}$$

de sorte que dans une (et donc toute) base de M *on ait*

$$\sigma_q^{\mathrm{M}} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = A(q,T) \cdot \begin{pmatrix} \sigma_q(f_1) \\ \vdots \\ \sigma_q(f_n) \end{pmatrix}\ , \tag{2.2.1.2}$$

où la matrice $A(q,T)$ *vérifie les propriétés* 1, 2, 3 *du théorème 2.1.5. Les morphismes entre* $\sigma-$*modules analytiques sont les morphismes* $\mathcal{A}_K(I)-$*linéaires qui commutent à* σ_q *pour tout* $q \in U$*. On note par*

$$\sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{an}} \tag{2.2.1.3}$$

la catégorie des $\sigma-$*modules analytiques sur l'ouvert* U*. Si le module* M *vérifie dans une base (et alors dans toute base) la proprieté* 4 *du théorème 2.1.5, alors nous dirons que* M *est* constant *sur* U*. Nous dénotons par* $\sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{an,const}}$ *la sous-catégorie pleine de* $\sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{an}}$ *dont les objets sont constants. Nous dirons que* M *est* admissible *s'il est* constant *sur* U *et si sa résolvante converge dans un voisinage de type* $\mathcal{U}_R$*. Nous notons*

$$\sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \ \subseteq\ \sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{an,const}} \tag{2.2.1.4}$$

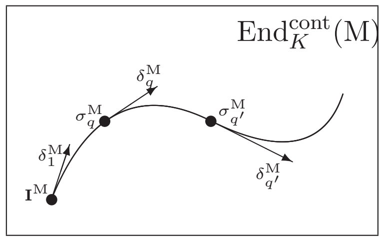
la sous-catégorie pleine de $\sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{an,const}}$ *des objets admissibles.*

2.3 Opérateurs $q-$tangents et $(\sigma, \delta)-$modules

Définition 2.3.1. *Pour tout $q \in \mathcal{Q}$ l'*opérateur $q-$tangent *sur $\mathcal{A}_K(I)$ est défini par*

$$\delta_q := \lim_{q' \to q} \frac{\sigma_{q'} - \sigma_q}{q' - q} = \sigma_q \circ \delta_1 \ , \tag{2.3.1.1}$$

où la limite est prise au sens de la topologie de la convergence simple de $\mathrm{End}_K^{\mathrm{cont}}(\mathcal{A}_K(I))$.



Lemme 2.3.2. On a $\delta_1 = T\frac{d}{dT}$ et $\delta_q := \sigma_q \circ \delta_1 = \delta_1 \circ \sigma_q$. En particulier, l'opérateur δ_q vérifie

$$\delta_q(fg) = \sigma_q(f) \cdot \delta_q(g) + \delta_q(f) \cdot \sigma_q(g) \tag{2.3.2.1}$$

pour tout $f, g \in \mathcal{A}_K(I)$.

2.3.1 $(\sigma, \delta)-$modules analytiques

Définition 2.3.3. *Soit $q \in \mathcal{Q}$. Nous notons*

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I)) \tag{2.3.3.1}$$

la catégorie dont les objets sont des $\mathcal{A}_K(I)-$modules libres de type fini munis d'un opérateur σ_q-semilinéaire $\sigma_q^{\mathrm{M}} : \mathrm{M} \xrightarrow{\sim} \mathrm{M}$ et d'un opérateur différentiel $\delta_1^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ avec la proprieté $\sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{M}} \circ \sigma_q^{\mathrm{M}}$. Les morphismes sont ceux qui commutent à σ_q^{M} et δ_1^{M}.

Un $(\sigma_q, \delta_q)-$module est dit admissible *si la résolvante de l'opérateur différentiel δ_1^{M} est égale à la résolvante de l'opérateur σ_q^{M} et si elle converge dans un voisinage de la diagonale de type $\mathcal{U}_R$. La catégorie des $(\sigma_q, \delta_q)-$modules admissibles sera note*

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} \ . \tag{2.3.3.2}$$

Remarque 2.3.4. *Comme $\delta_q = \sigma_q \circ \delta_1$ (cf. 2.3.1.1), la donnée du couple $(\sigma_q^{\mathrm{M}}, \delta_1^{\mathrm{M}})$ sur le module* M *est équivalente à la donnée du couple $(\sigma_q^{\mathrm{M}}, \delta_q^{\mathrm{M}})$, où*

$$\delta_q^{\mathrm{M}} := \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}} \ . \tag{2.3.4.1}$$

À partir d'un $\sigma-$module analytique (constant ou non) sur U on peut construire, pour tout $q \in U$, un $(\sigma_q, \delta_q)-$module, par les formules

$$\delta_q^{\mathrm{M}} := q \cdot \lim_{q' \to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q' - q} \ . \tag{2.3.4.2}$$

En particulier δ_1^{M} est égal à $(\sigma_q^{\mathrm{M}})^{-1} \circ \delta_q^{\mathrm{M}}$ et est une connexion sur M. On démontre que, si M est un $\sigma-$module analytique constant, alors la résolvante $Y(x, y)$ qui, par hypothèse, est solution de chaque σ_q^{M}, est aussi solution de l'opérateur δ_1^{M}.

Notation 2.3.5. Par la suite nous utiliserons aussi la notation

$$(\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \tag{2.3.5.1}$$

pour indiquer la catégorie $\sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}}$. Nous appelons ses objets les $(\sigma, \delta)-$modules analytiques admissibles sur U.

On obtient un foncteur

$$(\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \xrightarrow{\mathrm{Res}_q^U} (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} \ , \tag{2.3.5.2}$$

qui est pleinement fidèle. Donc si $U' \subset U$ le foncteur de restriction

$$(\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \xrightarrow{\mathrm{Res}_{U'}^U} (\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))_{U'}^{\mathrm{adm}} \ , \tag{2.3.5.3}$$

est aussi pleinemen fidèle. Par le théorème 2.1.5, on obtient alors une équivalence de catégories

$$\bigcup_U (\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \xrightarrow[\sim]{\bigcup_U \mathrm{Res}_q^U} (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} \ . \tag{2.3.5.4}$$

où U parcourt les ouverts qui contiennent q. Par ailleurs, on a les restrictions naturelles

$$\sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \xrightarrow{\mathrm{Res}_q^U} \sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} \ , \tag{2.3.5.5}$$

qui est pleinement fidèle seulement si q n'est pas racine de l'unité. Dans ce cas on obtient une équivalence de catégories

$$\bigcup_U \sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \xrightarrow[\sim]{\bigcup_U \mathrm{Res}_q^U} \sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} \ , \tag{2.3.5.6}$$

si et seulement si q n'est pas une racine de l'unité. On obtient alors le diagramme

$$\begin{array}{ccc} \bigcup_U \sigma - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} & = & \bigcup_U (\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))_U^{\mathrm{adm}} \\ {\scriptstyle \bigcup_U \mathrm{Res}_q^U} \downarrow & \odot & \wr \downarrow {\scriptstyle \bigcup_U \mathrm{Res}_q^U} \\ \sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} & \xleftarrow[\text{Oublide } \delta_q]{} & (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}. \end{array} \tag{2.3.5.7}$$

Si q n'est pas une racine de l'unité, alors toutes les flêches de ce diagramme sont des équivalences. Dans ce cas on peut oublier la donnée de l'operateur δ_q.

Par contre si $q = \xi$ est une racine de l'unité, alors la restriction de droite reste une équivalence et la restriction de gauche ne l'est pas. Dans ce cas l'information "au voisinage de ξ" est préservée par l'opérateur δ_ξ. On peut avoir l'impression que le foncteur "Oubli δ_ξ" est intéressant. Par contre on montre que dans le cas très important des équations avec structure de Frobenius le foncteur ne donne aucune information, car tout $\sigma_\xi-$module avec structure de Frobenius est trivial (i.e. somme directe de l'objet unité).

Notation 2.3.6. Pour ces raisons, désormais on va travailler uniquement avec les $(\sigma_q, \delta_q)-$modules, en sachant que, si q n'est pas une racine de l'unité, alors

$$\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} \;=\; (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}\ . \tag{2.3.6.1}$$

2.4 Déformation et Confluence

Soit $U \subseteq \mathcal{Q}$ un sous groupe ouvert. Notons par $(\sigma_q, \delta_q)-\mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U$ la sous-catégorie pleine de $(\sigma_q, \delta_q)-\mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}$ qui est l'image du foncteur 2.3.5.2, de façon que le foncteur (2.3.5.2) induise une équivalence

$$(\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U \xrightarrow[\sim]{\mathrm{Res}^U_q} (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U\ . \tag{2.4.0.2}$$

Pour tout $q, q' \in U$ on obtient par composition l'équivalence de déformation :

$$\begin{tikzcd}
& (\sigma, \delta) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U \arrow[rr, dotted, "\mathrm{Def}_{q,q'}", "\sim"'] & & (\sigma_{q'}, \delta_{q'}) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U
\end{tikzcd}$$

Dans le cas $q' = 1$ on appelle ce foncteur *confluence* :

$$\mathrm{Conf}_q : (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U \xrightarrow{\sim} \delta_1 - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}_U\ . \tag{2.4.0.3}$$

En particulier, si q n'est pas une racine de l'unité, alors le théorème 2.1.5, et l'identification 2.3.6.1, garantissent l'existence d'un tel sous-groupe. Donc pour tout q different d'une racine de l'unité on a un foncteur pleinement fidèle

$$\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}} \xrightarrow{\mathrm{Conf}_q} \delta_1 - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{adm}}\ . \tag{2.4.0.4}$$

2.5 Equations sur l'anneau de Robba

On adapte sans difficulté la théorie précédente a l'étude des équations sur l'anneau de Robba $\mathcal{R}_K$ et l'anneau $\mathcal{H}_K^{\dagger}$.

Soit $U \subseteq \mathrm{D}^-(1,1)$ un sous-groupe, et soit $\mathrm{M} \in \sigma - \mathrm{Mod}(\mathcal{R}_K)_U^{\mathrm{an,const}}$. Par définition M provient par extension des scalaires d'un module M_ε sur $\mathcal{A}_K(]1-\varepsilon,1[)$. De façon complètement analogue au cas des équations différentielles, on définit, pour $1-\varepsilon < \rho < 1$, la *rayon générique* $Ray(\mathrm{M}_\varepsilon, \rho)$ de M_ε.

Lemme 2.5.1. Soit $r := \lim_{\rho \to 1^-} Ray(\mathrm{M}_\varepsilon, \rho)$. Alors M est admissible sur $\mathrm{D}^-(1,r)$ (i.e. quitte à diminuer ε, M_ε est admissible sur $\mathrm{D}^-(1,r)$).

Définition 2.5.2. *On note*

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \tag{2.5.2.1}$$

$$(\textit{resp.} \qquad (\sigma, \delta) - \mathrm{Mod}(\mathcal{R}_K)^{[r]}_{\mathrm{D}^-(1,1)}) \tag{2.5.2.2}$$

la sous catégorie pleine de

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{adm}} \tag{2.5.2.3}$$

$$(\textit{resp.} \qquad (\sigma, \delta) - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{adm}}_{\mathrm{D}^-(1,r)} \quad) \tag{2.5.2.4}$$

formée par les $(\sigma_q, \delta_q)-$modules (resp. analytiques constants) qui vérifient

$$\lim_{\rho \to 1^-} Ray(\mathrm{M}_\varepsilon, \rho) \geq r \; . \tag{2.5.2.5}$$

Le corollaire suivant est alors une conséquence immédiate de la section 2.4

Corollaire 2.5.3. Pour tout $q \in \mathrm{D}^-(1,r)$ on a une équivalence

$$(\sigma, \delta) - \mathrm{Mod}(\mathcal{R}_K)^{[r]}_{\mathrm{D}^-(1,r)} \xrightarrow[\sim]{\mathrm{Res}_q^{\mathrm{D}^-(1,r)}} (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \; . \tag{2.5.3.1}$$

De plus pour tous $q, q' \in \mathrm{D}^-(1,1)$ (resp. $q, q' \notin \boldsymbol{\mu}(\mathrm{D}^-(1,1))$), et tout r qui vérifient

$$\max(|q-1|, |q'-1|) < r \leq 1 \; , \tag{2.5.3.2}$$

on a l'équivalence de déformation

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\mathrm{Def}_{q,q'}} (\sigma_{q'}, \delta_{q'}) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \; . \tag{2.5.3.3}$$

Rappelons encore que, si q n'est pas une racine de l'unité, alors le foncteur

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\text{“Oublie } \delta_q\text{”}} \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \tag{2.5.3.4}$$

est une équivalence.

2.6 Quasi-unipotence des $\sigma-$modules admissibles

On généralise a tout $\sigma-$module (admissible) avec structure de Frobenius le théorème de monodromie locale $p-$adique. Ce théorème est obtenu par "*déformation*" du théorème de monodromie locale $p-$adique pour les équations différentielles.

Nous supposons désormais que K est de valuation discrète et que son corps résiduel k est parfait. Cette hypothèse intervient dans le théorème de monodromie locale $p-$adique pour les équations différentielles.

2.6.1 Monodromie locale : quasi unipotence des modules avec structure de Frobenius

Les extensions finies séparables du corps $k((t))$ se relèvent bijectivement, par henselianité, à des extensions finies non ramifiées du corps $\mathcal{E}_K^\dagger$. En tensorisant avec $\mathcal{R}_K$ on obtient une classe d'extension de l'anneau $\mathcal{R}_K$ dites spéciales.

$$\left\{ \begin{array}{l} \text{Extensions finies} \\ \text{séparables de } k((t)) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{Extensions finies} \\ \text{speciales de } \mathcal{R}_K \end{array} \right\} . \qquad (2.6.0.5)$$

Definition 2.6.1. Un module différentiel est dit *quasi-unipotent* s'il est *trivialisé* par une extension de $\mathcal{R}_K$ de la forme $\mathcal{R}'[\log(T)]$ où $\mathcal{R}'$ est une extension finie spéciale de $\mathcal{R}_K$.

Rappelons que M est "trivialisé" par l'anneau $\mathcal{R}'[\log(T)]$, si $\mathrm{M} \otimes_{\mathcal{R}_K} \mathcal{R}'[\log(T)]$ est isomorphe à une somme directe itérée de l'objet unité (dans la catégorie des modules différentiels sur $\mathcal{R}'[\log(T)]$).

On sait que les modules différentiels *quasi-unipotents* ont une structure de Frobenius, la réciproque est vraie quitte à permettre une extension finie des constantes :

Théorème 2.6.2 (monodromie locale $p-$adique)**.** *Si M est un module différentiel avec structure de Frobenius sur $\mathcal{R}_K$, alors il existe une extension finie K'/K telle que $\mathrm{M} \otimes_K K'$ est quasi-unipotent.*

De façon complètement analogue à la situation des équations différentielles, on donne la définition de *structure de Frobenius* et de *quasi-unipotence* pour les $(\sigma_q, \delta_q)-$modules et pour les $(\sigma, \delta)-$modules sur un sous groupe ouvert $U \subseteq \mathrm{D}^-(1,1)$. On généralise alors, sans peine, le théorème de la monodromie locale $p-$adique :

Théorème 2.6.3 (monodromie locale $p-$adique (généralisée))**.** *Si M est un $(\sigma, \delta)-$module sur $\mathrm{D}^-(1,1)$ avec structure de Frobenius sur $\mathcal{R}_K$, alors il existe une extension finie K'/K telle que $\mathrm{M} \otimes_K K'$ est quasi-unipotent. En particulier, si $q \in \mathrm{D}^-(1,1)$ n'est pas une racine de l'unité, alors tout σ_q-module avec structure de Frobenius est quasi-unipotent.*

Deuxième partie

Rank One Solvable $p-$adic Differential Equations and Finite Abelian Characters via Lubin-Tate Groups.

Abstract

We introduce a new class of exponentials of Artin-Hasse type, called $\boldsymbol{\pi}$-exponentials. These exponentials depend on the choice of a generator $\boldsymbol{\pi}$ of the Tate module of a Lubin-Tate group $\mathfrak{G}$ over $\mathbb{Q}_p$. They arise naturally as solutions of solvable differential modules over the Robba Ring. If $\mathfrak{G}$ is isomorphic to $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$, we develop methods to test their over-convergence, and get in this way a more strong version of the Frobenius structure theorem for differential equations. We define a natural transformation of the Artin-Schreier complex into the Kummer complex. This provides an explicit generator of the Kummer unramified extension of $\mathcal{E}^{\dagger}_{K_\infty}$, whose residue field is a given Artin-Schreier extension of $k(\!(t)\!)$, where k is the residue field of K. We compute then explicitly the group, under tensor product, of isomorphism classes of rank one solvable differential equations. Moreover, we get a canonical way to compute the rank one $\varphi-$module over $\mathcal{E}^{\dagger}_{K_\infty}$ attached to a rank one representation of $\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))$, defined by an Artin-Schreier character.

Chapitre 1

Introduction

The aim of this paper is to make the theory of rank one solvable differential equations over the Robba ring $\mathcal{R}_K$ (cf. 2.1.1) as explicit as possible, where $(K, |.|)$ is a complete ultrametric field with residue field k. It is known (cf. [And02], [Meb02] and [Ked04]) that, under some restrictions on K and k, a solvable $p-$adic differential module over $\mathcal{R}_K$ becomes unipotent, after pull back on a covering of $\mathcal{R}_K$, coming from a separable extension of $\mathrm{E} := k(\!(t)\!)$. In particular, in [Meb02] the aim is to express this module as extension of rank one modules, and get a $p-$adic analogous of the classical Turritin's theorem for $K(\!(T)\!)-$differential modules.

Let $\partial_T := T\frac{d}{dT}$. We shall answer to the following questions :

1. When a given differential equation
$$L = \partial_T + g(T) \ , \quad g(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K \tag{1.0.3.1}$$
is solvable ?
2. What is its irregularity ?
3. Can we describe explicitly the group (under tensor product) $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)$ of isomorphism classes of rank one solvable differential equations over $\mathcal{R}_K$?
4. How this differential equations changes by Artin-Schreier extensions ? In particular what is the family of rank one solvable modules becoming trivial after a given separable extension of $\mathrm{E} = k(\!(t)\!)$?
5. Let $\mathrm{E}^{\mathrm{sep}}$ be the separable closure of E. What is explicitly the rank one $\varphi-$module attached to an Artin-Schreier character (or rank one representation) of $\mathrm{G_E} := \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$ via the theory of Fontaine-Katz ? In particular what is the solvable equation attached to this $\varphi-$module ?

1.1 Robba exponentials

The first example of irregular solvable differential equation was given by Dwork with the function

$$\exp(\pi T^{-1}) \ , \tag{1.1.0.2}$$

which is the Taylor solution at ∞ of the irregular operator $\partial_T + \pi T^{-1}$, where π is a solution of the equation $X^{p-1} = -p$. Dwork shows that the exponential

$$\vartheta(T^{-1}) := \exp(\pi(T^{-p} - T^{-1})) \tag{1.1.0.3}$$

is over-convergent (i.e. converges for $|T| > 1 - \varepsilon$, for some $\varepsilon > 0$). This provides the so called "Frobenius structure" isomorphism between $\partial_T + \pi T^{-1}$, and $\partial_T + \pi T^{-p}$.

1.1.1 The limits of the work of Robba

In [Rob85], Robba generalizes the example of Dwork by producing a class of exponentials, here called $E_m(T)$, commonly known as Robba's exponentials. Namely Robba shows that, for all number π_0 such that $|\pi_0| = |p|^{\frac{1}{p-1}}$, there exists a sequence $\alpha_1, \alpha_2, \ldots$ such that, for all $m \geq 1$, the exponential

$$E_m(T^{-1}) = \exp\Big(\pi_0\Big(\frac{T^{-p^m}}{p^m} + \alpha_1 \frac{T^{-p^{m-1}}}{p^{m-1}} + \cdots + \alpha_m T^{-1}\Big)\Big) \tag{1.1.0.4}$$

converges in the disk $|T| > 1$, and hence the operator

$$L = \partial_T + \pi_0(T^{-p^m} + \alpha_1 T^{-p^{m-1}} + \cdots + \alpha_m T^{-1}) \ , \tag{1.1.0.5}$$

with $E_m(T^{-1})$ as solution, is solvable at $\rho = 1$. Moreover Robba shows the necessity of the condition $|\pi_0 \alpha_i| = |p|^{\frac{1}{p^i - 1}}$, for all $i \geq 0$. This construction leads Robba to define the $p-$adic irregularity of a solvable differential equation as the slope at 1^- of the radius of convergence (cf. 2.4.5).

But the Robba's construction is not sufficient for two reasons. The first one is that the numbers α_i are obtained as intersection of a decreasing sequence of disks, and then the field K must be spherically complete. The second reason is that Robba was not able to prove the over-convergence of $E_m(T^{-p})/E_m(T^{-1})$, since the α_i's are essentially unknown.

1.1.2 Matsuda's progress

These problems are solved by S.Matsuda in [Mat95]. He simplifies remarkably the proof of Robba by using the Artin-Hasse exponential. The idea of introducing the Artin-Hasse exponential is due to Dwork (cf. [Dwo82, 21.1]), and Robba (cf. [Rob85, 10.12]) itself. Matsuda shows that, if ξ_{m+1} is a primitive $p^{m+1}-$th root of 1, and if $\xi_{m+1-j} := \xi_{m+1}^{p^j}$, then we can choose $\pi_0 = \xi_1 - 1$ and $\alpha_i = (\xi_{i+1} - 1)/(\xi_1 - 1)$, for all $i \geq 1$. Then

$$E_m(T^{-1}) = \exp\Big((\xi_1 - 1)\frac{T^{-p^m}}{p^m} + (\xi_2 - 1)\frac{T^{-p^{m-1}}}{p^{m-1}} + \cdots + (\xi_{m+1} - 1)T^{-1}\Big) . \tag{1.1.0.6}$$

Matsuda proves also that, if $p \neq 2$, then the exponential $E_m(T^{-p})/E_m(T^{-1})$ is over-convergent. He obtains these results by a quite complicate, but elementary, explicit estimation of the valuation of the coefficients of this exponential.

For the first time we see, in the paper of Matsuda, the algebraic nature of these analytic exponentials. Indeed, if $\alpha : \mathrm{G_E} \to \Lambda^\times$ is a character of $\mathrm{G_E}$ into a finite extension $\Lambda/\mathbb{Q}_p$, such that $\alpha(\mathrm{G_E})$ is finite, then Matsuda shows that the irregularity of the differential equation, attached to the $\varphi - \nabla-$module over $\mathcal{E}_K^\dagger$ defined by α, is equal to the Swan conductor of α.

1.1.3 Chinellato's algorithm

Independently from Matsuda, D.Chinellato, under the direction of Dwork, gets a new algorithm showing the existence of the α_is (cf. [Chi02]).

1.1.4 Our contribution

Even with the great progress given by Matsuda, (André, Kedlaya, Krew, Mebkhout, Tsuzuki and others...) the questions given in 1 are still open, and are the object of this paper.

We generalize, and improve, the technics of Matsuda and Chinellato, by introducing the Lubin-Tate theory. We recall that the Artin-Hasse exponential $E(-,T)$ is the group morphism $E(-,T) : \mathbf{W}(B) \to 1 + TB[[T]]$, functorial on the ring B, sending the Witt vector $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(B)$ into the series

$$E(\boldsymbol{\lambda}, T) = \exp\Big(\phi_0 T + \phi_1 \frac{T^p}{p} + \phi_2 \frac{T^{p^2}}{p^2} + \cdots\Big) , \tag{1.1.0.7}$$

where $\langle \phi_0, \phi_1, \ldots \rangle \in B^{\mathbb{N}}$ is the phantom vector of $\boldsymbol{\lambda}$ (cf. 2.7.0.2). If $B = \mathcal{O}_{K^{\mathrm{alg}}}$, then this exponential has bounded coefficients, and hence converges at least for $|T| < 1$. Given a Frobenius automorphism of $\mathbb{Z}_p[[X]]$, that is a series $P(X) \in X\mathbb{Z}_p[[X]]$ lifting $X^p \in \mathbb{F}_p[[X]]$, we consider a sequence $\{\pi_j\}_{j \geq 0}$ in $\mathcal{O}_{K^{\mathrm{alg}}}$, such that $P(\pi_0) = 0$, and $P(\pi_{j+1}) = \pi_j$, for all $j \geq 0$. Then we provide, for all $m \geq 0$, a Witt vector $[\pi_m] \in \mathbf{W}(\mathcal{O}_{K^{\mathrm{alg}}})$, whose phantom vector is $\langle \pi_m, \ldots, \pi_0, 0, 0, \ldots \rangle$. In this way, we obtain a large class of exponentials of type "Robba" :

$$E_m(T) := E([\pi_m], T) = \exp\Big(\pi_m T + \pi_{m-1} \frac{T^p}{p} + \cdots + \pi_0 \frac{T^{p^m}}{p^m}\Big) . \tag{1.1.0.8}$$

We show then that *the radius of convergence of these exponentials is* 1 *if and only if* $P(X)$ *is a Lubin-Tate series (cf. 2.9.0.3), and defines then a Lubin-Tate group* $\mathfrak{G}_P$. In this case $\boldsymbol{\pi} := (\pi_j)_{j\geq 0}$ is a generator of the Tate module of $\mathfrak{G}_P$ (cf. 2.9.5). If $\mathfrak{G}_P$ is the formal multiplicative group $\widehat{\mathbb{G}}_m$, that is if $P(X) = (X+1)^p - 1$, then we recover the Matsuda's exponentials 1.1.0.6. On the other hand, if $P(X) = pX + X^p$, we recover, for $m = 0$, the Dwork's exponential. Observe that, in the case considered by Dwork, the formal group $\mathfrak{G}_P$ is isomorphic, but not equal, to $\widehat{\mathbb{G}}_m$.

Furthermore, we show that $E_m(T^p)/E_m(T)$ *is over-convergent, for all* $m \geq 0$, *if and only if* $\mathfrak{G}_P$ *is isomorphic (but not necessary equal) to* $\widehat{\mathbb{G}}_m$. This is the reason of the over-convergence of the exponentials $E_m(T^p)/E_m(T)$ of Matsuda and Dwork.

From this starting point we develop the explicit link between abelian characters of $\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))$ and rank one solvable differential equations over $\mathcal{R}_K$, and examine various applications.

1.2 Organization of the paper

In Sections 2.1, 2.2, 2.7, 2.8, and 2.9 we get the definitions and recall some facts used in the sequel.

In Section 3.1 we define some canonical Witt vectors with coefficients in $\mathbb{Z}_p[[X]]$, and show their properties with respect to the Artin-Hasse exponential. In section 3.3 we introduce a new class of exponentials called $\boldsymbol{\pi}-$exponentials (cf. 3.3.1.1), and show their main properties with respect to the convergence/over-convergence.

1.2.1 An explicit Kummer generator

In Section 4, we give the first application. Fix a Lubin-Tate group $\mathfrak{G}_P$ isomorphic to $\widehat{\mathbb{G}}_m$, and a generator $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$ of the Tate module. Let L be a complete discrete valued field, with residue field k_L, and let $\varphi : L \to L$ be a lifting of the Frobenius $x \mapsto x^p$ of k_L. Let $L_m := L(\xi_m)$, where ξ_m is a primitive $p^{m+1}-$th root of 1. It is well known that we have the henselian bijection

$$\{\text{Finite unramified extensions of } L\} \xrightarrow{\sim} \{\text{Finite separable extensions of } k_L\}.$$

We shall describe an inverse of this map. Let k'/k_L be a finite cyclic abelian extension of degree d, and let L'/L be the corresponding unramified extension. If $(d,p) = 1$, and if k_L contains the $d-$th roots of 1, then k'/k_L is of Kummer type, and hence $L' = L(\theta)$, where θ is the Teichmüller of a Kummer generator $\bar{\theta} \in k'$.

On the other hand, if $d = p^m$, then k' is of Artin-Schreier type (cf. 2.8.3), and it is generated, over k_L, by (the entries of) a Witt vector $\bar{\boldsymbol{\nu}} \in \mathbf{W}_m(k_L^{\mathrm{sep}})$, which is solution of an equation of the type $\bar{\mathrm{F}}(\bar{\boldsymbol{\nu}})-\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\lambda}}$, where $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_L)$ is a so called Witt vector “defining” k'. In this case L'_m/L_m is again Kummer, since all cyclic extensions of L_m whose degree is p^m are Kummer. Now choose an arbitrary lifting $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$ of $\bar{\boldsymbol{\lambda}}$, and solve the equation $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$, $\boldsymbol{\nu} \in \mathbf{W}_m(\widehat{L}^{\mathrm{unr}})$. Then a Kummer generator θ of L'_m is given by the value at $T = 1$ of a certain $\boldsymbol{\pi}-$exponential, called $\theta_{p^m}(\boldsymbol{\nu}, T)$, (cf. 4.1.11).

The Artin-Schreier theory and Kummer theory are given by some *complexes* computing the Galois cohomology. Roughly speaking, we shall obtain a natural transformation of functors which “deforms” the Artin-Schreier complex into the Kummer complex and induces a quasi isomorphism (functorially on the unramified extensions of L) :

$$\begin{tikzcd}
1 \arrow[r] & (L_m)^\times \arrow[r, "x \mapsto x^{p^{m+1}}"] & (L_m)^\times \arrow[r] & 1 \\
0 \arrow[r] & \mathbf{W}_m(k_L) \arrow[r, "\bar{\mathrm{F}}-1"'] \arrow[u, dotted, "\theta"] & \mathbf{W}_m(k_L) \arrow[r] \arrow[u, dotted, "e^{p^m}"] & 0\,.
\end{tikzcd} \tag{1.2.0.9}$$

Actually, such a natural transformation can not exist, because the Artin-Schreier complex is in characteristic p, and the Kummer complex is in characteristic 0. As a matter of fact, we lift the Artin-Schreier complex in characteristic 0 and deform it into the Kummer complex, by using *the value at $T = 1$ of some over-convergent $\boldsymbol{\pi}-$exponentials called $\theta_{p^m}(-, T)$ and $\mathrm{e}_{p^m}(-, T)^{p^{m+1}}$* (see diagram 4.1.9.1). This provides a well defined morphism between the cohomologies.

Under some assumptions on K (cf. 4.2.1.1), even if the field $L = \mathcal{E}^\dagger_K$ is not complete, we show that this diagram exists for $\mathcal{E}^\dagger_K$ and its finite unramified extensions (cf. 4.2.5). The commutative diagram is then :

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_{p^{m+1}} \arrow[r] & (\mathcal{E}^\dagger_{K_m})^\times \arrow[r, "f \mapsto f^{p^{m+1}}"] & (\mathcal{E}^\dagger_{K_m})^\times \arrow[r, "\delta_{\mathrm{Kum}}"] & \mathrm{H}^1(\mathrm{G}_{\mathcal{E}^\dagger_{K_m}}, \boldsymbol{\mu}_{p^{m+1}}) \arrow[r] & 1 \\
 & \mathbf{W}_m(\mathcal{O}_K^{\sigma=1}) \arrow[r, hook] \arrow[u, two heads] \arrow[d, two heads] & \mathbf{W}_m(\mathcal{O}^\dagger_K) \arrow[r, "\varphi-1"'] \arrow[u, "\theta_{p^m}(-,1)"] \arrow[d, two heads] & \mathbf{W}_m(\mathcal{O}^\dagger_K) \arrow[u, "\mathrm{e}_{p^m}(-,1)^{p^{m+1}}"] \arrow[d, two heads] & & \\
0 \arrow[r] & \mathbb{Z}/p^{m+1}\mathbb{Z} \arrow[r] \arrow[uu, bend left, dotted, "\wr"] & \mathbf{W}_m(\mathrm{E}) \arrow[r, "\bar{\mathrm{F}}-1"'] & \mathbf{W}_m(\mathrm{E}) \arrow[r, "\delta"'] & \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \arrow[r] \arrow[uu, dotted, "\bar{\mathrm{e}}:=\overline{\mathrm{e}_{p^m}(-,1)^{p^{m+1}}}"'] & 0
\end{tikzcd} \tag{1.2.0.10}$$

where $E := k((t))$, $\mathrm{G}_{\mathrm{E}} := \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$ and $\mathrm{G}_{\mathcal{E}^\dagger_{K_m}} := \mathrm{Gal}(\mathcal{E}^{\dagger,\mathrm{alg}}_{K_m}/\mathcal{E}^\dagger_{K_m})$. We precise the kernel and the image of the morphism $\bar{\mathrm{e}}$ between the cohomologies. If $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_m(k((t)))$ is a Witt vector defining an Artin-Schreier separable extension of $k((t))$, then (up to add the $p^{m+1}-$th roots of 1) a generator of the corresponding unramified extension of $\mathcal{E}^\dagger_{K_m}$ is given by $\theta_{p^m}(\boldsymbol{\nu}, 1)$, where $\boldsymbol{\nu} \in \mathbf{W}_m(\widehat{\mathcal{E}}_K^{\mathrm{unr}})$ is a solution of the equation $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{f}(T)$, and $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathcal{O}_K^\dagger)$ is an arbitrary lifting of $\overline{\boldsymbol{f}}(t)$.

1.2.2 The Kummer generator over $\mathcal{E}_K^\dagger$ relations with differential equations

Let $K_m := K(\pi_m) = K(\xi_m)$, $K_\infty := \cup_m K_m$, and let k_m be the residue field of K_m. In Sections 5, 5.2, and 5.4 we classify all solvable rank one differential equations over $\mathcal{R}_{K_\infty}$. The key point is the following equality, arising from the diagram 4.2.5.1, and useful to describe the Kummer generator $\theta_{p^m}(\boldsymbol{\nu}, 1)$:

$$\theta_{p^m}(\boldsymbol{\nu}, 1)^{p^{m+1}} = \mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)^{p^{m+1}}. \tag{1.2.0.11}$$

The expression $\mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)$ has no meaning, because $\mathrm{e}_{p^m}(-, Z)$ is not overconvergent as function of Z. We make sense to this symbol in some cases : in Sections 5.2 we define a class of exponentials of the form

$$\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1) = \exp\left(\pi_m \phi_0^-(T) + \pi_{m-1}\frac{\phi_1^-(T)}{p} + \cdots + \pi_0 \frac{\phi_m^-(T)}{p^m}\right) , \tag{1.2.0.12}$$

where $\boldsymbol{f}^-(T) \in \mathbf{W}_m(T^{-1}\mathcal{O}_K[T^{-1}])$, and $\langle \phi_0^-(T), \cdots, \phi_m^-(T)\rangle \in (T^{-1}\mathcal{O}_K[T^{-1}])^{m+1}$ is its phantom vector. This exponential is $T^{-1}-$adically convergent and defines a series in $1 + T^{-1}\mathcal{O}_{K_m}[[T^{-1}]]$, whose $p^{m+1}-$th power lies in $\mathcal{R}_{K_m}$.

These Witt vectors correspond to totally ramified Artin-Schreier extensions of $\mathrm{E} := k(\!(t)\!)$. The exponential 1.2.0.12 is then the desired Kummer generator $\theta_{p^m}(\boldsymbol{\nu}, 1)$ of the extension of $\mathcal{E}_K^\dagger$ corresponding to this totally ramified extension of $k(\!(t)\!)$. Moreover $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)$ is at the same times the solution of a solvable differential equation.

1.2.3 Explicit correspondence between Abelian characters of $\mathrm{G_E}$ and differential equations

We state then the explicit bijection between the abelian Galois theory for $\mathrm{E} = k(\!(t)\!)$, and the theory of rank one differential equations over $\mathcal{R}_{K_\infty}$. Matsuda, in [Mat95], has pointed out, under some restrictions, that such a correspondence should exist. We go further by removing any restrictions, improving his methods, and by making the correspondence more explicit (cf. 5.2.2, 5.2.4). Namely we introduce the fundamental exponential $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)$. We show that every rank one differential module M over $\mathcal{R}_{K_\infty}$ comes, by scalar extension, from a module $M_{]0,\infty]}$, over $K_\infty[T^{-1}]$, and the Taylor solution, at ∞, is (in some basis) of the form

$$T^{a_0} \cdot \mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1) , \tag{1.2.0.13}$$

for some $m \geq 0$, $a_0 \in \mathbb{Z}_p$, and $\boldsymbol{f}^-(T) \in \mathbf{W}_m(T^{-1}\mathcal{O}_{K_m}[T^{-1}])$. Moreover, the isomorphism class of M depends only on the class of a_0 in $\mathbb{Z}_p/\mathbb{Z}$, and on the

Artin-Schreier character α defined by the reduction of $\boldsymbol{f}^-(T)$ in $\mathbf{W}_m(k_m(\!(t)\!))$. Suppose that a_0 belongs to $\mathbb{Z}_{(p)} := \mathbb{Q} \cap \mathbb{Z}_p$. Then a_0 corresponds to the moderate extensions of $\mathrm{E} = k(\!(t)\!)$, generated by t^{a_0}. On the other hand, $\boldsymbol{f}^-(T)$ corresponds to the Artin-Schreier extension given by (the kernel of) the Artin-Schreier character defined by the reduction $\boldsymbol{f}^-(T)$. We recover in this way the well known bijection

$$\left\{ \begin{array}{c} \text{Rank one} \\ \text{characters of } \mathcal{I}_{k_\infty(\!(t)\!)} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Isomorphism classes of rank one} \\ \text{solvable differential equations} \\ \text{over } \mathcal{R}_{K_\infty} \text{with rational residue} \end{array} \right\} , \tag{1.2.0.14}$$

where $\mathcal{I}_{k_\infty(\!(t)\!)}$ is the inertia subgroup of $\mathrm{Gal}(k_\infty(\!(t)\!)^{\mathrm{sep}}/k_\infty(\!(t)\!))$.

1.2.4 Frobenius structure for $\boldsymbol{\pi}-$exponentials

The central point is that the following $\boldsymbol{\pi}-$exponential is over-convergent

$$\frac{\mathrm{e}_{p^m}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T),1)}{\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)} = \mathrm{e}_{p^m}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T) - \boldsymbol{f}^-(T),1) , \tag{1.2.0.15}$$

where $\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T)$ is an arbitrary lifting of the reduction $\bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t)) \in \mathbf{W}_m(k_m(\!(t)\!))$. If a lifting of the $p-$th power map $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ is given, then this result implies the usual Frobenius structure theorem. Observe that we do not need the existence of φ (cf. 3.6.2), because actually the isomorphism class of a given module M depends only on the reduction of $\boldsymbol{f}^-(T)$ in characteristic p. This represents a progress in two directions, with respect to the analogous theorem of [CC96] : firstly we do not suppose k perfect, and secondly we get a precise description of the isomorphism class of M. In particular, we precise that, if $a_0 = 0$, then "the order" (cf. 2.5.8) of the Frobenius structure is 1 (cf. 6.2.3).

In section 5.4 we prove these theorems essentially by reducing the study to the "elementary" $\boldsymbol{\pi}-$exponentials called $s-$co-monomials. These exponentials are studied in detail in sections 3.3.

1.2.5 Complements

We give then some complements (Sections 6.1,6.2,6.3,6.4,6.5). In particular, in the Section 6.2, we compute the group of rank one solvable equations killed by a given Artin-Schreier extension and answer than to the question (4) of 1. In Section 6.3 we extend the definition of our $\boldsymbol{\pi}-$exponentials to a more large class of differential equations, and we provide an algorithm (see proof of 6.3.9), which gives a *criterion of solvability*, (cf. 6.3.15). In particular we show that *there is no irregular rank one equations if $K/\mathbb{Q}_p$ is unramified* (cf. 6.3.18). This answers to the question (1) of 1. Then we *compute the irregularity* in some classical cases (cf. 6.4). We describe the tannakian group of the category whose objects are successive extension of rank one solvable modules. We remove the hypothesis "K is spherically complete" present in the literature. In section 6.6 we compute the $\phi - \nabla -$module attached to a character with finite image of $\mathrm{G_E}$. This answer to question (5) of 1.

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Chapitre 2

Definitions and Notations.

2.1 General notations

Let $p > 0$ be a fixed prime number. Let $(K, |.|)$ be a complete valued field containing $(\mathbb{Q}_p, |.|)$. For every valued extension field L/K, we denote by $\mathcal{O}_L = \{x \in L \mid |x| \leq 1\}$ the ring of integers of L, by $\mathfrak{p}_L = \{x \in L \mid |x| < 1\}$ its maximal ideal and by $k_L = \mathcal{O}_L/\mathfrak{p}_L$ its residue field. We set $k := k_K$. K^{alg} will be a fixed algebraic closure of K, and $k^{\mathrm{alg}} := k_{K^{\mathrm{alg}}}$ will be its residue field. Ω/K will be a spherically complete extension field containing K^{alg}, satisfying $|\Omega| = \mathbb{R}_{\geq}$, and whose residue field k_Ω/k is not algebraic. We set

$$\omega := |p|^{\frac{1}{p-1}}.$$

We denote by $\partial_T := T\frac{d}{dT}$ the usual derivation. For all ring R we denote by $R^\times$ the group of invertible elements in R.

2.1.1 Analytic functions and the Robba ring.

For every (non vacuous) interval $I \subseteq \mathbb{R}_{\geq}$ we denote by $\mathcal{A}_K(I)$ (or simply $\mathcal{A}(I)$) the ring of convergent analytic functions on I, that is the ring formed by the series $f(T) = \sum_{i\in\mathbb{Z}} a_i T^i$, $a_i \in K$, such that $\lim_{i\to\pm\infty} |a_i|\rho^i = 0$, for all $\rho \in I$. The topology of $\mathcal{A}_K(I)$ is defined by the family of absolute values

$$|f(T)|_\rho := \max_{i\in\mathbb{Z}} |a_i|\rho^i \ , \qquad \forall\, \rho \in I. \tag{2.1.0.1}$$

The Robba's ring $\mathcal{R}_K$ (or simply $\mathcal{R}$) is the ring of germs of convergent analytic functions at 1^-, that is $\mathcal{R}_K = \cup_\varepsilon \mathcal{A}_K(]1-\varepsilon, 1[)$, with the limit topology. Let $\mathcal{E}_K$ be the Amice ring. The elements of $\mathcal{E}_K$ are series $f(T) := \sum_{i\in\mathbb{Z}} a_i T^i$, such that $\lim_{i\to-\infty} |a_i| = 0$, and $\sup_i |a_i| < +\infty$. If the valuation on K is not discrete we may have $|a_i| < \sup_i |a_i|$, for all $i \in \mathbb{Z}$. $\mathcal{E}_K$ is complete with respect to the topology given by the absolute value $|f(T)|_1 := \sup_i |a_i|$. We denote by $\mathcal{O}_{\mathcal{E}_K} := \{f \in \mathcal{E} \mid |f|_1 \leq 1\}$ its ring of integers.

Definition 2.1.1. For all algebraic extension H/K we set

$$\mathcal{A}_H(I) := \mathcal{A}_K(I) \otimes_K H \ , \quad \mathcal{R}_H := \mathcal{R}_K \otimes_K H \ , \quad \mathcal{E}_H := \mathcal{E}_K \otimes_K H \ . \quad (2.1.1.1)$$

Since K is algebraically closed in $\mathcal{A}_K(I)$ (resp. $\mathcal{R}_K$, $\mathcal{E}_K$) then $\mathcal{A}_H(I)$ (resp. $\mathcal{R}_H$, $\mathcal{E}_H$) is a domain. All $p-$adic differential equations over $\mathcal{A}_H(I)$ (resp. $\mathcal{R}_H, \mathcal{E}_H$) come, by scalar extension, from an equation over $\mathcal{A}_L(I)$ (resp. $\mathcal{R}_L$, $\mathcal{E}_L$) with L/K finite. This will justify the definition 2.4.7.

Definition 2.1.2. For all formal series $f(T) = \sum_{i\in\mathbb{Z}} a_i T^i$ we define

$$f^-(T) := \sum_{i\leq -1} a_i T^i \quad , \qquad f^+(T) := \sum_{i\geq 1} a_i T^i \ , \qquad (2.1.2.1)$$

we have $f(T) = f^-(T) + a_0 + f^+(T)$.

Definition 2.1.3. For all algebraic extension H/K, let

$$\mathcal{E}^\dagger_{H,T} := \mathcal{R}_H \cap \mathcal{E}_H.$$

We denote by $\mathcal{O}^\dagger_{H,T} := \mathcal{O}_{\mathcal{E}_H} \cap \mathcal{R}_H$. If no confusion is possible we will write $\mathcal{E}^\dagger_H$ (resp.$\mathcal{O}^\dagger_H$) instead of $\mathcal{E}^\dagger_{H,T}$ (resp. $\mathcal{O}^\dagger_{H,T}$).

Remark 2.1.4. The quotients $\mathcal{O}_{\mathcal{E}_K}/\{f \in \mathcal{O}_{\mathcal{E}_K} : |f|_1 < 1\}$ or $\mathcal{O}^\dagger_K/\{f \in \mathcal{O}^\dagger_K : |f|_1 < 1\}$ are reduced to $k(\!(t)\!)$ if and only if the valuation on K is discrete. Nevertheless, if the valuation is not discrete, the rings $\mathcal{O}_{\mathcal{E}_K}$ and $\mathcal{O}^\dagger_K$ are always local, their maximal ideals $\mathfrak{p}_{\mathcal{O}_{\mathcal{E}_K}}$ and $\mathfrak{p}_{\mathcal{O}^\dagger_K}$ are formed by series $f = \sum_i a_i T^i$ such that $|a_i| < 1$, for all $i \in \mathbb{Z}$, observe that, since the valuation is not discrete, this condition do not implies that $|f|_1 < 1$. The residue fields $\mathcal{O}^\dagger_K/\mathfrak{p}_{\mathcal{O}^\dagger_K}$, and $\mathcal{O}_{\mathcal{E}_K}/\mathfrak{p}_{\mathcal{O}_{\mathcal{E}_K}}$ are actually always equals to $k(\!(t)\!)$.

2.2 Generalities on rank one differential equations

2.2.1 The strategy of Manin

Let $D : \mathrm{B} \to \mathrm{B}$ be a derivation on the ring B. Let $\mathrm{B}[D] = \{\sum_i a_i D^i \mid a_i \in \mathrm{B}\}$ be the ring of differential polynomials on D, we recall that $Da = aD + D(a)$, for all $a \in \mathrm{B}$. Let $\mathrm{B}[D] - \mathrm{mod}$ be the category of left $\mathrm{B}[D]-$modules. The category MLC(B) of (free) differential modules over B is the full subcategory of $\mathrm{B}[D] - \mathrm{mod}$ formed by modules who are *free and of finite rank* as B$-$modules.

2.2.2 Operators and differential modules

Let $G = (g_{i,j})_{i,j} \in \mathrm{M}_n(\mathrm{B})$. The operator

$$D - G : \mathrm{B}^n \to \mathrm{B}^n$$

is the map $(b_i)_i \mapsto (D(b_i))_i - G \cdot (b_i)_i$. Let $M \in \mathrm{MLC}(\mathrm{B})$ and let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ be a basis of M. We will say that M is defined by the operator $D - G$, in the basis $\mathbf{e}$ if $D(\mathrm{e}_i) = -\sum_{j=1}^{n} g_{j,i} \cdot \mathrm{e}_j$.

$$\begin{tikzcd} M \arrow[r, "D\cdot"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\odot"] & M \arrow[d, "\wr"] \\ \mathrm{B}^n \arrow[r, "D-G"'] & \mathrm{B}^n \end{tikzcd} \tag{2.2.0.1}$$

We will call G the matrix of the $K-$linear map D in the basis $\mathbf{e}$. For all $s \geq 1$, the matrix of D^s in this basis is called $G_s \in \mathrm{M}_n(\mathrm{B})$ and satisfy $D^s(\mathbf{e}) = -\mathbf{e} \cdot G_s$. We have

$$G_{s+1} = D(G_s) + G \cdot G_s, \qquad G_0 := \mathrm{Id}\ . \tag{2.2.0.2}$$

In all the sequel B will be equal to $\mathcal{A}(I)$, $\mathcal{R}$, $\mathcal{E}$, or $\mathcal{E}^\dagger$, and we will work with both derivations $\frac{d}{dT}$ and $\partial_T = T\frac{d}{dT}$. We will denote by

$$G_s(T) = \text{the matrix of } \partial_T^s \qquad ; \qquad G_{[s]}(T) = \text{the matrix of } (d/dT)^s. \tag{2.2.0.3}$$

We have $G_1(T) = TG_{[1]}(T) \in \mathrm{M}_n(\mathrm{B})$. We will put $G(T) = G_1(T)$ for brevity. If M is of rank one, we will write $g(T)$, $g_s(T)$, $g_{[s]}(T)$ instead of $G(T)$, $G_s(T)$, $G_{[s]}(T)$.

2.2.3 The classification of Manin

Let $M \in \mathrm{MLC}(\mathrm{B})$ be the rank one module defined by $D - g(T)$. If $f \in \mathrm{B}^\times$, then M is defined, in the basis $f \cdot \mathbf{e}$, by the operator $D - (g - D(f)/f)$. Let N be another rank one B$-$differential module defined by the operator $D - \widetilde{g}$ in the basis $\widetilde{\mathbf{e}}$. The tensor product $M \otimes N$ is defined, in the basis $\mathbf{e} \otimes \widetilde{\mathbf{e}}$, by the operator

$$D - (g + \widetilde{g})\ . \tag{2.2.0.4}$$

Let us call Pic(B) the group (under tensor product) of isomorphisms classes of rank one differential modules over B. Following [Man65], we will identify

$$\mathrm{Pic}(\mathrm{B}) = \mathrm{B}/D_{\log}(\mathrm{B}^\times), \tag{2.2.0.5}$$

where $D_{\log} : \mathrm{B}^\times \to \mathrm{B}$ is the additive map $f \mapsto \frac{D(f)}{f}$.

Definition 2.2.1. Let C be a B$-$differential algebra, and let M be a rank one differential B$-$module defined in some basis by the operator $D - g$. A solution of M with values in C is an element $c \in \mathrm{C}$ satisfying $D(c) = g \cdot c$.

2.3 Taylor solution and radius of convergence

Let $I \subseteq \mathbb{R}_{\geq 0}$ be some interval. In this subsection, M will be a *rank one* $\mathcal{A}_K(I)-$differential module defined by the operator $\partial_T - g(T)$.

Let $x \in \Omega$, $|x| \in I$. We look at $\Omega[[T-x]]$ as an $\mathcal{A}_K(I)-$differential algebra by the Taylor map

$$f(T) \mapsto \sum_{k\geq 0} (\frac{d}{dT})^k (f)(x) \frac{(T-x)^k}{k!} \ : \ \mathcal{A}_K(I) \longrightarrow \Omega[[T-x]] \ .$$

The Taylor solution of $\partial_T - g(T)$ at x is (recall that $g(T) = Tg_{[1]}(T)$)

$$s_x(T) := \sum_{k\geq 0} g_{[k]}(x) \frac{(T-x)^k}{k!} \ . \tag{2.3.0.1}$$

Indeed $\partial_T(s_x(T)) = g(T)s_x(T)$. The radius of convergence of $s_x(T)$ at x is, by the usual definition,

$$Ray(M, x) = \liminf_s (|g_{[k]}(x)|/|k!|)^{-\frac{1}{k}} \ . \tag{2.3.0.2}$$

Definition 2.3.1. The radius of convergence of M at $\rho \in I$ is

$$\begin{aligned} Ray(M,\rho) \ &:= \ \min\Big(\rho \ , \ \liminf_k (|g_{[k]}|_\rho / |k!|)^{-1/k}\Big) \\ &= \ \min\Big(\rho \ , \ \omega\big[\limsup_k (|g_{[k]}|_\rho)^{1/k}\big]^{-1}\Big) \ . \end{aligned} \tag{2.3.1.1}$$

The second equality follows from the fact that the sequence $|k!|^{1/k}$ is convergent to ω, and $|g_{[k]}|_\rho^{1/k}$ is bounded by $\max(|g|_\rho, \rho^{-1})$. The presence of ρ in the minimum makes this definition invariant under change of basis in M.

Theorem 2.3.2 (Transfer)**.** *For all $\rho \in I$ we have*

$$Ray(M,\rho) = \min\big(\ \rho \ , \inf_{x\in\Omega, |x|=\rho} Ray(M,x) \ \big). \tag{2.3.2.1}$$

Assume now that $I = [0, \rho]$. Then

$$Ray(M,\rho) = \min(\rho, \min_{x\in\Omega, |x|\leq\rho} Ray(M,x)) \tag{2.3.2.2}$$

In particular $Ray(M,\rho) \leq \min(\rho, Ray(M,0))$.

Proof : Since for $\rho = |x|$ we have $|g_{[s]}(T)|_\rho \geq |g_{[s]}(x)|$, hence by definition 2.3.1, $Ray(M,\rho) \leq \min(\rho, Ray(M,x))$. Let $t_\rho \in \Omega$ be such that $\{x \in \Omega \mid |x - t_\rho| < \rho\} \cap K = \emptyset$, then $|g_{[s]}|_\rho = |g_{[s]}(t_\rho)|$, for all $s \geq 0$ ([CR94, 9.1]), hence $Ray(M,\rho) = \min(\rho, Ray(M,t_\rho))$. The last assertion follow the same line. $\square$

Lemma 2.3.3 (Small radius)**.** *Let $\rho \in I$. Then*

$$Ray(M,\rho) \geq \omega\rho \cdot \min(1, |g(T)|_\rho^{-1}). \tag{2.3.3.1}$$

Moreover $Ray(M,\rho) < \omega\rho$ if and only if $|g(T)|_\rho > 1$, and in this case we have

$$Ray(M,\rho) = \omega\rho \cdot |g(T)|_\rho^{-1}. \tag{2.3.3.2}$$

Proof : By induction on 2.2.0.2 one has (cf. 2.2.0.3)

$$|g_{[s]}|_\rho \leq \max(\rho^{-1}, |g_{[1]}|_\rho)^s = \rho^{-s} \max(1, |g|_\rho)^s \tag{2.3.3.3}$$

and equality holds if $|g_{[1]}|_\rho > \rho^{-1}$. Then apply definition 2.3.1. □

Definition 2.3.4. M is called *solvable* at $\rho \in I$, if $Ray(M,\rho) = \rho$.

Theorem 2.3.5 ([CD94])**.** *The map $\rho \mapsto Ray(M,\rho) : I \to \mathbb{R}_{\geq}$ is continuous and locally of the form $r \cdot \rho^{\beta+1}$, for suitable $r \in \mathbb{R}_{\geq}$, and $\beta \in \mathbb{N}$. More precisely there exist a partition $I = \cup_{n\in\mathbb{Z}} I_n$, $\sup I_n = \inf I_{n+1}$, and two sequences $\{r_n\}_{n\in\mathbb{Z}}$, $\{\beta_n\}_{n\in\mathbb{Z}}$, such that $\beta_n \in \mathbb{Z}$,*

$$Ray(M,\rho) = r_n \rho^{(\beta_n+1)} \ , \ \ \forall\ \rho \in I_n \ , \tag{2.3.5.1}$$

and (cf. 2.4.3)

$$\beta_n \geq \beta_{n+1}. \tag{2.3.5.2}$$

We will call this property the $\log-$concavity *of the function $\rho \mapsto Ray(M,\rho)$.*

Proof : The existence of the partition follows from the small radius lemma 2.3.3 and theorem 2.5.5 (below). For more details see [CM02, 8.6] and [CD94, 2.5]. □

Definition 2.3.6. We will call β_n the slope of M in the interior of I_n. More generally if $\rho = \sup I_n = \inf I_{n+1}$, we set

$$\mathrm{sl}^-(M,\rho) := \beta_n \quad , \qquad \mathrm{sl}^+(M,\rho) := \beta_{n+1}. \tag{2.3.6.1}$$

2.3.1 Solution of the tensor product

The Taylor solution of $M \otimes N$ is the product of the Taylor solutions of M and N. Hence, by 2.3.2

$$Ray(M \otimes N, \rho) \geq \min(Ray(M,\rho), Ray(N,\rho)). \tag{2.3.6.2}$$

Proposition 2.3.7. *If $Ray(M,\rho) \neq Ray(N,\rho)$, then we have*

$$Ray(M \otimes N, \rho) = \min(Ray(M,\rho), Ray(N,\rho)). \tag{2.3.7.1}$$

Proof : Let $s_M, s_N, s_{M\otimes N}$ be the Taylor solutions of $M, N, M \otimes N$ at the generic point $t_\rho \in \Omega$ of valuation ρ. Suppose that $Ray(M,\rho) < Ray(N,\rho)$. By a wronskian argument, it is well known that a solution of a rank one differential operator has no zeros in its disk of convergence. The inverse of such a power series solution is then convergent in the same disk. We can write $s_M = s_{M\otimes N} \cdot s_N^{-1}$, and then we get the contradiction

$$Ray(M, t_\rho) \geq Ray(N^\vee, t_\rho) = Ray(N, t_\rho) \ . \ \square \tag{2.3.7.2}$$

2.4 Solvability, Slopes and Irregularities

In this subsection, M is a rank one differential module over $\mathcal{R}_K$, defined by the operator

$$\partial_T + g(T) \quad , \quad g(T) := \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K \; . \tag{2.4.0.3}$$

Lemma 2.4.1 (Algebraicity). *There exists $d > 0$ such that M is isomorphic to the module defined by $\partial_T + \sum_{i\geq -d} a_i T^i$. In other words there exists $f(T) \in \mathcal{R}_K^\times$ such that $\partial_{T,\log}(f) = \sum_{i<-d} a_i T^i$.*

Proof : By hypothesis $g(T) \in \mathcal{A}_K(]1-\varepsilon, 1[)$, for some $\varepsilon > 0$. Then $\sum_{i\neq 0} a_i T^i/i \in \mathcal{A}_K(]1-\varepsilon, 1[)$. In particular $\lim_{i\to -\infty} |a_i/i|\rho^i = 0$, for all $\rho \in]1-\varepsilon, +\infty[$. Let $d > 0$ be such that $\sup_{i<-d}(|a_i/i|\rho^i) < \omega$ for all $]1-\varepsilon, 1[$. Then $f(T) = \exp(-\sum_{i<-d} a_i T^i/i)$ lies in $\mathcal{R}_K$.$\square$

Definition 2.4.2. Let M be a differential module over $\mathcal{R}_K$. The module M is called *solvable* if and only if

$$\lim_{\rho\to 1^-} Ray(M,\rho) = 1. \tag{2.4.2.1}$$

We will denote the category of solvable differential modules over $\mathcal{R}_K$ by $\mathrm{MLS}(\mathcal{R}_K)$.

Lemma 2.4.3. *Let $M \in \mathrm{MLS}(\mathcal{R}_K)$ be defined in some basis by the operator $\partial_T - g(T)$, $g(T) \in \mathcal{R}_K$. Then*

1. *There exist $0 < \varepsilon < 1$ and a last slope $\beta := \mathrm{sl}^-(M,1) \geq 0$ such that*

$$Ray(M,\rho) = \rho^{\beta+1} \; , \quad \text{for all } \rho \in]1-\varepsilon, 1[\; . \tag{2.4.3.1}$$

2. *There exists ε' such that $|g(T)|_\rho \leq 1$, for all $\rho \in]1-\varepsilon', 1[$.*
3. *If $g(T) = \sum_{-d}^{\infty} a_i T^i$, $d > 0$, then $|a_{-d}| \leq \omega$ and, for ρ close to 0,*

$$Ray(M,\rho) = \omega |a_{-d}|^{-1} \rho^{d+1} \quad (\text{for } \rho \text{ close to } 0). \tag{2.4.3.2}$$

4. *Moreover, if $d > 0$, and if $|a_{-d}| = \omega$, then $\beta = d$.*
5. *If $d \leq 0$, then $Ray(M,\rho) = \rho$, for all $\rho \in]0,1[$ and $\beta = 0$.*

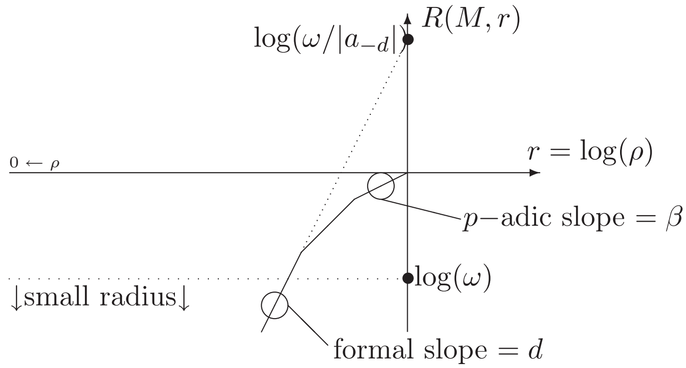
Proof : The slopes are positive natural numbers, hence the decreasing sequence $\{\beta_n\}_n$ becomes constant for $n \to \infty$. Then $\beta = \min_{n\in\mathbb{Z}}\{\beta_n\}$. The second assertion follows from the small radius lemma 2.3.3. Let now $g(T) = \sum_{i\geq -d} a_i T^i$, with $d > 0$. We study the function $\rho \mapsto Ray(M,\rho)/\rho$. Let

$$R(M,r) := \log(Ray(M,\rho)) - \log(\rho) \quad , \quad r := \log(\rho) \; . \tag{2.4.3.3}$$

Then $R(M,r) \leq 0$, for all $r \leq 1$, and the function

$$r \mapsto R(M,r) :]-\infty, 1[\; \longrightarrow \;]-\infty, 0] \tag{2.4.3.4}$$

is of the following form



Since $d > 0$, then $|g(T)|_\rho = |a_{-d}|\rho^{-d} > 1$, for $\rho \to 0$. Hence, near 0, we can apply the small radius lemma (cf. 2.4.4) : we have $Ray(M,\rho) = \omega|a_{-d}|^{-1}\rho^{d+1}$. Since $\lim_{\rho\to 1} Ray(M,\rho) = 1$, hence by $\log-$concavity and continuity, we must have $\omega|a_{-d}|^{-1} \geq 1$ (or equivalently $\log(\omega/|a_{-d}|) \geq 0$ as in the picture) and if $|a_{-d}| = \omega$, then, again by continuity and log-concavity, this graphic is a line, and $\beta = d$. If $d \leq 0$, then $|g(T)|_\rho \leq 1$, for all $\rho < 1$, hence the small radius lemma gives $R(M,r) \geq \log(\omega)$ for all $r \leq 0$. Since $R(M,r) \to 0$ for $r \to 0$ (solvability), then by $\log-$concavity and continuity this implies $R(M,r) = 0$, $\forall\, r \leq 0$. □

Remark 2.4.4. In the notation of lemma 2.4.3 part (3), let us call

$$\mathrm{Irr}_F(M) = \min(0,d) \tag{2.4.4.1}$$

be the Formal slope of M (i.e. the classical slope of M as $K((T))-$differential module), then we have

$$\mathrm{Irr}_F(M) := \mathrm{sl}^+(M,0)\ . \tag{2.4.4.2}$$

We recall that, in the higher dimensional case, $\mathrm{sl}_F(M)$ is the largest slope of the formal Newton polygon of M, and in the rank one case it is equal to the formal irregularity of M. In the rank one case, by definition $\mathrm{sl}_F(M) := \max(0\,,-v_T(g))$, where $v_T(g)$ is the $T-$adic valuation of $g(T)$. We recall that the formal Newton polygon of an operator $\sum_{s=0}^{\mu} a_s(T)\partial^s$, $a_\mu(T) = 1$, is the convex hull of the set formed by the points of the form $(s, v_T(a_s))$, and the two additional points $(-\infty, 0)$, $(0, +\infty)$. In particular the last point is always $(\mu, 0)$. This remark can be easily generalized to all ranks (cf. [Pul04, 3.2]).

The following definition is then natural

Definition 2.4.5. Let M be a solvable rank one differential module over $\mathcal{R}$. The $p-$adic *irregularity* of M is the natural number

$$\mathrm{Irr}(M) := \mathrm{sl}^-(M,1). \tag{2.4.5.1}$$

Remark 2.4.6. If M is the solvable module defined by an operator $\partial_T + g(T)$, $g(T) \in \mathcal{A}_K([0,1[)[1/T] \subset \mathcal{R}_K$, then by log-concavity and continuity we have

$$\mathrm{Irr}_F(M) \geq \mathrm{Irr}(M). \tag{2.4.6.1}$$

Definition 2.4.7. If K'/K is a finite extension, then we denote by $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K'})$ the group, under tensor product, of isomorphism classes of solvable rank one differential modules over $\mathcal{R}_{K'}$. For all algebraic extensions H/K, we set

$$\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_H) := \bigcup_{K\subset K'\subset H\ ,\ K'/K \text{ finite}} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K'}). \tag{2.4.7.1}$$

Corollary 2.4.8. *Let $M, N \in \mathrm{MLS}(\mathcal{R}_K)$. We have*

$$\mathrm{Irr}(M \otimes N) \leq \max(\mathrm{Irr}(M), \mathrm{Irr}(N))\ . \tag{2.4.8.1}$$

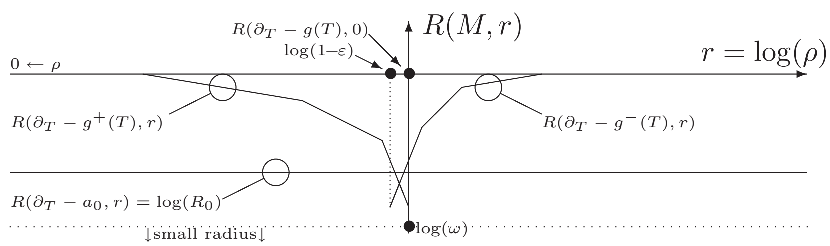
If $\mathrm{Irr}(M) \neq \mathrm{Irr}(N)$*, then*

$$\mathrm{Irr}(M \otimes N) = \max(\mathrm{Irr}(M), \mathrm{Irr}(N))\ . \tag{2.4.8.2}$$

Proof : This results immediately from 2.3.7.□

Proposition 2.4.9. *Let $\partial_T - g(T)$, $g(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K$, be a solvable differential equation. Then $\partial_T - g^-(T)$, $\partial_T - a_0$, $\partial_T - g^+(T)$ are all solvable (cf.2.1.2).*

Proof : Let us call $M_{]1-\varepsilon,\infty]}$, M_0, $M_{[0,1[}$ the differential modules defined by $\partial_T - g^-(T)$, $\partial_T - a_0$, $\partial_T - g^+(T)$ respectively. Then $M = M_{]1-\varepsilon,\infty]} \otimes M_0 \otimes M_{[0,1[}$. By the small radius lemma 2.3.3, the equation $\partial_T - g^-(T)$ (resp. $\partial_T - g^+(T)$) has a convergent solution at ∞ (resp. at 0), hence $Ray(M_{]1-\varepsilon,\infty]}, \rho) = \rho$, for large values of ρ and $Ray(M_{[0,1[}, \rho) = \rho$, for small values of ρ. While $Ray(M_0, \rho) = R_0 \cdot \rho$, for all ρ (cf. 2.6.3). Hence the slopes of $M_{]1-\varepsilon,\infty]}$ (resp. $M_{[0,1[}$, M_0) in the interval $]1-\varepsilon, 1[$ are strictly positive (resp. strictly negative, equal to 0) as in the picture (cf. 2.4.3.3)



By 2.3.6.2, we have

$$Ray(M, \rho) = \inf(Ray(M_{]1-\varepsilon,\infty]}, \rho), Ray(M_{[0,1[}, \rho), Ray(M_0, \rho)) \tag{2.4.9.1}$$

for all $1 - \varepsilon < \rho < 1$, with the exception of a finite numbers of ρ. By continuity of the radius, we have equality even for these isolated values of ρ.

Since $\lim_{\rho\to 1} Ray(M,\rho) = 1$, this implies $Ray(M_{[0,1[},\rho) = \rho$ for all $\rho < 1$, $Ray(M_0,\rho) = \rho$ for all ρ, and $Ray(M_{]1-\varepsilon,\infty]},\rho) = \rho$ for all $\rho \geq 1$. $\Box$

The classification of the equations of the type $\partial_T - a_0$ is well known (see 2.6), while the solvable equations of the form $\partial_T - g^+(T)$ are always trivial :

Proposition 2.4.10. *Let $\partial_T - g^+(T)$, with*

$$g^+(T) = \sum_{i\geq 1} a_i T^i \in \mathcal{A}_H([0,1[) \subset \mathcal{R}_K \tag{2.4.10.1}$$

be solvable at 1^- (cf. 2.4.2). Let M be the module attached to $\partial_T - g^+(T)$, then

1. *We have $g^+(T) \in T\mathcal{O}_H[[T]]$. Hence M comes, by scalar extension, from a differential module $M_{[0,1[}$ over $\mathcal{O}_H[[T]]$;*
2. *We have $Ray(M_{[0,1[}\ ,\ \rho) = \rho$, for all $\rho < 1$;*
3. *$M_{[0,1[}$ is trivial as $\mathcal{O}_H[[T]]-$module ;*
4. *The exponential $\exp(\sum_{i\geq 1} a_i T^i/i)$ lies in $1 + T\mathcal{O}_H[[T]]$.*

Proof : We have $|a_i| \leq 1$, because the small radius lemma 2.3.3. Since $\partial_T - g^+(T)$ has a convergent solution at 0 (namely this Taylor solution is $\exp(\sum_{i\geq 1} a_i T^i/i)$), then $Ray(M_{[0,1[},\rho) = \rho$ for all ρ close to 0. Since $\lim_{\rho\to 1^-} Ray(M_{[0,1[},\rho) = 1$, then by log-concavity we must have $Ray(M_{[0,1[},\rho) = \rho$, for all $\rho < 1$. By transfer theorem 2.3.2 the Taylor solution $\exp(\sum_{i\geq 1} a_i T^i/i)$ converge in the disk $|T| < 1$ and it belong to $\mathcal{O}_K[[T]]$ (because a non trivial solution of a differential equation has no zeros in its disk of convergence).$\Box$

Corollary 2.4.11. *Every rank one solvable differential module over $\mathcal{R}_K$ has a basis in which the matrix lies in $\mathcal{O}_K[T^{-1}]$.*

Proof : By 2.4.10 there exists a basis in which the matrix lies in $\mathcal{R}_K \cap \mathcal{O}_K[[T^{-1}]]$. The base change matrix to obtain this basis is an exponential convergent in $[0,1[$. Now, by 2.4.1 we recover the good basis, this last base change matrix is again an exponential convergent in $]1-\varepsilon,\infty[$.$\Box$

2.5 Frobenius structure and $p-$th ramification

Definition 2.5.1. An *absolute Frobenius on K* is a $\mathbb{Q}_p-$endomorphism $\sigma : K \to K$ such that $|\sigma(x) - x^p| < 1$, for all $x \in \mathcal{O}_K$.
If an absolute Frobenius $\sigma : K \to K$ is given, an *absolute Frobenius on $\mathcal{R}_K$* is then a continue endo-morphism of rings $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ extending σ, and such that

$$\varphi(T) - T^p = \sum a_i(\varphi)T^i\ ,\ \text{with } |a_i(\varphi)| < 1 \text{ for all } i \in \mathbb{Z}\ ,\ a_i(\varphi) \in K\ . \tag{2.5.1.1}$$

Remark 2.5.2. By continuity φ is given by σ and by the choice of $\varphi(T)$, indeed

$$\varphi(f(T)) = \varphi(\sum a_i T^i) = \sum \sigma(a_i)\varphi(T)^i = f^\sigma(\varphi(T)) \ , \qquad (2.5.2.1)$$

where $f(T) := \sum a_i T^i$ and $f^\sigma(T) := \sum \sigma(a_i)T^i$.

Let $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ be an absolute Frobenius. By scalar extension (and change of derivation), we have a functor :

$$\varphi^* : \mathrm{MLC}(\mathcal{R}) \rightsquigarrow \mathrm{MLC}(\mathcal{R}) \qquad (2.5.2.2)$$

If $M \in \mathrm{MLC}(\mathcal{R})$ is defined by the operator $\partial_T - G(T)$, $G(T) \in \mathrm{M}_n(\mathcal{R}_K)$, then $\varphi^*(M)$ is defined by the operator $\partial_T - (\frac{\partial_T(\varphi(T))}{\varphi(T)} \cdot G^\sigma(\varphi(T)))$.

The simplest absolute Frobenius is given by $\varphi(T) := T^p$, and we denote it by φ_σ :

$$\varphi_\sigma(\sum a_i T^i) := \sum \sigma(a_i)T^{pi} \ . \qquad (2.5.2.3)$$

Lemma 2.5.3 ([CM02, 7.1])**.** *Let $M \in \mathrm{MLC}(\mathcal{R})$. Let $\varphi, \widetilde{\varphi} : \mathcal{R} \to \mathcal{R}$ be two absolute Frobenius. If there exists $0 < \varepsilon < 1$ such that*

$$|\varphi(T) - \widetilde{\varphi}(T)|_\rho < Ray(M,\rho) \ , \qquad (2.5.3.1)$$

for all $\rho \in]1-\varepsilon, 1[$, then $\varphi^*(M)$ is isomorphic to $\widetilde{\varphi}^*(M)$ over $\mathcal{R}$.

Corollary 2.5.4 (Independence on φ)**.** *If $M \in \mathrm{MLS}(\mathcal{R})$, then $\varphi^*(M) \xrightarrow{\sim} \varphi_\sigma^*(M)$, for all absolute Frobenius $\varphi : \mathcal{R}_K \to \mathcal{R}_K$.*

Proof : We look at 2.5.1.1. Since K is not discrete valued (cf. 2.1.4), we observe that we could have $\sup_i |a_i(\varphi)|_1 = 1$. This will not be a problem because, in this case, for all $\beta = \mathrm{sl}^+(M,1) \geq 0$ there exists $\varepsilon < 1$ such that $|\varphi(T) - T^p|_\rho < \rho^{\beta+1}$, for all $\rho \in]1-\varepsilon, 1[$. Then we can apply 2.5.3. $\square$

2.5.1 $p-$th ramification

Let σ be an absolute Frobenius on K. For all analytic functions $f(T) := \sum_i a_i T^i \in \mathcal{A}(I)$, we set $f^\sigma(T) := \sum_i \sigma(a_i)T^i$, and

$$\varphi_p(f(T)) := f(T^p). \qquad (2.5.4.1)$$

Observe that φ_p is not an absolute Frobenius. We set $\varphi_\sigma(f(T)) := f^\sigma(T^p)$. The p-th ramification map $\varphi_p : \mathcal{A}(I^p) \to \mathcal{A}(I)$ defines, as before, a functor denoted by $\varphi_p^* : \mathrm{MLC}(\mathcal{A}_K(I^p)) \rightsquigarrow \mathrm{MLC}(\mathcal{A}_K(I))$.

Theorem 2.5.5. *Let $M \in \mathrm{MLC}(\mathcal{A}_K(I^p))$. Then for all $\rho \in I$*

$$Ray(\varphi_\sigma^*(M),\rho) = Ray(\varphi_p^*(M),\rho) \geq \min\big(Ray(M,\rho^p)^{1/p} \ , \ |p|^{-1}\rho^{1-p}Ray(M,\rho^p)\big),$$

and equality holds if $Ray(M,\rho) \neq \omega^p\rho$.

Proof : Since $f(T) \mapsto f^{\sigma}(T)$ is an isometry, we have the first equality. The second one follows from a quite complex, but elementary, explicit computation (see [Pul04, 5.7] for a complete proof of this well known theorem).□

Example 2.5.6. The radius of the operator $\partial_T - \frac{1}{p}$ is equal to $\omega|p|\rho = \omega^p\rho$ (cf. 2.3.3), but its image by Frobenius is the trivial module.

Corollary 2.5.7. *Let* $M \in \mathrm{MLS}(\mathcal{R}_K)$*, let* $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ *be an absolute Frobenius, then* $\mathrm{sl}^-(\varphi^*(M), 1) = \mathrm{sl}^-(M, 1)$.

Proof : Apply theorem 2.5.5 and corollary 2.5.4.□

Definition 2.5.8 (Frobenius structure)**.** Let M be a rank one $p-$adic differential module over $\mathcal{R}_K$. We will say that M has a *Frobenius structure of order h*, if there exists an $\mathcal{R}_K-$isomorphism

$$M \xrightarrow{\sim} \varphi_h^*(M)\ , \tag{2.5.8.1}$$

where $\varphi_1^*(M) := \varphi^*(M)$, and inductively $\varphi_{h+1}^*(M) := \varphi^*(\varphi_h^*(M))$.

Remark 2.5.9. If M has a Frobenius structure, then it is solvable by theorem 2.5.5 applied to "antecedents" of M. (see [CM02, 8.6 and 7.7 infra]).

Remark 2.5.10. By equation 2.5.5 we have

$$\mathrm{Irr}(\varphi^*(M)) = \mathrm{Irr}(\varphi_p^*(M)) = \mathrm{Irr}(M)\ . \tag{2.5.10.1}$$

2.6 Moderate characters

Definition 2.6.1. Let $a_0 \in K$. We denote by

$$\mathrm{M}(a_0, 0) \tag{2.6.1.1}$$

the module defined by the constant operator $\partial_T - a_0$ (cf. 2.2.2). We will call *moderate* every *solvable* differential module (over $\mathcal{R}_K$) of the form $\mathrm{M}(a_0, 0)$.

Remark 2.6.2. By [Rob85, 5.4], $\mathrm{M}(a_0, 0)$ is solvable if and only if $a_0 \in \mathbb{Z}_p$. Moreover the equation $\partial_{T,\log}(f(T)) = a_0$ has a solution $f(T) \in \mathcal{R}_K^{\times}$ if and only if $a_0 \in \mathbb{Z}$, and in this case $f(T) = T^{a_0}$. This shows that the group under tensor product of moderate differential modules is isomorphic to $\mathbb{Z}_p/\mathbb{Z}$.

Lemma 2.6.3. *Let* $\alpha(a_0) := \limsup_s(|a_0(a_0-1)(a_0-2)\cdots(a_0-s+1)|^{\frac{1}{s}})$. *Then*

$$\mathit{Ray}(\mathrm{M}(a_0, 0), \rho) = \rho \cdot R_0 \le \rho\ , \quad \textit{for all } \rho > 0\ , \tag{2.6.3.1}$$

with $R_0 := \min(1, \omega \cdot \alpha(a_0)^{-1})$.

Proof : A direct computation gives $g_{[s]}(T) = \alpha_s(a_0)T^{-s}$, with $\alpha_s(a_0) := a_0(a_0-1)\cdots(a_0-s+1)$ (cf.2.2.0.3). Then apply 2.3.1.□

Lemma 2.6.4. *Let $\varphi : \mathcal{R} \to \mathcal{R}$ be a Frobenius, then* $\mathrm{M}(a_0, 0)$ *has a Frobenius structure if and only if* $a_0 \in \mathbb{Z}_{(p)}$.

Proof : By 2.5.4 we can suppose $\varphi = \varphi_\sigma$. Suppose that $\mathrm{M}(a_0, 0)$ has a Frobenius structure of order h. Since $\mathrm{M}(a_0, 0)$ is solvable (cf. 2.5.9), hence $a_0 \in \mathbb{Z}_p$. By definition 2.5.8, $p^h \cdot a_0 - a_0 \in \mathbb{Z}$, hence $a_0 \in \mathbb{Q}$. Conversely, let $a_0 = a/b \in \mathbb{Z}_{(p)}$, $b > 0$. Let $b = \prod_i q_i^{r_i}$ be the factorization of b in positive prime numbers. For all $q, r \in \mathbb{Z}$, $r > 0$, we define

$$[q]_r := q^{q^{\cdots q}} , \qquad (2.6.4.1)$$

$r-$times, (i.e. $[q]_1 = q$ and $[q]_{r+1} = q^{[q]_r}$). We have $p^{[q]_r - 1} \equiv 1 \pmod{q^r}$. Then if $h = \prod_i([q_i]_{r_i} - 1)$ we have $(p^h - 1)a_0 \in \mathbb{Z}$.$\Box$

2.7 Notations on Witt Vectors and covectors

Let R be a ring. Notations concerning the ring $\mathbf{W}(R)$ of Witt vectors will follow [Bou83a], except for the indexation "m" of the ring $\mathbf{W}_m(R)$ of Witt vector of finite length. We set $\mathbf{W}_m(R) := \mathbf{W}(R)/\mathrm{V}^{m+1}\mathbf{W}(R)$ (see 2.7.3.5). We denote by

$$\phi_n(X_0, \ldots, X_n) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n \qquad (2.7.0.2)$$

the Witt polynomial. For all $\boldsymbol{\nu} := (\nu_0, \nu_1, \ldots) \in \mathbf{W}(R)$, we will call $\phi_j(\nu_0, \ldots, \nu_j) \in R$ the $j-$th phantom component of $\boldsymbol{\nu}$. We will write ϕ_j, or also $\phi_j(\boldsymbol{\nu})$, instead of $\phi_j(\nu_0, \ldots, \nu_j)$. We recall that, by definition of $\mathbf{W}$, for all $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \mathbf{W}(R)$ we have

$$\phi_j(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2) = \phi_j(\boldsymbol{\nu}_1) + \phi_j(\boldsymbol{\nu}_2) \qquad , \qquad \phi_j(\boldsymbol{\nu}_1 \cdot \boldsymbol{\nu}_2) = \phi_j(\boldsymbol{\nu}_1) \cdot \phi_j(\boldsymbol{\nu}_2) \, .$$

Vectors in $R^{\mathbb{N}}$ and in R^{m+1} will be distinguished from Witt vectors by the notation $\langle \phi_0, \phi_1, \ldots \rangle$ instead of $(\phi_0, \phi_2, \ldots)$. The vector $\langle \phi_0, \phi_1, \ldots \rangle \in R^{\mathbb{N}}$ (resp. $\langle \phi_0, \ldots, \phi_m \rangle \in R^{m+1}$) will be called the phantom vector of $(\nu_0, \nu_1, \ldots) \in \mathbf{W}(R)$ (resp. $(\nu_0, \ldots, \nu_m) \in \mathbf{W}_m(R)$).

Lemma 2.7.1 ([Bou83a, Lemme 3 §1, $N^0 2$])**.** *Let $\boldsymbol{\lambda} \mapsto \phi(\boldsymbol{\lambda}) : \mathbf{W}(R) \xrightarrow{\phi} R^{\mathbb{N}}$ be the phantom component map.*

- *If $p \in R$ is not a zero divisor, then ϕ is injective.*
- *If $p \in R$ is invertible then ϕ is bijective.* $\Box$

Lemma 2.7.2 ([Bou83a, Lemme 2 §1, $N^0 2$])**.** *Let $\sigma : R \to R$ be a ring morphism satisfying $\sigma(a) \equiv a^p \pmod{pR}$, for all $a \in R$. Then a vector $\langle \phi_0, \ldots, \phi_m \rangle \in R^{m+1}$ is the phantom vector of a Witt vector if and only if*

$$\phi_i \equiv \sigma(\phi_{i-1}) \mod p^i R \, , \quad \textit{for all } \; i = 1, \ldots, m \, . \, \Box \qquad (2.7.2.1)$$

Remark 2.7.3. All assertions concerning relations between Witt vectors or properties of $\boldsymbol{\pi}-$exponentials (see below) will be proved by translating these relations or properties in terms of phantom components.

Let

$$\psi : R_1 \to R_2$$

be a ring morphism. We denote again by

$$\psi : \mathbf{W}_m(R_1) \to \mathbf{W}_m(R_2)$$

the morphism obtained by functoriality. The phantom vector of

$$\psi(\nu_0, \ldots, \nu_m) = (\psi(\nu_0), \ldots, \psi(\nu_m)) \tag{2.7.3.1}$$

is $\langle \psi(\phi_0), \ldots, \psi(\phi_m) \rangle$. If $R_1 = R_2$ we will put also $\boldsymbol{\nu}^{\psi} := \psi(\boldsymbol{\nu})$.

2.7.1 Frobenius and Verschiebung

We denote by $\mathrm{F} : \mathbf{W}(R) \to \mathbf{W}(R)$ and $\mathrm{V} : \mathbf{W}(R) \to \mathbf{W}(R)$ the usual Frobenius and Verschiebung morphisms. We have

$$\mathrm{V}(\nu_0, \nu_1, \ldots) = (0, \nu_0, \nu_1, \ldots) \ . \tag{2.7.3.2}$$

We recall that if $\langle \phi_0, \phi_1, \ldots \rangle$ is the phantom vectors of $\boldsymbol{\nu} \in \mathbf{W}(R)$, then the phantom vectors of $\mathrm{F}(\boldsymbol{\nu})$ and $\mathrm{V}(\boldsymbol{\nu})$ are

$$\phi(\mathrm{F}(\boldsymbol{\nu})) = \langle \phi_1, \phi_2, \ldots \rangle \tag{2.7.3.3}$$

$$\phi(\mathrm{V}(\boldsymbol{\nu})) = \langle 0, p\cdot\phi_0, p\cdot\phi_1, \ldots \rangle \tag{2.7.3.4}$$

We recall that $\mathrm{F}(\boldsymbol{\nu}) \equiv \boldsymbol{\nu}^p \mod p\mathbf{W}(R)$, and that $\mathrm{FV}(\boldsymbol{\nu}) = p \cdot \boldsymbol{\nu}$. We denote by

$$\mathrm{F} : \mathbf{W}_{m+1}(R) \to \mathbf{W}_m(R) \qquad ; \qquad \mathrm{V} : \mathbf{W}_m(R) \to \mathbf{W}_{m+1}(R) \tag{2.7.3.5}$$

the reduction of the Frobenius and Verschiebung morphism on $\mathbf{W}_m(R)$. We have again $\mathrm{FV}(\boldsymbol{\nu}) = p \cdot \boldsymbol{\nu}$ in $\mathbf{W}_m(R)$.

If R has characteristic p, then $\mathrm{F}(\nu_0, \nu_1, \ldots) = (\nu_0^p, \nu_1^p, \ldots)$. Hence it is possible to reduce the morphism F of $\mathbf{W}(R)$ to a morphism of $\mathbf{W}_m(R)$ into itself, by setting

$$\bar{\mathrm{F}}(\nu_0, \ldots, \nu_m) = (\nu_0^p, \ldots, \nu_m^p) \ . \tag{2.7.3.6}$$

We denote this morphism by

$$\bar{\mathrm{F}} : \mathbf{W}_m(R) \to \mathbf{W}_m(R). \tag{2.7.3.7}$$

2.7.2 Completeness

Let R be a topological ring. Let us identify topologically $\mathbf{W}_m(R)$ with R^{m+1}, via the function $(\nu_0, \ldots, \nu_m) \mapsto \langle \nu_0, \ldots, \nu_m \rangle$. Then the operations on $\mathbf{W}_m(R)$ are continuous, because defined by polynomials.

Lemma 2.7.4. *If R has a basis $\mathcal{U}_R$ of neighborhood of 0 formed by ideals, then R is complete if and only if $\mathbf{W}_m(R)$ is complete for all $m \geq 0$.*

Proof : It is evident for $m = 0$. Let $m \geq 1$ and $\{\boldsymbol{\nu}_n\}_n, \boldsymbol{\nu}_n := (\nu_{n,0}, \ldots, \nu_{n,m})$ be a Cauchy sequence in $\mathbf{W}_m(R)$. The sequence $\nu_{0,n}$ is Cauchy in R and we denote by $\nu_0 := \lim_n \nu_{0,n}$. The translate sequence $\boldsymbol{\nu}_n^1 := \boldsymbol{\nu}_n - (\nu_0, 0, \ldots, 0)$ is Cauchy, so we can suppose $\nu_n^1 = 0$. For every ideal $I \in \mathcal{U}_R$ there exists n_I such that $\boldsymbol{\nu}_{n_1}^1 - \boldsymbol{\nu}_{n_2}^1 = (S_{0,n_1,n_2}, \ldots, S_{m,n_1,n_2}) \in \mathbf{W}_m(I)$, for all $n_1, n_2 \geq n_I$. Let us write $S_{1,n_1,n_2} = \nu_{1,n_1}^1 - \nu_{1,n_2}^1 + P(\nu_{0,n_1}^1, \nu_{0,n_2}^1)$. By [Bou83a, §1 $n^0 3$ a)] the polynomial S_{k,n_1,n_2} is isobaric without constant term. Since $\nu_{0,n}^1 \in I$, for $n \geq n_I'$, sufficiently large and since I is an ideal, hence $\nu_{1,n_1}^1 - \nu_{1,n_2}^1 \in I$, for all $n_1, n_2 \geq n_I'$. So the sequence $\nu_{1,n}^1$ is Cauchy and converges to $\nu_1 \in R$. Moreover the sequence $\boldsymbol{\nu}_n^2 := \boldsymbol{\nu}_n^1 - (0, \nu_1, 0, \ldots, 0)$ is such that both $\nu_{0,n}^2$ and $\nu_{1,n}^2$ go to 0. Now we can restart this process and conclude by induction.$\Box$

Corollary 2.7.5. *If $(R, |.|)$ is an ultrametric valued ring, then R is complete if and only if $\mathbf{W}_m(R)$ is complete for all $m \geq 0$.*

Proof : $R^0 := \{r \in R \mid |r| \leq 1\}$ satisfies the hypothesis of 2.7.4. Following the proof of 2.7.4, the sequence $n \mapsto \boldsymbol{\nu}_n - \boldsymbol{\nu}_{n_1}$ lies in $\mathbf{W}_m(I) \subset \mathbf{W}_m(R^0)$, for $n >> 0$. $\Box$

2.7.3 Length

Let R be a ring of characteristic p. If the vector $\boldsymbol{\nu} = (\nu_0, \ldots, \nu_m) \in \mathbf{W}_m(R)$ is such that $\nu_0 = \ldots = \nu_{r-1} = 0$ and $\nu_r \neq 0$, then we define the length of $\boldsymbol{\nu}$ as

$$\ell(\boldsymbol{\nu}) := m - r \ , \tag{2.7.5.1}$$

and $\ell(\mathbf{0}) := -\infty$. If R is not of characteristic p, then we will define $\ell(\boldsymbol{\nu})$ as the length of the image of $\boldsymbol{\nu}$ in $\mathbf{W}_m(R/pR)$.

2.7.4 Covectors

We recall that the covectors module $\mathbf{CW}(R)$ is the additive group defined by the following inductive limit ([Bou83a, §1 ex. 23 p.47]) :

$$\mathbf{CW}(R) := \varinjlim(\mathbf{W}_m(R) \xrightarrow{\mathrm{V}} \mathbf{W}_{m+1}(R) \xrightarrow{\mathrm{V}} \cdots) \ . \tag{2.7.5.2}$$

2.7.5 Skew covectors

In the sequel we must work with a slightly different sequence. Let R be a ring of characteristic p. Then $\mathrm{V}\bar{\mathrm{F}} = \bar{\mathrm{F}}\mathrm{V}$ and $\mathrm{V}\bar{\mathrm{F}}(\nu_0, \ldots, \nu_m) = (0, \nu_0^p, \ldots, \nu_m^p)$. We define $\widetilde{\mathbf{CW}}(R)$ as the following inductive limit :

$$\widetilde{\mathbf{CW}}(R) := \varinjlim(\mathbf{W}_m(R) \stackrel{\mathrm{V}\bar{\mathrm{F}}}{\to} \mathbf{W}_{m+1}(R) \stackrel{\mathrm{V}\bar{\mathrm{F}}}{\to} \cdots) \ . \tag{2.7.5.3}$$

If R is a perfect field of characteristic p, then $\mathbf{CW}(R)$ is isomorphic to $\widetilde{\mathbf{CW}}(R)$. This results from the following commutative diagram :

$$\begin{tikzcd} R \arrow[r, "\mathrm{V}"] \arrow[d, equal] \arrow[dr, phantom, "\odot"] & \mathbf{W}_1(R) \arrow[r, "\mathrm{V}"] \arrow[d, "\bar{\mathrm{F}}", "\wr"'] \arrow[dr, phantom, "\odot"] & \mathbf{W}_2(R) \arrow[r, "\mathrm{V}"] \arrow[d, "\bar{\mathrm{F}}^2", "\wr"'] & \cdots \arrow[r] & \mathbf{CW}(R) \arrow[d, "\wr"'] \\ R \arrow[r, "\mathrm{V}\bar{\mathrm{F}}"] & \mathbf{W}_1(R) \arrow[r, "\mathrm{V}\bar{\mathrm{F}}"] & \mathbf{W}_2(R) \arrow[r, "\mathrm{V}\bar{\mathrm{F}}"] & \cdots \arrow[r] & \widetilde{\mathbf{CW}}(R) \ . \end{tikzcd} \tag{2.7.5.4}$$

Remark 2.7.6. If R is a field in characteristic p, then

$$\widetilde{\mathbf{CW}}(R) = \widetilde{\mathbf{CW}}(R^p) \ . \tag{2.7.6.1}$$

while $\mathbf{CW}(R^p) \subseteq \mathbf{CW}(R)$.

2.8 Notations in Artin-Schreier theory

Let R be a field of characteristic $p > 0$ and let R^{sep}/R be a fixed separable closure of R.

Definition 2.8.1. We denote by $\mathrm{G}_R = \mathrm{Gal}(R^{\mathrm{sep}}/R)$. If R is a complete discrete valuation field, we denote by $\mathcal{I}_R$ the inertia and by $\mathcal{P}_R$ the pro$-p-$Sylow of $\mathcal{I}_R$.

We have (cf. [Ser62, Ch.X, §3])

$$\mathrm{H}^1(\mathrm{G}_R, \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Z}/p^m\mathbb{Z}) \ . \tag{2.8.1.1}$$

The situation is then expressed by the following commutative diagram :

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/p^{m+1}\mathbb{Z} \arrow[r] \arrow[d, "\imath"] \arrow[dr, phantom, "\odot"] & \mathbf{W}_m(R) \arrow[r, "\bar{\mathrm{F}}-1"] \arrow[d, "\mathrm{V}"] \arrow[dr, phantom, "\odot"] & \mathbf{W}_m(R) \arrow[r, "\delta"] \arrow[d, "\mathrm{V}"] \arrow[dr, phantom, "\odot"] & \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Z}/p^{m+1}\mathbb{Z}) \arrow[r] \arrow[d, "\jmath"] & 0 \\ 0 \arrow[r] & \mathbb{Z}/p^{m+2}\mathbb{Z} \arrow[r] & \mathbf{W}_{m+1}(R) \arrow[r, "\bar{\mathrm{F}}-1"] & \mathbf{W}_{m+1}(R) \arrow[r, "\delta"] & \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Z}/p^{m+2}\mathbb{Z}) \arrow[r] & 0 \end{tikzcd} \tag{2.8.1.2}$$

where $\imath : 1 \mapsto p$ is the usual inclusion, and $\jmath$ is the composition with $\imath$. Let $\boldsymbol{\lambda} \in \mathbf{W}_m(R)$, the character $\alpha = \delta(\boldsymbol{\lambda})$ sends the automorphism γ in the element $\alpha(\gamma) := \gamma(\boldsymbol{\nu}) - \boldsymbol{\nu} \in \mathbb{Z}/p^{m+1}\mathbb{Z}$, where $\boldsymbol{\nu} \in R^{\mathrm{sep}}$ is a solution of the equation $\bar{\mathrm{F}}(\boldsymbol{\nu})-\boldsymbol{\nu}=\boldsymbol{\lambda}$. Taking the inductive limit, we get the following exact sequence :

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbf{CW}(R) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{CW}(R) \to \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Q}_p/\mathbb{Z}_p) \to 0\ , \tag{2.8.1.3}$$

where the word "cont" means that all characters $\mathrm{G}_R \to \mathbb{Q}_p/\mathbb{Z}_p$ factorize on a finite quotient of G_R. Indeed $\varinjlim_m \mathrm{Hom}(\mathrm{G}_R, \mathbb{Z}/p^m\mathbb{Z})$ can be seen as the subset of $\mathrm{Hom}(\mathrm{G}_R, \mathbb{Q}_p/\mathbb{Z}_p)$ formed by the elements killed by a power of p.

Remark 2.8.2. If the vertical arrows V are replaced by $\mathrm{V}\bar{\mathrm{F}}$ in the diagram 2.8.1.2, then the morphisms $\imath$ and $\jmath$ remain the same. Indeed

$$\delta(\boldsymbol{\lambda}) = \delta(\bar{\mathrm{F}}(\boldsymbol{\lambda})), \tag{2.8.2.1}$$

because $\bar{\mathrm{F}}(\boldsymbol{\lambda}) = \boldsymbol{\lambda} + (\bar{\mathrm{F}}-1)(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda} \in \mathbf{W}_s(R)$. Hence we have also

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \widetilde{\mathbf{CW}}(R) \xrightarrow{\bar{\mathrm{F}}-1} \widetilde{\mathbf{CW}}(R) \to \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Q}_p/\mathbb{Z}_p) \to 0\ . \tag{2.8.2.2}$$

Remark 2.8.3. Let $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m) \in \mathbf{W}_m(R)$. The kernel of $\alpha := \delta(\boldsymbol{\lambda})$ is the subgroup of G_R whose corresponding extension field is

$$R(\{\nu_0, \dots, \nu_m\}),$$

(i.e. the smallest field containing the set $\{\nu_0, \dots, \nu_m\}$), where $\boldsymbol{\nu} = (\nu_0, \dots, \nu_m) \in \mathbf{W}_m(R^{\mathrm{sep}})$ is solution of $\bar{\mathrm{F}}(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$. All cyclic extensions of R, whose degree is a power of p, are of this form for a suitable $m \geq 0$.

Let κ be a field of characteristic $p > 0$, and let $R := \kappa(\!(t)\!)$. The Galois group of an abelian extension of $\kappa(\!(t)\!)$ is the product of its $p-$torsion part (controlled by the Artin-Schreier theory) and its moderate part (controlled by Kummer theory).

Definition 2.8.4. We put

$$\mathbf{P}(\kappa) := \mathrm{Hom}^{\mathrm{cont}}(\mathcal{P}_R\ ,\ \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}^{\mathrm{cont}}(\mathcal{I}_R\ ,\ \mathbb{Q}_p/\mathbb{Z}_p). \tag{2.8.4.1}$$

Remark 2.8.5. We will see that

$$\mathbf{P}(\kappa) \cong \frac{\mathbf{CW}(t^{-1}\kappa[t^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(t^{-1}\kappa[t^{-1}])}\ . \tag{2.8.5.1}$$

On the other hand

$$\mathbf{CW}(\kappa)/(\bar{\mathrm{F}}-1)\mathbf{CW}(\kappa) = \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R/\mathcal{I}_R, \mathbb{Q}_p/\mathbb{Z}_p)\ , \tag{2.8.5.2}$$

and describes the abelianized of a pro-p-Sylow of the quotient $\mathrm{G}_R/\mathcal{I}_R$.

2.9 Notations in Lubin-Tate theory

For notations and results on Lubin-Tate theory we refer to [LT65]. In this paper we will treat only Lubin-Tate groups over the field $\mathbb{Q}_p$. We recall briefly only the facts used in this paper. Let

$$\mathrm{w} := p \cdot u \in p\mathbb{Z}_p \ , \quad u \in \mathbb{Z}_p^{\times} ,$$

be a uniformizing element. We recall that $\mathfrak{F}_{\mathrm{w}}$ is the family of formal power series $P(X) \in \mathbb{Z}_p[[X]]$ satisfying

$$P(X) \equiv \mathrm{w}X \pmod{X^2\mathbb{Z}_p[[X]]} \quad , \quad P(X) \equiv X^p \pmod{\mathrm{w}\mathbb{Z}_p[[X]]}. \tag{2.9.0.3}$$

A series in $\mathfrak{F}_{\mathrm{w}}$ will be called a Lubin-Tate series. For all $P \in \mathfrak{F}_{\mathrm{w}}$, there exists an unique formal group law $\mathfrak{G}_P(X,Y) \in \mathbb{Z}_p[[X,Y]]$ such that

$$P(\mathfrak{G}_P(X,Y)) = \mathfrak{G}_P(P(X),P(Y)) \tag{2.9.0.4}$$

that is $P(X)$ is an endomorphism of $\mathfrak{G}_P(X,Y)$.

Lemma 2.9.1. *Let $P, \widetilde{P} \in \mathfrak{F}_{\mathrm{w}}$. For all $a \in \mathbb{Z}_p$ there exists a unique formal series $[a]_{P,\widetilde{P}}(X) \in \mathbb{Z}_p[[X]]$ such that*

1. *$[a]_{P,\widetilde{P}}(X) \equiv aX \pmod{X^2\mathbb{Z}_p[[X]]}$,*
2. *$[a]_{P,\widetilde{P}}(\mathfrak{G}_P(X,Y)) = \mathfrak{G}_{\widetilde{P}}([a]_{P,\widetilde{P}}(X),[a]_{P,\widetilde{P}}(Y))$.*

In other words, $[a]_{P,\widetilde{P}}(X)$ is an homomorphism of group laws. We set

$$[a]_P(X) := [a]_{P,P}(X) \ . \ \square \tag{2.9.1.1}$$

By the uniqueness, we have that $P(X) = [\mathrm{w}]_P(X)$, and that the map $a \mapsto [a]_P(X) : \mathbb{Z}_p \to \mathrm{End}(\mathfrak{G}_P)$ is an injective morphism of rings which makes $\mathfrak{G}_P(X,Y)$ a $\mathbb{Z}_p-$module. Since $\mathfrak{G}_P(X,Y) = X+Y+$(monomials of degree $\geq$ 2), hence the bounded series $\mathfrak{G}_P(x,y)$ converges for $|x|,|y| < 1$, and $|\mathfrak{G}_P(x,y)| \leq \max(|x|,|y|) < 1$. For all complete extension fields $L/\mathbb{Q}_p$, the position $x*y := \mathfrak{G}_P(x,y)$ defines on $\mathfrak{p}_L$ a new group law. We denote by $\mathfrak{G}_P(\mathfrak{p}_L)$ this group. The group $\mathfrak{G}_P(\mathfrak{p}_L)$ becomes a $\mathbb{Z}_p-$module by setting $ax := [a]_P(x)$, $x \in \mathfrak{p}_L$.

Let $P^{(k)}$ denote the series $P \circ P \circ \cdots \circ P$, $k-$times. Following [LT65] let

$$\Lambda_{P,m} = \mathrm{Ker}(P^{(m)}) = \mathrm{Ker}([\mathrm{w}^m]_P) = \{x \in \mathbb{C}_p \mid P^{(m)}(x) = 0 \text{ and } |x| < 1\} \tag{2.9.1.2}$$

the set of w^m-torsion points of $\mathfrak{G}_P(\mathfrak{p}_{\mathbb{C}_p})$, and

$$\Lambda_P := \cup_m \Lambda_{P,m} \ . \tag{2.9.1.3}$$

We observe that $\Lambda_P \subset \mathbb{Q}_p^{\mathrm{alg}}$, because zeros of convergent analytic functions are algebraic (cf. [CR94, 5.4.7]). Since $[1]_{P,\widetilde{P}}(\Lambda_{P,m}) = \Lambda_{\widetilde{P},m}$, hence the field

$\mathbb{Q}_p(\Lambda_{P,m})$ depends only on w. Since $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/\mathbb{Q}_p)$ acts continuously on $\mathbb{Q}_p^{\mathrm{alg}}$, hence

$$\gamma([a]_P(x)) = [a]_P(\gamma(x)) \ , \tag{2.9.1.4}$$

for all $\gamma \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/\mathbb{Q}_p)$, and all $x \in \mathfrak{G}_P(\mathfrak{p}_{\mathbb{C}_p})$. Then $\mathbb{Q}_p(\Lambda_{P,m})/\mathbb{Q}_p$ is Galois.

Theorem 2.9.2 ([LT65, Th.2])**.** *We have the following properties :*

1. *For all $m \geq 0$, we have the following isomorphisms of $\mathbb{Z}_p$−modules*

$$\Lambda_{P,m} \cong \mathbb{Z}_p/p^m\mathbb{Z}_p \ , \tag{2.9.2.1}$$

$$\Lambda_P \cong \mathbb{Q}_p/\mathbb{Z}_p \ . \tag{2.9.2.2}$$

2. *Let $\gamma \in \mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p)$. There exists an unique unit $u_\gamma \in \mathbb{Z}_p^\times$ such that*

$$\gamma(x) = [u_\gamma]_P(x) \ , \quad \forall x \in \Lambda_P.$$

3. *The map $\gamma \mapsto u_\gamma$ is an isomorphism of $\mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p)$ onto the group $\mathbb{Z}_p^\times$. The same map gives an isomorphism*

$$\mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p(\Lambda_{P,m})) \xrightarrow{\sim} 1 + \mathrm{w}^m\mathbb{Z}_p \ , \quad \forall\, m \geq 1. \tag{2.9.2.3}$$

4. *If $u \in \mathbb{Z}_p^\times$, then*

$$[u]_P(x) = (u^{-1}, \mathbb{Q}_p(\Lambda_{P,m})/\mathbb{Q}_p)(x) \ , \tag{2.9.2.4}$$

for all $x \in \Lambda_{P,m}$, where $(u^{-1}, \mathbb{Q}_p(\Lambda_{P,m})/\mathbb{Q}_p) \in \mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p)$ is the norm residue symbol.

Remark 2.9.3. The simplest Lubin-Tate series is $P(X) = \mathrm{w}X + X^p$. If $\mathrm{w} = p$, then a non trivial zero π_0 of P is called a "π" of Dwork. If again $\mathrm{w} = p$ and $P(X) = (X+1)^p - 1$, then $\mathfrak{G}_P \cong \widehat{\mathbb{G}}_m$, and all torsion points are of the form $\xi - 1$, with $\xi^{p^k} = 1$, $\exists\, k \geq 0$. This was the choice made by Matsuda [Mat95].

Theorem 2.9.4 ([Haz78, Prop. 8.3.22])**.** *Let $\mathfrak{G}$ and $\widetilde{\mathfrak{G}}$ be two Lubin-Tate groups relative to the uniformizers* w *and* $\tilde{\mathrm{w}}$ *respectively. Then $\mathfrak{G}$ is isomorphic to $\widetilde{\mathfrak{G}}$ (as formal groups over $\mathbb{Z}_p$) if and only if* $\mathrm{w} = \tilde{\mathrm{w}}$. □

2.9.1 Tate module

Definition 2.9.5. The Tate module of $\mathfrak{G}_P$ is

$$\mathrm{T}(\mathfrak{G}_P) := \varprojlim_m (\Lambda_{P,m} \xrightarrow{[\mathrm{w}]} \Lambda_{P,m-1}) \ . \tag{2.9.5.1}$$

A generator

$$\boldsymbol{\pi} = (\pi_{P,j})_{j\geq 0}$$

of the Tate module $\mathrm{T}(\mathfrak{G}_P)$ is a family $(\pi_{P,j})_{j\geq 0}$, $\pi_j \in \Lambda_P$, such that

$$P(\pi_{P,0}) = 0\ , \quad \pi_{P,0} \neq 0\ , \quad \text{and} \quad P(\pi_{P,j+1}) = \pi_{P,j}\ ,$$

for all $j \geq 0$. If no confusion is possible, we will write π_j instead of $\pi_{P,j}$.

The Newton polygon of P shows that P has exactly $p-1$ non trivial zeros of value $\omega = |p|^{\frac{1}{p-1}}$, and inductively $P(X) - \pi_{j-1}$ has p zeros of valuation $\omega^{\frac{1}{p^j}}$. Hence

$$|\pi_j| = \omega^{1/p^j}\ , \quad \forall\, j \geq 0, \tag{2.9.5.2}$$

and then the Galois extension $\mathbb{Q}_p(\Lambda_{P,m}) = \mathbb{Q}_p(\pi_{m-1})$ is totally ramified. On the other hand the field $K(\pi_{m-1})$ is not always totally ramified.

Definition 2.9.6. We set $K_m := K(\pi_m)$ (resp. $K(\Lambda_P) := \cup_{m\geq 0} K_m$), and denote by k_m (resp. k_{w}) its residue field. Moreover, if $\mathrm{w} = p$, we put

$$K_\infty := K(\Lambda_P) \tag{2.9.6.1}$$

and $k_\infty := k_p$. For all algebraic extensions L/K, L_∞ will be the smallest field containing L and K_∞.

Example 2.9.7. If $P(X) = (X+1)^p - 1$, then $\mathfrak{G}_P = \widehat{\mathbb{G}}_m$, and $\Lambda_m = \{\xi_m - 1 \mid \xi_m^{p^{m+1}} = 1\}$ is the set of $p^{m+1}-$th root of 1 minus 1. A generator of $\mathrm{T}(\widehat{\mathbb{G}}_m)$ is a family $(\xi_j - 1)_{j\geq 0}$ satisfying $\xi_j^{p^i} = \xi_{j-i}$, for all $0 \leq i \leq j$.

Definition 2.9.8. Let $P, \widetilde{P} \in \mathfrak{F}_{\mathrm{w}}$ be two Lubin-Tate series. We will say that $x \in \Lambda_P$ and $y \in \Lambda_{\widetilde{P}}$ are equivalent if $y = [1]_{P,\widetilde{P}}(x)$ (cf. 2.9.1).

Remark 2.9.9. Since $[1]_{P,\widetilde{P}}(x) = x +$ (things divisible by x^2), hence

$$|x - [1]_{P,\widetilde{P}}(x)| \ \leq\ |x|^2\ . \tag{2.9.9.1}$$

In particular, if $\mathrm{w} = p$ and if π_m is fixed, then there exists a unique $p^{m+1}-$th root of 1, say ξ_m, such that $|(\xi_m - 1) - \pi_m| \leq \omega^{\frac{2}{p^m}}$ (cf. 2.9.3).

Chapitre 3

π−EXPONENTIALS

3.1 Construction of Witt vectors

Let $P(X) \in \mathbb{Z}_p[[X]]$ be a series, with $P(0) = 0$, satisfying

$$P(X) \equiv X^p \mod p\mathbb{Z}_p[[X]]. \tag{3.1.0.1}$$

We consider the Frobenius $\sigma_P : \mathbb{Z}_p[[X]] \to \mathbb{Z}_p[[X]]$ given by $\sigma_P(h(X)) := h(P(X))$.

Lemma 3.1.1 ([Bou83a, Ch.IX,§1,ex.14,a)])**.** *There is a unique ring morphism*

$$[-] : \mathbb{Z}_p[[X]] \xrightarrow[h(X)\mapsto[h(X)]]{} \mathbf{W}(\mathbb{Z}_p[[X]]) \tag{3.1.1.1}$$

such that $\phi_j \circ [-] = \sigma_P^j$. In other words, for all $h(X) \in \mathbb{Z}_p[[X]]$, the Witt vector $[h(X)]$ is the unique one whose phantom vector is equal to

$$\langle\ h(X)\ ,\ h(P(X))\ ,\ h(P(P(X)))\ ,\ \ldots\ \rangle. \tag{3.1.1.2}$$

Moreover $[-]$ is also the unique ring morphism satisfying the relation

$$\mathrm{F}([h(X)]) = [h(P(X))]. \tag{3.1.1.3}$$

Proof : By lemma 2.7.2, the ring morphism $h(X) \mapsto \langle h(X), h(P(X)), \ldots\rangle : \mathbb{Z}_p[[X]] \to (\mathbb{Z}_p[[X]])^{\mathbb{N}}$ has its values in the image of the phantom component map $\phi : \mathbf{W}(\mathbb{Z}_p[[X]]) \hookrightarrow (\mathbb{Z}_p[[X]])^{\mathbb{N}}$. Since, by 2.7.1, ϕ is injective, the lemma is proved. $\square$

Definition 3.1.2. Let B be a complete topologized $\mathbb{Z}_p$−ring, and let $b \in \mathrm{B}$ be a topologically nilpotent element. The specialization $X \mapsto b : \mathbb{Z}_p[[X]] \to \mathrm{B}$ gets, by functoriality, a morphism $\mathbf{W}(\mathbb{Z}_p[[X]]) \to \mathbf{W}(\mathrm{B})$. For brevity, we denote by $[h(b)]$ the image of $h(X)$ via the morphism

$$\mathbb{Z}_p[[X]] \xrightarrow{[-]} \mathbf{W}(\mathbb{Z}_p[[X]]) \xrightarrow{X\mapsto b} \mathbf{W}(\mathrm{B}). \tag{3.1.2.1}$$

We will denote again by $[h(b)]$ its image in $\mathbf{W}_m(\mathrm{B})$.

Remark 3.1.3. The phantom vector of $[h(b)]$ is

$$\langle\, h(b)\ ,\ h(P(b))\ ,\ h(P(P(b)))\ ,\ \ldots\,\rangle. \tag{3.1.3.1}$$

In general there is no morphism $\mathbb{Z}_p[b] \to \mathbf{W}(\mathrm{B})$ sending $h(b)$ into $[h(b)]$, the notation $[h(b)]$ is actually abusive, but more handy.

Lemma 3.1.4 (Key Lemma)**.** *Let* $(\mathrm{B}, |.|)$ *be a* $\mathbb{Z}_p-$*ring, complete with respect to an absolute value* $|.|$*, extending the absolute value of* $\mathbb{Z}_p$*. Let* $h(X) = \sum_{i\geq 0} a_i X^i \in \mathbb{Z}_p[[X]]$*, and let* $[h(b)] = (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathrm{B})$*, with* $|b| < 1$*. Then the following statements are equivalent :*

1. $|a_0| = |p|^r$,
2. $|\lambda_0|, \ldots, |\lambda_{r-1}| < 1$*, and* $|\lambda_r| = 1$.

Proof : Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots) = [h(b)]$. We denote by $\overline{\mathrm{B}}$ the residue field. The condition (2) is equivalents to $\bar{\lambda}_r \neq 0$, and $\bar{\lambda}_i = 0$, for all $i < r$, or, if $k \geq 0$ is given, it is equivalent to $\bar{\lambda}_r^{p^k} \neq 0$, and $\bar{\lambda}_i^{p^k} = 0$, for all $i < r$. This last condition is equivalent to the condition (2) for the vector $\mathrm{F}^k(\boldsymbol{\lambda})$. Now the phantom vector of $\mathrm{F}^k(\boldsymbol{\lambda})$ is

$$\langle h(P^{(k)}(b)), h(P^{(k+1)}(b)), \ldots\rangle \tag{3.1.4.1}$$

(cf. 2.9.1.2). Moreover $|P(b)| \leq \sup(|b|^p, |p||b|)$, hence, for all $\varepsilon > 0$, there exists $k \geq 0$ such that $|P^{(i)}(b)| < \varepsilon$, for all $i \geq k$. If ε is small enough, then $|h(P^{(i)}(b))| = |a_0|$, for all $i \geq k$. Let $(\nu_0, \nu_1, \ldots) := \mathrm{F}^k(\boldsymbol{\lambda})$, then, since

$$p^j \nu_j = h(P^{(j)}(b)) - (\nu_0^{p^j} + \cdots + p^{j-1}\nu_{j-1}^p)\ , \tag{3.1.4.2}$$

we see, by induction, that $|a_0| = |p|^r$ if and only if $|\nu_r| = 1$ and $|\nu_j| = |p|^{r-j}$, for all $j \leq r-1$.$\square$

Definition 3.1.5. We fix now a sequence $\boldsymbol{\varpi} := \{\varpi_j\}_{j\geq 0}$ in $\mathbb{Q}_p^{\mathrm{alg}}$ satisfying $|\varpi_0| < 1$, $P(\varpi_0) = 0$ and $P(\varpi_{j+1}) = \varpi_j$, for all $j \geq 1$.

Remark 3.1.6. The ring $\mathbb{Z}_p[\varpi_m]$ is complete, for all $m \geq 0$. Indeed ϖ_m is algebraic and integral over $\mathbb{Z}_p$, hence $\mathbb{Z}_p[\varpi_m]$ is a free module over $\mathbb{Z}_p$.

Remark 3.1.7. If P is a Lubin-Tate series, then $\boldsymbol{\varpi}$ is a generator of the Tate module $\mathrm{T}(\mathfrak{G}_P)$, while if $P(X) \equiv X^p \mod X^{p+1}\mathbb{Z}_p[[X]]$, then $\varpi_j = 0$ for all $j \geq 0$.

Observe that, for all $m \geq 0$, $[\varpi_m] \in \mathbf{W}(\mathbb{Z}_p[\varpi_m])$ is the unique Witt vector whose phantom vector is $\langle \varpi_m, \varpi_{m-1}, \ldots, \varpi_0, 0, \ldots\rangle$. The uniqueness follows from the injectivity of the phantom map $\phi : \mathbf{W}(\mathbb{Z}_p[\varpi_m]) \hookrightarrow (\mathbb{Z}_p[\varpi_m])^{\mathbb{N}}$.

Proposition 3.1.8. *For all* $\mathbb{Z}_p[\varpi_m]-$*algebra* B *of characteristic* 0*, we have*

$$[\varpi_j]\mathbf{W}(\mathrm{B}) \subset [\varpi_{j+1}]\mathbf{W}(\mathrm{B})\ , \qquad j = 0, \ldots, m-1\ . \tag{3.1.8.1}$$

Moreover, for all $\boldsymbol{\lambda} \in \mathbf{W}(\mathrm{B})$*, and all* $j = 0, \ldots, m-1$*, we have*

$$\mathrm{F}([\varpi_{j+1}]) = [\varpi_j] \qquad ; \qquad \mathrm{V}([\varpi_j] \cdot \boldsymbol{\lambda}) = [\varpi_{j+1}] \cdot \mathrm{V}(\boldsymbol{\lambda}). \tag{3.1.8.2}$$

Hence $\mathrm{F}([\varpi_{j+1}]\mathbf{W}(\mathrm{B})) \subset [\varpi_j]\mathbf{W}(\mathrm{B})$ *and* $\mathrm{V}([\varpi_j]\mathbf{W}(\mathrm{B})) \subset [\varpi_{j+1}]\mathbf{W}(\mathrm{B})$.

If now $\varpi_0 \neq 0$*, then the kernel of the morphism* $\boldsymbol{\lambda} \mapsto [\varpi_m]\boldsymbol{\lambda}$ *is the ideal* $\mathrm{V}^{m+1}\mathbf{W}(\mathrm{B})$*. The induced morphism* $\mathbf{W}_m(\mathrm{B}) \to \mathbf{W}(\mathrm{B})$ *is a functorial isomorphism of* $\mathbf{W}_m(\mathrm{B})$ *into the ideal* $[\varpi_m]\mathbf{W}(\mathrm{B})$ *(as* $\mathbf{W}(\mathrm{B})-$*modules), which commutes with* $\mathrm{V} : \mathbf{W}_m(\mathrm{B}) \to \mathbf{W}_{m+1}(\mathrm{B})$ *and* $\mathrm{F} : \mathbf{W}_{m+1}(\mathrm{B}) \to \mathbf{W}_m(\mathrm{B})$

$$\begin{tikzcd} \mathbf{W}(\mathrm{B}) \arrow[r, "\boldsymbol{\lambda}\mapsto[\varpi_m]\boldsymbol{\lambda}"] \arrow[d, two heads] & \mathbf{W}(\mathrm{B}) \\ \mathbf{W}_m(\mathrm{B}) \arrow[r, "\sim"] & {[\varpi_m] \cdot \mathbf{W}(\mathrm{B})\ .} \arrow[u, hook] \end{tikzcd} \tag{3.1.8.3}$$

Proof : Let $h(X) := P(X)/X$, then $[\varpi_j]\boldsymbol{\lambda} = [P(\varpi_{j+1})]\boldsymbol{\lambda} = [\varpi_{j+1} \cdot h(\varpi_{j+1})]\boldsymbol{\lambda} = [\varpi_{j+1}] \cdot [h(\varpi_{j+1})]\boldsymbol{\lambda}$. This shows the inclusion 3.1.8.1. All other assertions are easily verified on the phantom components. □

Corollary 3.1.9. *Let* $\langle \phi_0, \ldots, \phi_m \rangle \in \mathbb{Z}_p[\varpi_m]^{m+1}$*. If there exists a formal series* $h(X) = \sum_{i \geq 0} a_i X^i \in \mathbb{Z}_p[[X]]$ *satisfying*

$$h(\varpi_{m-j}) = \phi_j\ , \quad \textit{for all } 0 \leq j \leq m, \tag{3.1.9.1}$$

then $\langle \phi_0, \ldots, \phi_m \rangle$ *is the phantom vector of* $[h(\varpi_m)] := (\nu_0, \ldots, \nu_m) \in \mathbf{W}_m(\mathbb{Z}_p[\varpi_m])$*. Moreover,* $|a_0| = |p|^r$*, for some* $r \geq 0$*, if and only if* $|\nu_0|, \ldots, |\nu_{r-1}| < 1$ *and* $|\nu_r| = 1$. □

3.2 Artin-Hasse exponential and Robba exponentials

Definition 3.2.1 ([Bou83a, ex.58])**.** Let B be a $\mathbb{Z}_{(p)}-$ring, and let

$$E(T) := \exp(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots) \in 1 + T\mathbb{Z}_{(p)}[[T]]\ . \tag{3.2.1.1}$$

For all $\boldsymbol{\lambda} := (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathrm{B})$, the Artin-Hasse exponential relative to $\boldsymbol{\lambda}$ is

$$E(\boldsymbol{\lambda}, T) := \prod_{j \geq 0} E(\lambda_j \cdot T^{p^j}) = \exp(\phi_0 T + \phi_1 \frac{T^p}{p} + \phi_2 \frac{T^{p^2}}{p^2} + \cdots) \in 1 + T\mathrm{B}[[T]]\ , \tag{3.2.1.2}$$

where $\langle \phi_0, \phi_1, \ldots \rangle$ is the phantom vector of $\boldsymbol{\lambda}$.

Remark 3.2.2. The Artin-Hasse exponential is then a group morphism

$$E(-,T) : \mathbf{W}(\mathrm{B}) \to 1 + T\mathrm{B}[[T]], \tag{3.2.2.1}$$

functorial on the $\mathbb{Z}_{(p)}-$ring B.

Proposition 3.2.3. *Let $[\varpi_m] \in \mathbf{W}(\mathbb{Z}_p[\varpi_m])$ be as in 3.1.5. The exponential*

$$E_m(T) := E([\varpi_m], T) = \exp(\varpi_m T + \varpi_{m-1}\frac{T^p}{p} + \cdots + \varpi_0 \frac{T^{p^m}}{p^m}) \tag{3.2.3.1}$$

converges exactly in the disk $|T| < 1$, for all $m \geq 0$, if and only if $P(X)$ is a Lubin-Tate series, and $\boldsymbol{\varpi} := (\varpi_j)_{j \geq 0}$ is a generator of the Tate module $\mathrm{T}(\mathfrak{G}_P)$.

Proof : Assume that the radius of convergence of $E([\varpi_m], T)$ is equal to 1, for all $m \geq 0$. Then, for $m = 0$, the radius of convergence of $\exp(\varpi_0 T)$ is 1, hence $|\varpi_0| = \omega$. The Newton polygon of $P(X)$ implies that $P(X) \equiv \mathrm{w}X \mod X\mathbb{Z}_p[[X]]$, with $|\mathrm{w}| = |p|$, hence $P(X)$ is a Lubin-Tate series. Conversely, assume that $P(X)$ is a Lubin-Tate series, and that $\boldsymbol{\varpi} := (\varpi_j)_{j \geq 0}$ is a generator of $\mathrm{T}(\mathfrak{G}_P)$. Consider the differential operator

$$L := \partial_T + \varpi_m T^{-1} + \varpi_{m-1} T^{-p} + \cdots + \varpi_0 T^{-p^m} . \tag{3.2.3.2}$$

Then $E_m(T^{-1})$ is the Taylor solution at $+\infty$ of L. Since $|\varpi_0| = \omega$, by lemma 2.4.3, we have $Ray(L, \rho) = \rho^{p^m+1}$, for all $\rho < 1$. In particular, the irregularity of L is p^m. Then $E_m(T^{-1})$ is not convergent for $|T| < 1$, because otherwise, by transfer at ∞, $E_m(T^{-1}) \in \mathcal{R}$, and L will be trivial. □

Theorem 3.2.4. *Let $P(X) = \mathrm{w}X + \cdots$ be a Lubin-Tate series, and let $\boldsymbol{\varpi} := (\varpi_j)_{j \geq 0}$ be a generator of $\mathrm{T}(\mathfrak{G})$. Then the formal series $E_m(T^p)/E_m(T)$ is over-convergent (i.e. convergent for $|T| < 1 + \varepsilon$, for some $\varepsilon > 0$) if and only if*

$$|\mathrm{w} - p| \leq |p|^{m+2} .$$

In particular, $E_m(T^p)/E_m(T)$ is over-convergent for all $m \geq 0$ if and only if $\mathfrak{G}_P$ is isomorphic to the formal multiplicative group $\widehat{\mathbb{G}}_m$ (cf. theorem 2.9.4).

Proof : This theorem will follow easily from the theory of $\boldsymbol{\pi}-$exponentials (cf. the proof of 3.6.1 infra), and is placed here for expository reasons.□

3.3 $\boldsymbol{\pi}-$exponentials

We preserve the notations of section 3.1. In this section we fix an uniformizing element w of $\mathbb{Z}_p$, a $\mathbb{Q}_p-$Lubin-Tate series $P \in \mathbb{Z}_p[[X]]$, $P \in \mathfrak{F}_{\mathrm{w}}$, and a generator $\boldsymbol{\pi} = (\pi_j)_{j \geq 0}$ of the Tate module. We fix three natural numbers n, m, d such that

$$d = n \cdot p^m > 0 \ , \quad \text{and } (n, p) = 1. \tag{3.3.0.1}$$

Definition 3.3.1. Let B be a $\mathbb{Z}_p[\pi_m]-$algebra. Let $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathrm{B})$, and let $\langle \phi_0, \ldots, \phi_m \rangle \in \mathrm{B}^{m+1}$ be its phantom vector. We set

$$\mathrm{e}_d(\boldsymbol{\lambda}, T) := E([\pi_m]\boldsymbol{\lambda}, T^n) = \exp\bigl(\pi_m \phi_0 T^n + \pi_{m-1}\phi_1 \frac{T^{np}}{p} + \cdots + \pi_0 \phi_m \frac{T^d}{p^m}\bigr) \ . \tag{3.3.1.1}$$

We will call $\mathrm{e}_d(\boldsymbol{\lambda}, T) \in 1 + \pi_m T\mathrm{B}[[T]]$ the $\boldsymbol{\pi}-$*exponential* attached to $\boldsymbol{\lambda}$.

Proposition 3.3.2. *The map $\boldsymbol{\lambda} \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T)$ defines a group morphism*

$$\mathbf{W}_m(\mathrm{B}) \longrightarrow 1 + \pi_m T\mathrm{B}[[T]] \ . \tag{3.3.2.1}$$

Moreover, for all $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathbf{W}_m(\mathrm{B})$, we have

$$\mathrm{e}_d(\boldsymbol{\lambda}, T) = \prod_{j=0}^{m} E_{m-j}(\lambda_j T^{np^j}), \tag{3.3.2.2}$$

$$\begin{array}{rclcrcl} E_m(T) & = & \mathrm{e}_{p^m}((1,0,\ldots,0),T) & , & \mathrm{e}_d(\boldsymbol{\lambda},T) & = & \mathrm{e}_{p^m}(\boldsymbol{\lambda},T^n) \ , \\ \mathrm{e}_d(\boldsymbol{\lambda},T^p) & = & \mathrm{e}_{p\cdot d}(\mathrm{V}(\boldsymbol{\lambda}),T) & , & \mathrm{e}_d(\boldsymbol{\lambda}+\boldsymbol{\nu},T) & = & \mathrm{e}_d(\boldsymbol{\lambda},T)\cdot \mathrm{e}_d(\boldsymbol{\nu},T) \ . \end{array}$$

Furthermore, if $\mathrm{B} = \mathcal{O}_L$ is the ring of integers of some finite extension L/K, and if, for some $r \geq 1$, there exists a Frobenius σ on $\mathcal{O}_L$ lifting the p^r-th power map $x \mapsto x^{p^r}$ of k_L, and satisfying $\sigma(\pi_j) = \pi_j$, $\forall\ 0 \leq j \leq m$, then we have

$$\mathrm{e}_d^{\sigma}(\boldsymbol{\lambda}, T) = \mathrm{e}_d(\sigma(\boldsymbol{\lambda}), T) \ , \tag{3.3.2.3}$$

where $\sigma(\lambda_0, \ldots, \lambda_m) = (\sigma(\lambda_0), \ldots, \sigma(\lambda_m))$ (cf. 2.7) and, for all $f(T) = \sum a_i T^i$, we set $f^{\sigma}(T) = \sum \sigma(a_i) T^i$ (cf. 2.5.2.1).

Proof : All the assertions are easily verified on the phantom components.
□

3.4 Study of the differential module attached to a $\boldsymbol{\pi}-$exponential

We preserve the notations of section 3.3. As usual $d = np^m > 0$, with $(n, p) = 1$. In this subsection H/K is an algebraic extension (not necessary complete) and

$$H_m := H(\pi_m) \ . \tag{3.4.0.4}$$

Remark 3.4.1. The Witt vectors we are considering have a finite number of entries. Hence the exponential $\mathrm{e}_d(\boldsymbol{\lambda}, T)$ has its coefficients in a *finite* (hence complete) extension of K. This will solve all problems concerning the convergence.

Definition 3.4.2. Let $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$, and let $\langle \phi_0, \ldots, \phi_m \rangle \in \mathcal{O}_H^{m+1}$ be its phantom vector. We define

$$\mathrm{L}_d(\boldsymbol{\lambda}) := \partial_T - \partial_{T,\log}(\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})) = \partial_T + n \cdot \Big(\sum_{j=0}^{m} \pi_{m-j} \cdot \phi_j \cdot T^{-np^j} \Big) \ . \quad (3.4.2.1)$$

We denote by $\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})$ the differential module over $\mathcal{R}_{H_m}$ defined by $\mathrm{L}_d(\boldsymbol{\lambda})$.

Lemma 3.4.3. *$\mathrm{L}_d(\boldsymbol{\lambda})$ is solvable at $\rho = 1$, and hence $\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}) \in \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m})$.*

Proof : The Taylor solution at $+\infty$ of $\mathrm{L}_d(\boldsymbol{\lambda})$ is $\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1}) \in 1 + \pi_m T^{-1} \mathcal{O}_{H_m}[[T^{-1}]]$, which has bounded coefficients and converges then for $|T| > 1$. By transfer (cf. 2.3.2), $Ray(\mathrm{L}_d(\boldsymbol{\lambda}), \rho) = \rho$, for all $\rho > 1$. By continuity of the radius, $Ray(\mathrm{L}_d(\boldsymbol{\lambda}), 1) = 1.\Box$

Proposition 3.4.4. *The map $\boldsymbol{\lambda} \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})$ defines a group morphism*

$$\mathbf{W}_m(\mathcal{O}_H) \longrightarrow 1 + \pi_m T^{-1} \mathcal{O}_{H_m}[[T^{-1}]] \ . \quad (3.4.4.1)$$

More precisely, for all $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathbf{W}_m(\mathcal{O}_H)$, we have :

$$\varphi_p^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) \ = \ \widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda})) \quad , \quad \widetilde{\mathrm{M}}_d(\boldsymbol{\lambda} + \boldsymbol{\nu}) \ = \ \widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}) \otimes \widetilde{\mathrm{M}}_d(\boldsymbol{\nu}) \ , \quad (3.4.4.2)$$

where $\varphi_p(f(T)) = f(T^p)$ (cf. 2.5.4.1). Moreover, if there exists an absolute Frobenius σ on H_m (cf. 2.5.1) such that $\pi_j^\sigma = \pi_j$, for all $0 \leq j \leq m$, then we have

$$\varphi_\sigma(\mathrm{e}_d(\boldsymbol{\lambda}, T)) \ = \ \mathrm{e}_d(\varphi_\sigma(\boldsymbol{\lambda}), T^p) \quad , \qquad \varphi_\sigma^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) \ = \ \widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\sigma(\boldsymbol{\lambda}))) \ ,$$

where $\varphi_\sigma(f(T)) = f^\sigma(T^p)$, (cf. 2.5.2.3), and $\sigma(\lambda_0, \ldots, \lambda_m) = (\sigma(\lambda_0), \ldots, \sigma(\lambda_m))$.

Proof : The first part is a direct consequence of 3.3.2. The last assertion is a consequence of 3.5.4 and is placed here for expository reasons. Observe that, in the sequel, we do not suppose the existence of σ on H. Indeed, our "Frobenius structure theorem" does not need the existence of φ (cf. 3.6.3). $\Box$

Remark 3.4.5. In particular $\widetilde{\mathrm{M}}_d$ defines a morphism of groups

$$\widetilde{\mathrm{M}}_d : \mathbf{W}_m(\mathcal{O}_H) \longrightarrow \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m}) \ . \quad (3.4.5.1)$$

Theorem 3.4.6. *Let $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$ be a Witt vector, and let $\langle \phi_0, \ldots, \phi_m \rangle \in \mathcal{O}_H^{m+1}$ be its phantom vector. The following assertions are equivalent :*

1. *$\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})$ is trivial (i.e. isomorphic to $\mathcal{R}_{H_m}$);*
2. *The exponential $\mathrm{e}_d(\boldsymbol{\lambda}, T)$ is over-convergent (i.e. convergent in some disk $|T| < 1 + \varepsilon$, with $\varepsilon > 0$);*
3. *$|\lambda_0|, \ldots, |\lambda_m| < 1$.*

Moreover, if $|\lambda_0|,\dots,|\lambda_{r-1}|<1$ and $|\lambda_r|=1$, $r\leq m$, then we have (cf. 2.7.3)

$$\mathrm{Irr}(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}))=n\cdot p^{\ell(\boldsymbol{\lambda})}=d/p^r\ . \qquad (3.4.6.1)$$

Proof : The equivalence (1) $\Leftrightarrow$ (2) is evident. By 3.3.2.2 and by 3.2.3, the condition (3) implies that $\mathrm{e}_d(\boldsymbol{\lambda},T^{-1})\in\mathcal{R}_{H_m}$, then $\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})$ is trivial. The converse follows from the last assertion. Let then $|\lambda_0|,\dots,|\lambda_{r-1}|<1$, and $|\lambda_r|=1$, $r\leq m$. Clearly $|\phi_m(\boldsymbol{\lambda})|=1$ if and only if $|\lambda_0|=1$ (cf. 2.7.0.2). Then, if $r=0$, we can apply 2.4.3, and hence $\mathrm{Irr}(M)=\mathrm{Irr}_F(M)=d$. Let now $0<r\leq m$, then $E_{m-j}(\lambda_jT^{-p^j})$ belongs to $\mathcal{R}_{H_m}^{\times}$, for all $j=0,\dots,r-1$. Then we change basis by the function $f(T):=\prod_{j=0}^{r-1}E_{m-j}(\lambda_jT^{-p^j})^{-1}\in\mathcal{R}_{H_m}^{\times}$. By the rules of 3.3.2, the new solution is

$$f(T)\cdot\mathrm{e}_d(\boldsymbol{\lambda},T)=\mathrm{e}_d((0,\dots,0,\lambda_r,\dots,\lambda_m),T)=\mathrm{e}_{d/p^r}((\lambda_r,\dots,\lambda_m),T^{p^r}). \qquad (3.4.6.2)$$

In other words, we have $\widetilde{\mathrm{M}}_d(\lambda_0,\dots,\lambda_m)\xrightarrow{\sim}\varphi_p^*(\widetilde{\mathrm{M}}_{d/p^r}(\lambda_r,\dots,\lambda_m))$ (cf. 2.5.2.1 and 3.4.4.2). By 2.5.10, the theorem is proved by induction, since $|\lambda_r|=1$.□

Remark 3.4.7. In particular, $\widetilde{\mathrm{M}}_d$ passes to the quotient $\mathbf{W}_m(k_H)$, and induces an injective additive map called M_d :

$$\begin{tikzcd} \mathbf{W}_m(\mathcal{O}_H) \arrow[r, "\widetilde{\mathrm{M}}_d"] \arrow[d] & \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m}) \\ \mathbf{W}_m(k_H) \arrow[ur, hook, dashed, "\mathrm{M}_d"'] & \end{tikzcd} \qquad (3.4.7.1)$$

Corollary 3.4.8. *Consider the morphism of groups*

$$\mathbb{Z}_p[[X]]\xrightarrow{[-]}\mathbf{W}_m(\mathbb{Z}_p[\pi_m])\subset\mathbf{W}_m(\mathcal{O}_{H_m})\xrightarrow{\widetilde{\mathrm{M}}_d}\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m}). \qquad (3.4.8.1)$$

Let $h(X):=\sum_{i\geq0}a_iX^i\in\mathbb{Z}_p[[X]]$ be such that $|a_0|=|p|^r$ ($v_p(a_0)=r$). Then $\widetilde{\mathrm{M}}_d([h(\pi_m)])$ has irregularity d/p^r, and is trivial if and only if $r\geq m+1$. In particular, the kernel of the composite map is the ideal $p^{m+1}\mathbb{Z}_p[[X]]+X\mathbb{Z}_p[[X]]$.

Proof : Combine 3.1.9 and 3.4.6.□

3.5 Dependence on the Lubin-Tate group and on $\boldsymbol{\pi}$

We preserve the notations of sections 3.3 and 3.4. As usual

$$d=np^m>0\ ,\quad (n,p)=1\ . \qquad (3.5.0.2)$$

Theorem 3.5.1 (Dependence on the choice of $\boldsymbol{\pi}$)**.** *Let $\boldsymbol{\pi}=(\pi_j)_{j\geq0}$, $\boldsymbol{\pi'}=(\pi'_j)_{j\geq0}$ be two generators of $T(\mathfrak{G}_P)$. Denote by $\mathrm{M}'_d(-)$, $E'_j(T)$ and $\mathrm{e}'_d(-,T)$ the constructions attached to $\boldsymbol{\pi'}$. Then $\mathrm{M}_d(1,0,\dots,0)$ and $\mathrm{M}'_d(1,0,\dots,0)$ are isomorphic over $\mathcal{R}_{H_m}$ if and only if $\pi_m=\pi'_m$. Moreover, in this case, $\mathrm{M}_d(\boldsymbol{\lambda})$ and $\mathrm{M}'_d(\boldsymbol{\lambda})$ are isomorphic over $\mathcal{R}_{H_m}$, for all $\boldsymbol{\lambda}\in\mathbf{W}_m(k_H)$.*

Proof : The solution at ∞ of $\mathcal{R}_{H_m}$ is $\mathrm{e}_d((1,0,\dots,0),T^{-1}) = E_m(T^{-n})$. We shall show that $E_m(T^{-n})/E'_m(T^{-n}) \in \mathcal{R}^\times$, that is $E_m(T^{-1})/E'_m(T^{-1}) \in \mathcal{R}^\times$, if and only if $\pi_m = \pi'_m$. We have

$$E_m(T^{-1})/E'_m(T^{-1}) = \exp\Big(\sum_{j=0}^{m} \pi_{m-j}(1 - \frac{\pi'_{m-j}}{\pi_{m-j}})\frac{T^{-p^j}}{p^j}\Big)\ . \tag{3.5.1.1}$$

There exists $\gamma \in \mathrm{Gal}(\mathbb{Q}_p(\Lambda)/\mathbb{Q}_p)$ such that $\pi'_j = \gamma(\pi_j)$, for all $j \geq 0$ and, by the Lubin-Tate theorem 2.9.2, $\gamma(\pi_j) = [u_\gamma]_P(\pi_j)$, $u_\gamma \in \mathbb{Z}^\times$. We set [1]

$$h_\gamma(X) := 1 - [u_\gamma]_P(X)/X\ , \tag{3.5.1.2}$$

in order to have

$$E_m(T^{-1})/E'_m(T^{-1}) = \mathrm{e}_d([h_\gamma(\pi_m)], T^{-1}). \tag{3.5.1.3}$$

Indeed, by construction (cf. corollary 3.1.9 and definition 3.3.1.1), we have $\phi_j([h_\gamma(\pi_m)]) = 1 - \pi'_{m-j}/\pi_{m-j}$. Since $h_\gamma(0) = 1 - u_\gamma$, hence, by the reduction theorem 3.4.6 and lemma 3.4.8, the series $E_m(T^{-1})/E'_m(T^{-1})$ lies in $\mathcal{R}^\times$ if and only if $|1-u_\gamma| \leq |p|^{m+1}$, i.e. $u_\gamma \in 1+p^{m+1}\mathbb{Z}_p$. Then, again by the reciprocity law 2.9.2, the automorphism γ is the identity on $\mathbb{Q}_p(\Lambda_{P,m+1}) = \mathbb{Q}_p(\pi_m)$. Hence $\pi_m = \pi'_m$.$\square$

We recall that two Lubin-Tate groups are isomorphic (as formal groups over $\mathbb{Z}_p$) if and only if they are relative to the same uniformizer w (cf. theorem 2.9.4).

Theorem 3.5.2 (Independence on the Lubin-Tate group)**.** *Let $P, \widetilde{P} \in \mathfrak{F}_{\mathrm{w}}$ be two Lubin-Tate series, let $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$ and $\widetilde{\boldsymbol{\pi}} = (\pi_{\widetilde{P},j})_{j\geq 0}$ be a generator of $\mathrm{T}(\mathfrak{G}_P)$ and $\mathrm{T}(\mathfrak{G}_{\widetilde{P}})$ respectively. Let us denote by $\mathrm{M}_d^{(\widetilde{P})}(-)$, $E_m^{(\widetilde{P})}(T)$, $\mathrm{e}_d^{(\widetilde{P})}(-,T)$ the constructions attached to $\widetilde{\boldsymbol{\pi}}$, and denote in the usual way the constructions attached to $\boldsymbol{\pi}$. If $\pi_{\widetilde{P},m} = [1]_{P,\widetilde{P}}(\pi_{P,m})$, then $\mathrm{M}_d(\boldsymbol{\lambda}) \xrightarrow{\sim} \mathrm{M}_d^{(\widetilde{P})}(\boldsymbol{\lambda})$ over $\mathcal{R}_{H_m}$, for all $\boldsymbol{\lambda} \in \mathbf{W}_m(k_H)$.*

Proof : Let $\boldsymbol{\lambda} \in \mathbf{W}_m(k_H)$, and let $\tilde{\boldsymbol{\lambda}} \in \mathbf{W}_m(\mathcal{O}_H)$ be a lifting of $\boldsymbol{\lambda}$. We shall show that $e_d(\tilde{\boldsymbol{\lambda}},T)/e_d^{(\widetilde{P})}(\tilde{\boldsymbol{\lambda}},T)$ belongs to $\mathcal{R}_H$. By equation 3.3.2.2, we reduce us to show that $E_{m-j}(T^{-1})/E_{m-j}^{(\widetilde{P})}(T^{-1}) \in \mathcal{R}_H$, for all $0 \leq j \leq m$. Since $\pi_{\widetilde{P},m} = [1]_{P,\widetilde{P}}(\pi_{P,m})$, then $\pi_{\widetilde{P},j} = [1]_{P,\widetilde{P}}(\pi_{P,j})$, for all $0 \leq j \leq m$. We have

$$E_m(T^{-1})/E_m^{(\widetilde{P})}(T^{-1}) = \exp\big(\sum_{j=0}^{m} \pi_{m-j}(1 - \frac{\pi_{\widetilde{P},m-j}}{\pi_{P,m-j}})\frac{T^{-p^j}}{p^j}\big). \tag{3.5.2.1}$$

[1]Note that the symbol $[-]_P$ was defined in 2.9.1 and is different from $[-]$ defined at 3.1.2.

Let us set, as usual,

$$h_{P,\widetilde{P}}(X) := 1 - [1]_{P,\widetilde{P}}(X)/X \ , \tag{3.5.2.2}$$

in order to have (cf. 3.1.9 and definition 3.3.1.1)

$$E_m(T^{-1})/E_m^{(\widetilde{P})}(T^{-1}) = \mathrm{e}_d([h_{P,\widetilde{P}}(\pi_m)], T^{-1}) \ . \tag{3.5.2.3}$$

Since $[1]_{P,\widetilde{P}}(X) \equiv X \mod X^2\mathbb{Z}_p[[X]]$, hence $h_{P,\widetilde{P}}(X) \in X \cdot \mathbb{Z}_p[[X]]$, and, by the reduction theorem 3.4.6 and lemma 3.4.8, this exponential lies in $\mathcal{R}_{H_m}$. By the way, its inverse lies also in $\mathcal{R}_{H_m}$, then $\mathrm{M}_d(\boldsymbol{\lambda}) \xrightarrow{\sim} \mathrm{M}_d^{(\tilde{P})}(\boldsymbol{\lambda})$, over $\mathcal{R}_{H_m}$.□

Remark 3.5.3. If $\mathrm{w} = p$, and if P is given, then, by 2.9.8 and 2.9.9, the isomorphism class of $\mathrm{M}_d(\boldsymbol{\lambda})$ is determined by the choice of a sequence $\{\xi_j\}_{j\geq 0}$ of $p^{j+1}-$th roots of 1 such that $\xi_m^{p^j} = \xi_{m-j}$.

Corollary 3.5.4. *Let $\gamma : H(\Lambda_P) \to H(\Lambda_P)$ be a continue endo-morphism of fields. Then $\gamma(E_m(T^{-1}))/E_m(T^{-1}) \in \mathcal{R}_{H_m}$ if and only if γ is the identity on $\mathbb{Q}_p(\pi_m)$, and in this case, for all $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_{H(\Lambda_P)})$, we have*

$$\mathrm{e}_d^{\gamma}(\boldsymbol{\lambda}, T) = \mathrm{e}_d(\gamma(\boldsymbol{\lambda}), T) \ , \tag{3.5.4.1}$$

where, for all $f(T) = \sum a_i T^i$, we set $f^{\gamma}(T) := \sum \gamma(a_i) T^i$.

Proof : The proof follows the same line of the proof of 3.5.1.□

3.6 Frobenius structure for $\boldsymbol{\pi}-$exponentials

Theorem 3.6.1. *Let $r \geq 0$ and let $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$. Let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_H)$ be a lifting of $\bar{\boldsymbol{\lambda}}$, and let $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})} \in \mathbf{W}_m(\mathcal{O}_H)$ be an arbitrary lifting of $\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}) \in \mathbf{W}_m(k_H)$. The following statements are equivalent :*

1. *The power series*
$$\frac{\mathrm{e}_d(\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}, T^p)}{\mathrm{e}_d(\boldsymbol{\lambda}, T)} \tag{3.6.1.1}$$
is over-convergent, for all choices of $\boldsymbol{\lambda}$, $\bar{\boldsymbol{\lambda}}$ and $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}$;
2. *The modules $\mathrm{M}_d(\bar{\boldsymbol{\lambda}})$ and $\mathrm{M}_{pd}(\mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}))$ are isomorphic over $\mathcal{R}_{H_m}$, for all $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$;*
3. *The power series $E_m(T^p)/E_m(T)$ is over-convergent.*
4. *We have the inequality $|\mathrm{w} - p| \leq |p|^{m+2}$.*

Proof : (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are evident. Let us show (3) $\Leftrightarrow$ (4). Write

$$\begin{aligned} E_m(T^p)/E_m(T) &= \exp\Big(\big(\sum_{j=0}^{m} \pi_{m-j} \frac{T^{p^{j+1}}}{p^j}\big) - \big(\sum_{j=0}^{m} \pi_{m-j} \frac{T^{p^j}}{p^j}\big) \Big) \\ &= \exp(-p\pi_{m+1}T) \cdot \exp\Big(\sum_{j=0}^{m+1} \pi_{m+1-j} \big(p - \frac{\pi_{m-j}}{\pi_{m-j+1}}\big) \frac{T^{p^j}}{p^j} \Big) , \end{aligned}$$

where $\pi_{-1} := P(\pi_0) = 0$. Let

$$h_{\mathrm{Frob}}(X) := p - P(X)/X \ , \tag{3.6.1.2}$$

in order to have (cf. 3.1.9 and definition 3.3.1.1)

$$E_m(T^p)/E_m(T) = \exp(-p\pi_{m+1}T) \cdot \mathrm{e}_{p^{m+1}}([h_{\mathrm{Frob}}(\pi_{m+1})], T). \tag{3.6.1.3}$$

Since the function $\exp(-p\pi_{m+1}T)$ is over-convergent, hence $E_m(T^p)/E_m(T)$ is over-convergent if and only if $\mathrm{e}_{p^{m+1}}([h_{\mathrm{Frob}}(\pi_{m+1})], T)$ is. The constant term of $h_{\mathrm{Frob}}(X)$ is $p - \mathrm{w}$. Hence, as usual, by the reduction theorem 3.4.6 and lemma 3.1.9, $E_m(T^p)/E_m(T)$ is over-convergent if and only if $|p - \mathrm{w}| \leq |p|^{m+2}$. Let us now show (3) $\Rightarrow$ (1). Since (3) and (4) are equivalent, we see that $E_j(T^p)/E_j(T)$ is over-convergent, for all $j = 0, \ldots, m$. Let $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m)$ and $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})} := (\lambda_0^{(\bar{\mathrm{F}})}, \ldots, \lambda_m^{(\bar{\mathrm{F}})})$. We can suppose $\lambda_j^{(\bar{\mathrm{F}})} = \lambda_j^p$, $\forall\ j = 0, \ldots, m$. Indeed, the Witt vector $\boldsymbol{\eta} := \boldsymbol{\lambda}^{(\bar{\mathrm{F}})} - (\lambda_0^p, \ldots, \lambda_m^p) = (\eta_0, \ldots, \eta_m) \in \mathbf{W}_m(\mathcal{O}_H)$ satisfies $|\eta_j| < 1$, $\forall\ j = 0, \ldots, m$. Hence

$$\mathrm{e}_d(\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}, T^p) = \mathrm{e}_d(\boldsymbol{\eta}, T^p) \cdot \mathrm{e}_d((\lambda_0^p, \ldots, \lambda_m^p), T^p) \ , \tag{3.6.1.4}$$

and the function $\mathrm{e}_d(\boldsymbol{\eta}, T^p)$ is over-convergent by the reduction theorem 3.4.6. Now, by the equation 3.3.2.2, we have

$$\frac{\mathrm{e}_d((\lambda_0^p, \ldots, \lambda_m^p), T^p)}{\mathrm{e}_d(\boldsymbol{\lambda}, T)} = \prod_{j=0}^{m} \frac{E_{m-j}(\lambda_j^p T^{np^{j+1}})}{E_{m-j}(\lambda_j T^{np^j})}. \tag{3.6.1.5}$$

Since $E_{m-j}(T^p)/E_{m-j}(T)$ is over-convergent, for all $j = 0, \ldots, m$, then all factors $E_{m-j}(\lambda_j^p T^{np^{j+1}})/E_{m-j}(\lambda_j T^{np^j})$ are over-convergent. $\square$

Remark 3.6.2. In this theorem we do not need the existence of an absolute Frobenius on H. This is due to the fact that the isomorphism class of $\mathrm{M}_d(\boldsymbol{\lambda})$ depends only on the reduction $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$, and k_H is endowed naturally with the Frobenius given by the $p-$th power map.

Remark 3.6.3. We will generalize this theorem for all rank one differential equations (cf. theorem 5.2.2). Let us show how to recover, from 3.6.1, the Frobenius structure theorem in the usual sense. Let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_H)$ be a lift of $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$. Suppose that $\mathrm{w} = p$, in order to have the theorem 3.6.1. Suppose that $\sigma : H_\infty \to H_\infty$ is an absolute Frobenius (cf. 2.5.1) such that $\pi_j^\sigma = \pi_j$, for all $j \geq 0$, and such that $\sigma(H) \subseteq H$. By corollary 3.5.4, we have $\varphi_\sigma(\mathrm{e}_d(\boldsymbol{\lambda}, T)) = \mathrm{e}_d(\boldsymbol{\lambda}^\sigma, T^p)$, and hence $\varphi_\sigma^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) = \widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda}^\sigma))$. By 3.4.6, the isomorphism class of $\widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda}^\sigma))$ depends only on the reduction $\mathrm{V}(\overline{\boldsymbol{\lambda}^\sigma}) = \mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}) \in \mathbf{W}_{m+1}(k_{H_\infty})$, then $\widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda}^\sigma))$ is isomorphic to $\mathrm{M}_{pd}(\mathrm{V}\bar{\mathrm{F}}(\boldsymbol{\lambda}))$ over $\mathcal{R}_{H_\infty}$. Then theorem 3.6.1 gives us the usual Frobenius structure. Indeed,

$$\varphi_\sigma^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) \xrightarrow[\text{Cor.3.5.4}]{\sim} \widetilde{\mathrm{M}}_{p\cdot d}(\mathrm{V}(\boldsymbol{\lambda}^\sigma)) \xrightarrow[\text{Th.3.4.6}]{\sim} \mathrm{M}_{p\cdot d}(\mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}})) \xrightarrow[\text{Th.3.6.1}]{\sim} \widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}) \ .$$

Remark 3.6.4. Let φ_p^* be the $p-$th ramification map (cf. 2.5.4.1), and let $\lambda \in \mathcal{O}_H$. We observe that we can not have an isomorphism $\mathrm{M}_1(\lambda) \xrightarrow{\sim} \varphi_p^*(\mathrm{M}_1(\lambda))$, for all λ. For example, suppose that λ is such that $\bar{\lambda}^{p^r} \neq \bar{\lambda}$ in k_H, for all $r \geq 0$ (i.e. $\bar{\lambda} \notin \mathbb{F}_p^{\mathrm{alg}}$). Then $\exp(\pi_0\lambda T^p)/\exp(\pi_0\lambda T)$ is not over-convergent. Indeed, for all lifting $\lambda^{(\bar{\mathrm{F}}^r)} \in \mathcal{O}_H$ of $\bar{\lambda}^{p^r}$ we have $|\lambda^{(\bar{\mathrm{F}}^r)} - \lambda| = 1$, then

$$\frac{\exp(\pi_0\lambda T^p)}{\exp(\pi_0\lambda T)} = \frac{\mathrm{e}_1(\lambda, T^p)}{\mathrm{e}_1(\lambda, T)} = \frac{\mathrm{e}_1(\lambda^{(\bar{\mathrm{F}}^r)}, T^p)}{\mathrm{e}_1(\lambda, T)} \cdot \mathrm{e}_1(\lambda - \lambda^{(\bar{\mathrm{F}}^r)}, T^p) \ ,$$

and while $\mathrm{e}_1(\lambda^{(\bar{\mathrm{F}}^r)}, T^p)/\mathrm{e}_1(\lambda, T)$ is over-convergent, the function $\mathrm{e}_1(\lambda - \lambda^{(\bar{\mathrm{F}}^r)}, T^p)$ is not over-convergent, since the reduction of $\lambda^{(\bar{\mathrm{F}}^r)} - \lambda$ is not 0 in k_H (cf. 3.4.6).

Chapitre 4

A natural transformation of the Artin-Schreier complex into the Kummer complex, via Dwork's splitting functions.

Hypothesis 4.0.5. From now on, until the end of the paper, we will suppose $\mathrm{w} = p$ in order to have the theorem 3.6.1. Then $\mathfrak{G}_P \xrightarrow{\sim} \widehat{\mathbb{G}}_m$, the formal multiplicative group (cf. 2.9.4). We fix moreover a generator $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$ of the Tate module $\mathrm{T}(\mathfrak{G}_P)$.

4.1 The deformation from Artin-Schreier to Kummer

In this section L will be a complete valued field, containing $(\mathbb{Q}_p, |\cdot|)$, and endowed with an absolute Frobenius $\varphi : \mathcal{O}_L \to \mathcal{O}_L$ (i.e. a lifting the $x \mapsto x^p$ of k_L).

We set as usual $L_m := L(\pi_m)$ and $L_\infty := \cup_m L_m$. We denote by k_m (resp. k_∞) the residue field of L_m (resp. L_∞). We fix an algebraic closure L^{alg} of L, then $k_L^{\mathrm{alg}} := k_{L^{\mathrm{alg}}}$ is an algebraic closure of k_L. Let k_L^{sep} (resp. k_m^{sep}, k_∞^{sep}) be the separable closure of k_L (resp. k_m, k_∞) in k_L^{alg} (we recall that k_L is not supposed perfect). We denote by $\widehat{L}^{\mathrm{unr}}$ (resp. $\widehat{L}_m^{\mathrm{unr}}$) the completion of the maximal unramified extension of L (resp. L_m) in L^{alg}. We set

$$\mathrm{G}_{k_L} := \mathrm{Gal}(k_L^{\mathrm{sep}}/k_L), \qquad \mathrm{G}_{k_m} := \mathrm{Gal}(k_m^{\mathrm{sep}}/k_m)\ , \tag{4.1.0.1}$$

$$\mathrm{G}_L := \mathrm{Gal}(L^{\mathrm{alg}}/L), \qquad \mathrm{G}_{L_m} := \mathrm{Gal}(L_m^{\mathrm{alg}}/L_m)\ . \tag{4.1.0.2}$$

Remark 4.1.1. Let $k_L^0 = k_L^{\mathrm{sep}} \cap k_m$ be the separable closure of k_L in k_m and let $L_0 := \mathbf{W}(k_L^0) \otimes_{\mathbf{W}(k_L)} L = \widehat{L}^{\mathrm{unr}} \cap L_m$. The extension k_m/k_L^0 is purely inseparable (i.e. for all $x \in k_m$ there exists $r \geq 0$ such that $x^{p^r} \in k_L^0$), then $\mathrm{Gal}(k_m/k_L^0) = 1$, and we have a canonical identification $\mathrm{G}_{k_m} := \mathrm{Gal}(k_m^{\mathrm{sep}}/k_m) \xrightarrow{\sim} \mathrm{Gal}(k_L^{\mathrm{sep}}/k_L^0)$. Hence G_{k_m} is naturally contained in G_{k_L} :

$$\begin{array}{ccccc} & & L_m & \subseteq & \widehat{L}_m^{\mathrm{unr}} \\ & & \cup\!\!\mid & & \cup\!\!\mid \\ L & \subseteq & L^0 & \subseteq & \widehat{L}^{\mathrm{unr}} \end{array} \qquad \begin{array}{ccccc} & & k_m & \subseteq & k_m^{\mathrm{sep}} \\ & & \cup\!\!\mid & & \cup\!\!\mid \\ k_L & \subseteq & k_L^0 & \subseteq & k_L^{\mathrm{sep}} \end{array} . \tag{4.1.1.1}$$

All these extensions are Galois. We will identify G_{k_m} with $\mathrm{Gal}(\widehat{L}_m^{\mathrm{unr}}/L_m)$, and G_{k_L} with $\mathrm{Gal}(\widehat{L}^{\mathrm{unr}}/L)$. In this way G_{k_m} acts naturally on $\widehat{L}^{\mathrm{unr}}$.

Remark 4.1.2. The Frobenius φ extends uniquely to all unramified extensions of L, and hence it commutes with the action of G_{k_L}. One can shows that the Frobenius φ extends (not uniquely) to an absolute Frobenius φ_m of L_m. But in general the is no absolute Frobenius on L_m satisfying $\varphi_m(\pi_m) = \pi_m$. For this reason we do not suppose the existence of a Frobenius on $\mathcal{O}_{L_m}$. We need the existence of φ because the functor of Witt vectors of finite length $\mathbf{W}_m(-)$ is not endowed canonically with an additive functorial Frobenius morphism (see remark 4.1.12 to improve this situation).

Definition 4.1.3. For $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_L)$, we set

$$\theta_d^{(\varphi)}(\boldsymbol{\lambda}, T) := \frac{\mathrm{e}_d(\varphi(\boldsymbol{\lambda}), T^p)}{\mathrm{e}_d(\boldsymbol{\lambda}, T)}. \tag{4.1.3.1}$$

To simplify the notations, we will write $\theta_d(\boldsymbol{\lambda}, T)$ if no confusion is possible.

Example 4.1.4. Let $d = 1$ and $P(X) = pX + X^p$ (cf. 2.9.3). Then π_0 is the "π of Dwork", and $\theta_1(1, T) = \exp(\pi_0(T^p - T))$ is the usual Dwork's splitting function. While in general, if $\lambda \in \mathcal{O}_L$, we have $\theta_1(\lambda, T) = \exp(\pi_0(\varphi(\lambda)T^p - \lambda T))$.

The following theorem show that the over-convergent function $\boldsymbol{\lambda} \mapsto \theta(\boldsymbol{\lambda}, 1)$ is a splitting function in a generalized sense with respect to Dwork (cf. [Dwo82, §4, *a*),p.55]). In a paper in preparation we shall see in detail this kind of functions.

Definition 4.1.5. We set $\mathcal{O}_L^{\varphi=1} := \{\lambda \in \mathcal{O}_L \mid \varphi(\lambda) = \lambda\}$ and $\overline{\mathcal{O}_L^{\varphi=1}} := \mathcal{O}_L^{\varphi=1}/(\mathcal{O}_L^{\varphi=1} \cap \mathfrak{p}_L)$. We see that $\overline{\mathcal{O}_L^{\varphi=1}} = \mathbb{F}_p$.

Theorem 4.1.6. *Let $a^p = a \in \mathcal{O}_L$, and let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L^{\varphi=1})$. Then $\theta_d^{(\varphi)}(\boldsymbol{\lambda}, a)$ is a p^{m+1}−th root of 1. Moreover the group morphism*

$$\theta_d^{(\varphi)}(-, a) : \mathbf{W}_m(\mathcal{O}_L^{\varphi=1}) \longrightarrow \boldsymbol{\mu}_{p^{m+1}} \subset \mathbb{Z}_p[\pi_m]$$

factorizes on $\mathbf{W}_m(\overline{\mathcal{O}_L^{\varphi=1}}) = \mathbf{W}_m(\mathbb{F}_p) = \mathbb{Z}/p^{m+1}\mathbb{Z}$ *and defines an isomorphism*

$$\overline{\theta}_d^{(\varphi)}(-,a) : \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{m+1}}. \qquad (4.1.6.1)$$

More precisely the image of $1 \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ *is* the inverse *of the unique primitive* $p^{m+1}-$*th root of* 1*, say* ξ_m*, satisfying*

$$|a^n\pi_m - (\xi_m - 1)| < |a^n\pi_m|. \qquad (4.1.6.2)$$

In particular, if $a = 1$*, then* ξ_m *is the* $p^{m+1}-$*th root of* 1 *defined in remark 2.9.9.*

Proof : Let $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_L^{\varphi=1})$. Let us show that $\theta_d(\boldsymbol{\lambda}, a)^{p^{m+1}} = 1$. Indeed $T \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T)^{p^{m+1}}$ is over-convergent (cf. 4.1.7), then

$$\theta_d(\boldsymbol{\lambda}, a)^{p^{m+1}} = \mathrm{e}_d(\varphi(\boldsymbol{\lambda}), a^p)^{p^{m+1}} / \mathrm{e}_d(\boldsymbol{\lambda}, a)^{p^{m+1}} = 1 \;, \qquad (4.1.6.3)$$

since both numerator and denominator do make sense and are equal. If $|\lambda_j| < 1$, for all $j = 0, \ldots, m$, then $T \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T)$ is over-convergent (cf. reduction theorem 3.4.6), hence both numerator and denominator of the expression $\mathrm{e}_d(\boldsymbol{\lambda}, a^p)/\mathrm{e}_d(\boldsymbol{\lambda}, a)$ do make sense and are equals. Let us show the last assertion. By the equation 3.6.1.3, we have

$$\begin{aligned} \theta_d((1, 0 \ldots, 0), T) &= E_m(T^{np})/E_m(T^n) \qquad (4.1.6.4) \\ &= \exp(-p\pi_{m+1}T^n) \cdot \mathrm{e}_{pd}([h_{\mathrm{Frob}}(\pi_{m+1})], T). \end{aligned}$$

By 3.3.2.1 this series lies in $1 + \pi_{m+1}T\mathbb{Z}_p[\pi_{m+1}][[T]]$. To show that this root is ξ_m^{-1} it is sufficient to show that $|\theta_d((1, 0, \ldots, 0), a)^{-1} - \xi_m| < |\pi_m| = |\xi_m - 1|$. We work then modulo the following sub group

$$C := \{1 + \sum c_i T^i \mid c_i \in \mathbb{Z}_p[\pi_{m+1}] \;,\; |c_i| < |\pi_m| \;,\; \text{for all } i \geq 1\}.$$

We have $\exp(-p\pi_{m+1}T^n) \equiv 1 \mod C$. Let us consider (cf. the setting 3.6.1.2)

$$[h_{\mathrm{Frob}}(\pi_{m+1})] = [p - P(\pi_{m+1})/\pi_{m+1}] = (\nu_0, \ldots, \nu_{m+1}). \qquad (4.1.6.5)$$

Then $\nu_0 = p - (\pi_m/\pi_{m+1})$ and, since $p = \mathrm{w}$, by lemma 3.1.9, we have $|\nu_j| \leq |\pi_{m+1}|$, for all $j = 0, \ldots, m+1$. By the equation 3.3.2.2 we have

$$\mathrm{e}_{pd}([h_{\mathrm{Frob}}(\pi_{m+1})], T) = \prod_{j=0}^{m+1} E_{m+1-j}(\nu_j T^{np^j}) \;. \qquad (4.1.6.6)$$

Moreover, we know that (cf. equation 3.3.2.1)

$$E_{m+1-j}(\nu_j T^{np^j}) = 1 + (\text{things of valuation } \leq |\pi_{m+1-j} \cdot \nu_j|) \;, \qquad (4.1.6.7)$$

for all $j = 0, \ldots, m+1$. Then

$$\theta_d((1,0,\ldots,0),T)^{-1} \equiv E_{m+1}(\nu_0 T^n)^{-1} \mod C. \tag{4.1.6.8}$$

Since $|\nu_0^p| = |\pi_m|^{p-1}$, hence, by 3.3.2.1, only the first $p-1$ terms of $E_{m+1}(\nu_0 T^n)^{-1}$ are bigger than or equal to $|\pi_m|$, that is

$$E_{m+1}(\nu_0 T^n)^{-1} \equiv 1 + \pi_{m+1}\nu_0 T^n + \cdots + \frac{(\pi_{m+1}\nu_0 T^n)^{p-1}}{(p-1)!} \mod C. \tag{4.1.6.9}$$

Since $\pi_{m+1}\nu_0 = p \cdot \pi_{m+1} - \pi_m$, hence $\theta_d((1,0,\ldots,0),T)^{-1} \equiv 1 + \pi_m T^n \mod C$.$\square$

Remark 4.1.7. Observe that $T \mapsto \mathrm{e}_d(\boldsymbol{\lambda},T)^{p^{m+1}} = \mathrm{e}_d(p^{m+1}\boldsymbol{\lambda},T)$ is over-convergent for all $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$, because the reduction of $p^{m+1}\boldsymbol{\lambda}$ in $\mathbf{W}_m(k_L)$ is 0 (cf. theor.3.4.6).

Remark 4.1.8. Recall that there is no Frobenius on L_m (cf. Remark 4.1.2).

Theorem 4.1.9. *The following diagram is well-defined, commutative and functorial, on the complete (or algebraic) unramified extensions of L*

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_{p^{m+1}} \arrow[r] & (L_m)^\times \arrow[r, "f\mapsto f^{p^{m+1}}"] & (L_m)^\times \arrow[r, "\delta_{\mathrm{Kum}}"] & \mathrm{H}^1(\mathrm{G}_{L_m}, \boldsymbol{\mu}_{p^{m+1}}) \arrow[r] & 1 \\
 & \mathbf{W}_m(\mathcal{O}_L^{\varphi=1}) \arrow[u] \arrow[d, two heads] \arrow[r, hook] & \mathbf{W}_m(\mathcal{O}_L) \arrow[u, "\theta_{p^m}(-,1)"'] \arrow[d, two heads] \arrow[r, "\varphi-1"'] & \mathbf{W}_m(\mathcal{O}_L) \arrow[u, "\mathrm{e}_{p^m}(-,1)^{p^{m+1}}"'] \arrow[d, two heads] & & \\
0 \arrow[r] & \mathbb{Z}/p^{m+1}\mathbb{Z} \arrow[r] \arrow[uu, dotted, bend left, "\wr"] & \mathbf{W}_m(k_L) \arrow[r, "\bar{\mathrm{F}}-1"'] & \mathbf{W}_m(k_L) \arrow[r, "\delta"] & \mathrm{H}^1(\mathrm{G}_{k_L}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \arrow[r] \arrow[uu, dotted, "\overline{e}:=\overline{\mathrm{e}_{p^m}(-,1)^{p^{m+1}}}"'] & 0
\end{tikzcd} \tag{4.1.9.1}$$

where $\mathrm{G}_{L_m} := \mathrm{Gal}(L_m^{\mathrm{alg}}/L_m)$. *More explicitly $\theta_{p^m}(-,1)$ induces the identification*

$$1 \mapsto \xi_m^{-1} \ : \ \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \mu_{p^{m+1}} \qquad \textit{(cf. 4.1.6)}\,, \tag{4.1.9.2}$$

where ξ_m is the unique $p^{m+1}-$th root of 1 satisfying $|(\xi_m - 1) - \pi_m| < |\pi_m|$ (cf. 2.9.9). Moreover $\overline{e}$ sends $\mathrm{H}^1(\mathrm{G}_{k_L}, \mathbb{Z}/p^{m+1}\mathbb{Z})$ *in* $\mathrm{H}^1(\mathrm{G}_{k_m}, \boldsymbol{\mu}_{p^{m+1}}) \subseteq \mathrm{H}^1(\mathrm{G}_{L_m}, \boldsymbol{\mu}_{p^{m+1}})$ *via the canonical diagram*

$$\begin{tikzcd}
\mathrm{G}_{k_L} \arrow[d, "\alpha"'] & \mathrm{G}_{k_m} \arrow[l, hook'] \arrow[d, dotted, "\overline{e}(\alpha)"] \\
\mathbb{Z}/p^{m+1}\mathbb{Z} \arrow[r, "\sim", "1\mapsto\xi_m^{-1}"'] & \boldsymbol{\mu}_{p^{m+1}}
\end{tikzcd}\;. \tag{4.1.9.3}$$

In other words the Artin-Schreier character $\gamma \mapsto \alpha(\gamma) : \mathrm{G}_{k_L} \to \mathbb{Z}/p^{m+1}\mathbb{Z}$ is sent by $\overline{e}$ into the Kummer character $\gamma \mapsto \overline{e}(\alpha)(\gamma) = \xi_m^{-\alpha(\gamma)} : \mathrm{G}_{k_m} \to \boldsymbol{\mu}_{p^{m+1}}$. In particular $\overline{e}(\alpha) = 1$ if and only if $G_{k_m} \subseteq \mathrm{Ker}(\alpha)$.

Proof : Let L'/L be an unramified extension, and let $\boldsymbol{\lambda}' = (\lambda'_0, \ldots, \lambda'_m) \in \mathbf{W}_m(\mathcal{O}_{L'})$. If L'/L is not complete, but algebraic, then the series $\theta_{p^m}(\boldsymbol{\lambda}', T)$, and $\mathrm{e}_{p^m}(\boldsymbol{\lambda}', T)^{p^{m+1}}$, are convergent at $T = 1$, since the finite extension $L(\{\lambda'_i\}_i)/L$ is complete. By 4.1.6, to show the commutativity it is enough to prove that $\overline{e}$ is well-defined. Let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$ be such that $\delta(\bar{\boldsymbol{\lambda}}) = 0$ (cf. diagram 2.8.1.2). By definition, there exist $\boldsymbol{z}, \boldsymbol{\eta} \in \mathbf{W}_m(\mathcal{O}_L)$, such that $\boldsymbol{\eta} = (\eta_0, \ldots, \eta_m)$, with $|\eta_j| < 1$, for all $j = 0, \ldots, m$, and that $\boldsymbol{\lambda} = \varphi(\boldsymbol{z}) - \boldsymbol{z} + \boldsymbol{\eta}$. Hence

$$\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}} = \theta_{p^m}(\boldsymbol{z}, 1)^{p^{m+1}} \cdot \mathrm{e}_{p^m}(\boldsymbol{\eta}, 1)^{p^{m+1}} \;. \tag{4.1.9.4}$$

Then $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}} \in (\mathcal{O}_{L_m})^{p^{m+1}}$. In other words, even if the symbol $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)$ has no meaning, the number $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}$ is the $p^{m+1}-$th power of the number $\theta_{p^m}(\boldsymbol{z}, 1) \cdot \mathrm{e}_{p^m}(\boldsymbol{\eta}, 1)$ of L_m. Hence $\delta_{\mathrm{Kum}}(\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}) = 1$.

Let us show that the map $\overline{e}$ works as indicated in the diagram 4.1.9.3. Let $\alpha = \delta(\bar{\boldsymbol{\lambda}})$, and let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$ be an arbitrary lifting of $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_L)$. By lemma 4.1.10, an easy induction on m shows that there exists $\boldsymbol{\nu} \in \mathbf{W}_m(\mathcal{O}_{\widehat{L}^{\mathrm{unr}}})$ such that

$$\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda} \;. \tag{4.1.9.5}$$

By definition (cf. 2.8.1.2), for all $\gamma_1 \in \mathrm{G}_{k_L}$, we have $\alpha(\gamma_1) = \gamma_1(\overline{\boldsymbol{\nu}}) - \overline{\boldsymbol{\nu}} \in \mathbb{Z}/p^{m+1}\mathbb{Z}$. On the other hand, by definition, $\bar{e}(\alpha)$ is the Kummer character of G_{L_m} defined by $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}$, and is given by $\bar{e}(\alpha)(\gamma) = \gamma(y)/y$, for all $\gamma \in \mathrm{G}_{L_m}$, where y is an arbitrary root of the equation $Y^{p^{m+1}} = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}$. We let $y := \theta_{p^m}(\boldsymbol{\nu}, 1)$. Then

$$\overline{e}(\alpha)(\gamma) = \gamma(y)/y = \gamma(\theta_{p^m}(\boldsymbol{\nu}, 1))/\theta_{p^m}(\boldsymbol{\nu}, 1) = \theta_{p^m}(\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu}, 1) \in \boldsymbol{\mu}_{p^{m+1}}, \tag{4.1.9.6}$$

because $\gamma(\pi_m) = \pi_m$, since $\gamma \in \mathrm{G}_{L_m}$. Now $\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu} \in \mathcal{O}_L^{\varphi=1}$, because $\gamma(\boldsymbol{\nu})$ is again a solution of the equation 4.1.9.5. By 4.1.6, the root $\theta_{p^m}(\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu}, 1)$ depends only on the reduction of $\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu}$ in k_L^{sep}, and is equal to $\xi_m^{-\alpha(\gamma)}$. □

Lemma 4.1.10. *Let L be discrete valued, and let $\widehat{L}^{\mathrm{unr}}$ be the completion of the unramified extension of L. Then for all $\lambda \in \mathcal{O}_{\widehat{L}^{\mathrm{unr}}}$, the equation*

$$\varphi(\nu) - \nu = \lambda \tag{4.1.10.1}$$

has a solution in $\widehat{L}^{\mathrm{unr}}$.

Proof : The equation $\bar{v}^p - \bar{v} = \bar{\lambda}$ has a solution in k_L^{sep}, hence $|(\varphi(v) - v) - \lambda| < 1$, for all lift v of $\bar{v}$. Since L is discrete valuated, the lemma follows from an induction on the value of the "error" η, in the equation $\varphi(\nu) - \nu = \lambda + \eta$.□

Theorem 4.1.11. *Let L be discrete valued. Let $\alpha = \delta(\bar{\boldsymbol{\lambda}})$ be the Artin-Schreier character defined by $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_L)$ (cf. 2.8.1.2). Let k_α/k_L be the* *separable extension of k_L, defined by the kernel of α, and let L_α/L be the corresponding unramified extension. Then*

$$L_\alpha(\pi_m) = L_m(\theta_{p^m}(\boldsymbol{\nu},1)) \;, \tag{4.1.11.1}$$

where $\boldsymbol{\lambda}$ is an arbitrary lifting of $\bar{\boldsymbol{\lambda}}$ in $\mathbf{W}_m(\mathcal{O}_L)$, and $\boldsymbol{\nu} \in \mathbf{W}_m(\mathcal{O}_{\widehat{L}^{\mathrm{unr}}})$ is a solution of the equation $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$. In other words, up to change L with L_m, the extension L_α is generate by $\theta_{p^m}(\boldsymbol{\nu},1)$.

Proof : Since both $L_\alpha(\pi_m)$ and $L_m(\theta_{p^s}(\boldsymbol{\nu},1))$ contain L^0 (cf. 4.1.1), and since φ extends uniquely to L^0, hence we can suppose $L = L^0$. In this case $\overline{e}$ is injective, L_m/L is totally ramified, and G_{k_L} can be identified to G_{k_m}. Let us show the inclusion $L_m(\theta_{p^m}(\boldsymbol{\nu},1)) \subseteq L_\alpha(\pi_m)$. If $\mathrm{G}_{k_\alpha} := \mathrm{Gal}(k_L^{\mathrm{sep}}/k_\alpha) = \mathrm{Ker}(\alpha)$, then the inclusion follows from the fact that $\theta_{p^m}(\boldsymbol{\nu},1)$ is fixed by $\mathrm{G}_{k_\alpha}(\subseteq \mathrm{G}_{k_m} \xrightarrow{\sim} \mathrm{Gal}(\widehat{L}_m^{\mathrm{unr}}/L_m))$. Indeed, for all $\gamma \in \mathrm{Gal}(\widehat{L}_m^{\mathrm{unr}}/L_m)$, we have, as in the proof of 4.1.9,

$$\gamma\left(\theta_{p^m}(\boldsymbol{\nu},1)\right) = \theta_{p^m}(\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu},1)\cdot\theta_{p^m}(\boldsymbol{\nu},1) = \xi_m^{-\alpha(\gamma)}\cdot\theta_{p^m}(\boldsymbol{\nu},1) \;, \tag{4.1.11.2}$$

and if $\gamma \in \mathrm{G}_{k_\alpha}$, we have $\alpha(\gamma) = 0$. Then $L_m(\theta_{p^m}(\boldsymbol{\nu},1)) \subseteq L_\alpha(\pi_m)$. In particular,

$$[L_m(\theta_{p^m}(\boldsymbol{\nu},1)) : L_m] \leq [L_\alpha(\pi_m) : L_m] = [k_{\alpha,m} : k_m] \;, \tag{4.1.11.3}$$

where $k_{\alpha,m}$ is the smallest field in k_m^{sep} containing k_m and k_α (i.e. the sub-field of k_m^{sep} fixed by G_{k_α} acting on k_m^{sep}). The inclusion $L_\alpha(\pi_m) \subseteq L_m(\theta_{p^m}(\boldsymbol{\nu},1))$ follows from the equality $[L_m(\theta_{p^m}(\boldsymbol{\nu},1)) : L_m] = [k_{\alpha,m} : k_m]$. Indeed, since $L_0 = L$, then the map $\overline{e}$ is injective. Hence $[L_m(\theta_{p^m}(\boldsymbol{\nu},1)) : L_m] = [k_\alpha : k_L]$, because these two degrees are equal to the *cardinality* of the cyclic Galois groups generated by $\overline{e}(\alpha)$ and α respectively. On the other hand, since $k_L = k_L^0$, we have $[k_\alpha : k_L] = [k_{\alpha,m} : k_m]$.□

Remark 4.1.12. The hypothesis of discreteness of L, in theorem 4.1.11, and the hypothesis of existence of φ can be suppress as follows. Let $\mathrm{F}_p : \mathbf{W}_m \to \mathbf{W}_m$ be the map $(\lambda_0,\ldots,\lambda_m) \mapsto (\lambda_0^p,\ldots,\lambda_m^p)$. Replace φ by F_p, and define $\theta_d^{(\mathrm{F}_p)}(\boldsymbol{\lambda},T) := \mathrm{e}_d(\mathrm{F}_p(\boldsymbol{\lambda}),T^p)/\mathrm{e}_d(\boldsymbol{\lambda},T)$. Then F_p is defined for *all* extensions of L, and commutes with the Galois action. It is easy to recover theorems analogous of 4.1.6, 4.1.9, and 4.1.11. In particular the analogous of the diagram 4.1.9.1 is defined and functorial, on *all* complete (or algebraic) extensions of L. Observe that the map $\boldsymbol{\lambda} \mapsto \theta_d^{(\mathrm{F}_p)}(\boldsymbol{\lambda},T)$ is not a group morphism, but induces again the group morphism $1 \mapsto \xi_m^{-1} : \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{m+1}}$ (cf. 4.1.6.1), which is the reduction of *set* $\mathbf{W}_m(\mathcal{O}_L^{\mathrm{F}_p=1}) := \{\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L) \mid \mathrm{F}_p(\boldsymbol{\lambda}) = \boldsymbol{\lambda}\}$, formed by Witt vectors whose entries are $p-1$ roots of 1.

4.2 Application to the field $\mathcal{E}_K^\dagger$

Remark 4.2.1. These methods apply to obtain a description of the Kummer extensions of $\mathcal{E}_K$ (resp. $\mathcal{E}_K^\dagger$) coming by henselianity from an Artin-Schreier extension of $k((t))$ (see below). This description is really entirely explicit, since the Kummer generator $\theta_{p^m}(\boldsymbol{\nu}, 1)$ is explicitly and directly given by the vector $\boldsymbol{\lambda}$ (and not on $\boldsymbol{\nu}$). Indeed, we will give meaning to the expression $\theta_{p^m}(\boldsymbol{\nu}, 1) = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)$, and we do not need to find a solution of the equation $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$ (cf. definition 5.1.1, and Theorem 6.6.8-(3)).

The theory can be applied to the field $L = \mathcal{E}_K$, under the following assumptions on K :

$$\begin{cases} (1)\ K \text{ is discrete valued (used in 4.1.10).} \\ (2)\ \text{There exists an } \textit{absolute} \text{ Frobenius } \sigma : K \to K \\ \quad\ \text{(i.e. a lifting of the } p-\text{th power map of } k). \end{cases} \tag{4.2.1.1}$$

Fixing an absolute Frobenius of $\mathcal{E}_K$ the theory applies without problems. Recall that we can suppress these two hypothesis if necessary (cf. remark 4.1.12).

The situation is slightly different for the field $\mathcal{E}_K^\dagger$, because it is not complete. Nevertheless the precedent results are still true for $\mathcal{E}_K^\dagger$. Let K satisfy 4.2.1.1, and fix an absolute Frobenius $\varphi : \mathcal{O}_K^\dagger \to \mathcal{O}_K^\dagger$, extending σ, by choosing $\varphi(T)$ in $\mathcal{O}_K^\dagger$, lifting $t^p \in \mathrm{E} = k((t))$ (cf. 2.5.2.1).

Remark 4.2.2. Since $\varphi(T) \in \mathcal{O}_K^\dagger$ is a lifting of $t^p \in \mathrm{E}$, hence there exists $0 < \varepsilon_\varphi < 1$ such that $\varphi(\mathcal{A}_{K_m}(I^p)) \subseteq \mathcal{A}_{K_m}(I)$, where $I =]1 - \varepsilon_\varphi, 1[$.

The fact that K is discrete valued is used also in these two following results.

Theorem 4.2.3 ([Cre87, 4.2],[Mat95, 2.2])**.** *If K is discrete valued, then $\mathcal{O}_K^\dagger$ is Henselian, hence we have a bijection*

$$\{\textit{Finite unramified extensons of } \mathcal{E}_K^\dagger\} \xrightarrow{\sim} \{\textit{Finite separable extensions of } \mathrm{E} = k((t))\} .$$

Proposition 4.2.4. *Let $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathcal{O}_K^\dagger)$, then*

$$\theta_{p^m}(\boldsymbol{f}(T), 1)\ ,\ \mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)^{p^{m+1}} \in \mathcal{O}_{K_m}^\dagger\ . \tag{4.2.4.1}$$

Moreover if $\boldsymbol{u}(T) = (u_0(T), \ldots, u_s(T)) \in \mathbf{W}_s(\mathcal{O}_{K_s}^\dagger)$ is such that $|u_i(T)|_1 < 1$, then $\mathbf{e}_{p^s}(\boldsymbol{u}(T), 1)$ make sense (cf. 3.4.6), and belongs to $\mathcal{O}_{K_s}^\dagger$.

Proof : Let $\varepsilon > 0$ be such that $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{A}_K(]1-\varepsilon, 1[))$. For all compact $J \subset]1-\varepsilon, 1[$, the algebra $\mathcal{A}_K(J)$ is complete with respect to the absolute value $\|f(T)\|_J := \sup_{\rho \in J} |f(T)|_\rho$. Hence $\mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)^{p^{m+1}} \in \mathbf{W}_m(\mathcal{A}_{K_m}(J))$, for all compact $J \subset]1 - \varepsilon, 1[$, and then $\mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)^{p^{m+1}} \in \mathbf{W}_m(\mathcal{A}_{K_m}(]1 -$

$\varepsilon, 1[)$). On the other hand, $\theta_{p^m}(\boldsymbol{f}(T), Z) \in 1 + \pi_m Z \mathcal{O}_{\mathcal{E}_{K_m}}[[Z]]$ is a series in Z depending only on $\boldsymbol{f}(T)$ and $\varphi(\boldsymbol{f}(T))$. By 4.2.2, there exists ε' such that both $\boldsymbol{f}(T)$ and $\varphi(\boldsymbol{f}(T))$ lie in $\mathbf{W}_m(\mathcal{A}_K(]1-\varepsilon', 1[))$. Hence as before $\theta_{p^m}(\boldsymbol{f}(T), Z) \in 1 + \pi_m \mathcal{A}_{K_m}(J)[[Z]]$, for all compact $J \in]1-\varepsilon', 1[$, and hence

$$\theta_{p^m}(\boldsymbol{f}(T), Z) \in 1 + \pi_m Z \mathcal{A}_{K_m}(]1-\varepsilon', 1[)[[Z]].$$

The assertion on $\boldsymbol{u}(T)$ follows from 3.4.6, and the same considerations.□

Corollary 4.2.5. *The diagram 4.1.9.1 can be computed for $\mathcal{E}_K^\dagger$ instead of L, and the other assertions of theorems 4.1.9 and 4.1.11 remain true*

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_{p^{m+1}} \arrow[r] & (\mathcal{E}^\dagger_{L_m})^\times \arrow[r, "f \mapsto f^{p^{m+1}}"] & (\mathcal{E}^\dagger_{L_m})^\times \arrow[r, "\delta_{\mathrm{Kum}}"] & \mathrm{H}^1(\mathrm{G}_{\mathcal{E}^\dagger_{L_m}}, \boldsymbol{\mu}_{p^{m+1}}) \arrow[r] & 1 \\
& \mathbf{W}_m(\mathcal{O}_L^{\sigma=1}) \arrow[u] \arrow[d] \arrow[r, hook] & \mathbf{W}_m(\mathcal{O}_L^\dagger) \arrow[u, "\theta_{p^m}(-,1)"] \arrow[d] \arrow[r, "\varphi-1"'] & \mathbf{W}_m(\mathcal{O}_L^\dagger) \arrow[u, "\mathrm{e}_{p^m}(-,1)^{p^{m+1}}"'] \arrow[d] & & \\
0 \arrow[r] & \mathbb{Z}/p^{m+1}\mathbb{Z} \arrow[r] \arrow[uu, dotted, bend left, "\wr"] & \mathbf{W}_m(\mathrm{E}) \arrow[r, "\bar{\mathrm{F}}-1"'] & \mathbf{W}_m(\mathrm{E}) \arrow[r, "\delta"'] & \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \arrow[r] \arrow[uu, dotted, "\bar{\mathrm{e}} := \overline{\mathrm{e}_{p^m}(-,1)^{p^{m+1}}}"'] & 0
\end{tikzcd} \tag{4.2.5.1}$$

In particular, if $\alpha = \delta(\overline{\boldsymbol{f}}(t))$ is the Artin-Schreier character defined by some $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_m(\mathrm{E})$, and if $\mathrm{E}_\alpha/\mathrm{E}$ is the separable extension defined by the kernel of α, then the (Kummer) unramified extension of $\mathcal{E}^\dagger_{K_m}$, whose residue field is E_α, is given by

$$\mathcal{E}^\dagger_{K_m}\Big(\theta_{p^m}(\boldsymbol{\nu}, 1)\Big)\,, \tag{4.2.5.2}$$

where $\boldsymbol{\nu}$ is a solution of $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{f}(T)$, for an arbitrary lifting $\boldsymbol{f}(T)$ of $\overline{\boldsymbol{f}}(t)$.

Proof : Let $\mathrm{F}_\alpha/\mathrm{E}$ be the separable Artin-Schreier extension defined by $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathrm{E})$, and let $\mathcal{F}_\alpha^\dagger$ be the corresponding unramified extension of $\mathcal{E}_K^\dagger$. Let $\boldsymbol{\nu} \in \mathbf{W}_m(\widehat{\mathcal{E}}_K^{\mathrm{unr}})$ be a solution of $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{f}(T)$. The non trivial fact is that $\theta_{p^m}(\boldsymbol{\nu}, 1)$ lies in $\mathcal{F}_\alpha^\dagger(\pi_m)$ and not only in its completion, say $\mathcal{F}_\alpha(\pi_m)$. In other words, we shall show that $\mathcal{F}_\alpha^\dagger(\pi_m) = \mathcal{E}^\dagger_{K_m}(\theta_{p^m}(\boldsymbol{\nu}, 1))$. Both $\mathcal{E}^\dagger_{K_m}(\theta_{p^m}(\boldsymbol{\nu}, 1))$ and $\mathcal{F}_\alpha^\dagger(\pi_m)$ are unramified over $\mathcal{E}_K^{\dagger,0} = \mathcal{E}^\dagger_{K_m} \cap \mathcal{E}_K^{\dagger,\mathrm{unr}}$, since their completions are unramified. Moreover, by theorem 4.1.11, they have the same residue field, since this last coincides with that of their completions. By uniqueness (cf. 4.2.3), they are equal. □

Remark 4.2.6. We will see that the study of a generic Artin-Schreier character, given by $\boldsymbol{f}(T)$, can be reduced to the case in which $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[T^{-1}])$ (cf. 4.2.7, 5.1.2, and 5.4.10).

Lemma 4.2.7. *Let $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathcal{O}^{\dagger}_{K_m})$, then there exist $\widetilde{\boldsymbol{f}}(T) \in \mathbf{W}_m(\mathcal{O}_{K_m}[[T]][T^{-1}])$ and $\boldsymbol{u}(T) = (u_0(T), \ldots, u_m(T)) \in \mathbf{W}_m(\mathcal{O}^{\dagger}_{K_m})$ such that $|u_j(T)|_1 < 1$ for all $j = 0, \ldots, m$ and $\boldsymbol{f}(T) = \boldsymbol{u}(T) + \widetilde{\boldsymbol{f}}(T)$. In particular $\theta_{p^m}(\boldsymbol{\nu}, 1) = \mathrm{e}_{p^m}(\boldsymbol{u}(T), 1) \cdot \theta_{p^m}(\tilde{\boldsymbol{\nu}}, 1)$, where $\tilde{\boldsymbol{\nu}}$ is a solution of $\varphi(\tilde{\boldsymbol{\nu}}) - \tilde{\boldsymbol{\nu}} = \widetilde{\boldsymbol{f}}(T)$.*

Proof : It is evident for $m = 0$. By induction the lemma follow from the relation

$$(f_0(T), \ldots, f_m(T)) = (f_0(T), 0, \ldots, 0) + (0, f_1(T), \ldots, f_m(T)) \qquad (4.2.7.1)$$

valid for Witt vectors in general ([Bou83a, ch.10,§1,Lemme4]).□

Chapitre 5

Classification of rank one differential equations over $\mathcal{R}_{K_\infty}$ (or $\mathcal{E}^\dagger_{K_\infty}$)

Throughout this second application, we will not need the results of section 4. Namely, $(K, |.|)$ is only a complete valued field containing $(\mathbb{Q}_p, |\cdot|)$, we will not suppose that K verifies 4.2.1.1, nor that its residue field is perfect. We fix a Lubin-Tate group $\mathfrak{G}_P$, isomorphic to $\widehat{\mathbb{G}}_m$, and fix a generator $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$ of the Tate module $T(\mathfrak{G}_P)$.

We recall that $K_s = K(\pi_s)$, and that k_s is its residue field (cf. 2.9.6). For all algebraic extension H/K, we set $H_s := H(\pi_s)$. The residue fields of H and H_s are denoted by k_H and k_{H_s} respectively. We set $\mathrm{E}_s := k_s(\!(t)\!)$.

5.1 The settings

The starting point of the classification is the equation

$$\theta_{p^s}(\boldsymbol{\nu}, 1)^{p^{s+1}} = \mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)^{p^{s+1}} \ , \tag{5.1.0.1}$$

with the notations of corollary 4.2.5. In some cases the symbol $\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)$ does make sense, and the interesting "Kummer generator" $\theta_{p^s}(\boldsymbol{\nu}, 1)$ is equal to $\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)$. We will show that all rank one solvable differential equations over $\mathcal{R}_{K_m}$ admit, in some basis, such an exponential as solution.

Definition 5.1.1. Let $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$, then we set

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) := \exp\Big(\pi_s\phi_0^-(T) + \pi_{s-1}\frac{\phi_1^-(T)}{p} + \cdots + \pi_0\frac{\phi_s^-(T)}{p^s}\Big) \ , \tag{5.1.1.1}$$

where $\phi_j^-(T)$ is the $j-$th phantom component of $\boldsymbol{f}^-(T) = (f_0^-(T), \ldots, f_s^-(T))$.

Remark 5.1.2. Clearly $\phi_j^-(T)$ lies in $T^{-1}\mathcal{O}_K[T^{-1}]$, for all $j = 0, \dots, s$, and hence the expression 5.1.1 converges T^{-1}−adically. Moreover,

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) \;=\; \prod_{j=0}^{s} E_{s-j}(f_j^-(T)) \;\in\; 1 + \pi_s T^{-1}\mathcal{O}_{K_s}[[T^{-1}]] \,. \tag{5.1.2.1}$$

In particular, $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ is convergent for $|T| > 1$. As mentioned in the remark 4.2.6, lemmas 4.2.7, 5.4.5, and 6.3.15, will be useful to reduce the study of $\theta_{p^s}(\boldsymbol{\nu}, 1)$, with $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{f}(T)$, to this case in which $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[T^{-1}])$.

5.2 Survey of the Results

Remark 5.2.1. For all algebraic extensions H/K, the function

$$\boldsymbol{f}^-(T) \mapsto \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) \tag{5.2.1.1}$$

(cf. 5.1.1) defines a group morphism (as we can see on the phantom components)

$$\mathrm{e}_{p^s}(-, 1) : \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}]) \longrightarrow 1 + \pi_s T^{-1}\mathcal{O}_{H_s}[[T^{-1}]] \,. \tag{5.2.1.2}$$

Indeed $\boldsymbol{f}^-(T)$ involve only a finite numbers of coefficients of H, then the series $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ lies in a finite (and hence complete) extension of K.

Let $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$, we set

$$\mathrm{L}(0, \boldsymbol{f}^-(T)) = \partial_T - \partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)). \tag{5.2.1.3}$$

Observe that $1 + \pi_s T^{-1}\mathcal{O}_{H_s}[[T^{-1}]]$ is not contained in $\mathcal{E}_{H_s} = \mathcal{E}_K \otimes_K H_s$. However, every series in this multiplicative group is convergent for $|T| > 1$ (cf. 5.1.2). Then, by 2.3.2 and by continuity of the radius, $\mathrm{L}(0, \boldsymbol{f}^-(T))$ is solvable over $\mathcal{R}_{H_s}$.

Theorem 5.2.2 (main theorem)**.** *Let M be a rank one solvable differential module over $\mathcal{R}_{K_\infty}$ (i.e. over $\mathcal{R}_{K_m}$, for some $m \geq 0$, or over $\mathcal{R}_K$ (cf. 2.4.7)). Then there exists a basis of M such that*

1. *the 1×1 matrix of the derivation of M lies in $\mathcal{A}_K(]0,\infty]) \cap \mathcal{O}_K[[T]][T^{-1}]$;*
2. *there exist an $s \geq 0$, and a Witt vector $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$ such that the Taylor solution (cf. 2.3.0.1) of M, at ∞, is*

$$T^{a_0} \cdot \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) \,, \tag{5.2.2.1}$$

with $a_0 \in \mathbb{Z}_p$. In particular M is defined (in this basis) by the operator

$$\begin{aligned} \mathrm{L}(a_0, \boldsymbol{f}^-(T)) \;&:=\; \partial_T - \partial_{T,\log}\left(T^{a_0} \cdot \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)\right) && (5.2.2.2) \\ &=\; \partial_T + a_0 + \sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T)). \end{aligned}$$

Moreover the isomorphism class of M depends bijectively on

- *the class of a_0 in $\mathbb{Z}_p/\mathbb{Z}$;*
- *the Artin-Schreier character $\alpha := \delta(\overline{\boldsymbol{f}^-}(t))$ defined by the reduction $\overline{\boldsymbol{f}^-}(t) \in \mathbf{W}_s(E_s)$ of $\boldsymbol{f}^-(T)$.*

Definition 5.2.3. We will denote indifferently by $\mathrm{M}(a_0, \alpha)$, $\mathrm{M}(a_0, \overline{\boldsymbol{f}^-}(t))$ or $\mathrm{M}(a_0, \boldsymbol{f}^-(T))$, the differential module defined by $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$.

Assume the point (1) and (2) of the theorem 5.2.2. Then the last assertion can be translated in terms of $\boldsymbol{\pi}-$exponentials as follow. Recall that $p = \mathrm{w}$ (cf. 3.6.1).

Theorem 5.2.4. *Let $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$, and let $\overline{\boldsymbol{f}^-}(t) \in \mathbf{W}_s(t^{-1}k_s[t^{-1}])$ be its reduction. Then*

(3) If $\widetilde{\boldsymbol{f}}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$ is another lifting of $\overline{\boldsymbol{f}^-}(t)$, then

$$\frac{\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)}{\mathrm{e}_{p^s}(\widetilde{\boldsymbol{f}}^-(T), 1)} = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T), 1) \tag{5.2.4.1}$$

is convergent for $|T| > 1 - \varepsilon$, for some $\varepsilon > 0$ (i.e. lies in $\mathcal{R}_{K_s}$).

(4) If $\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$ is an arbitrary lifting of $\bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t))$, then

$$\frac{\mathrm{e}_{p^s}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T), 1)}{\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)} = \mathrm{e}_{p^s}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T) - \boldsymbol{f}^-(T), 1) \tag{5.2.4.2}$$

is convergent for $|T| > 1 - \varepsilon'$, for some $\varepsilon' > 0$ (i.e. lies in $\mathcal{R}_{K_s}$).

(5) Conversely the function $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ lies in $\mathcal{R}_{K_s}$ if and only if the equation

$$\bar{\mathrm{F}}\left(\bar{\boldsymbol{\nu}}^-\right) - \bar{\boldsymbol{\nu}}^- = \overline{\boldsymbol{f}^-}(t) \tag{5.2.4.3}$$

has a solution $\bar{\boldsymbol{\nu}}^- \in \mathbf{W}_s(t^{-1}k_s[t^{-1}])$.

Notation 5.2.5. The point (5) will be called the Frobenius structure theorem.

5.3 Description of $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})$ (or $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{E}^\dagger_{K_\infty})$)

By the main theorem 5.2.2, the definition 5.1.1 and by the rule 2.2.0.4, it follows that, for all $s \geq 0$, and for all algebraic extensions H/K, we have a exact sequence of abelian groups (functorial on the algebraic extensions H of K)

$$\mathbf{W}_s(t^{-1}k_H[t^{-1}]) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{W}_s(t^{-1}k_H[t^{-1}]) \xrightarrow{\mathrm{M}(0,-)} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_s}) \ . \tag{5.3.0.1}$$

On the other hand, it follows by the definition 5.1.1, that we have

$$\mathrm{e}_{p^{s+1}}(\mathrm{V}(\boldsymbol{f}^-(T)), 1) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) \ . \tag{5.3.0.2}$$

Hence, for all $s \geq 0$, we have the following functorial commutative diagram

$$\begin{tikzcd} \mathbf{W}_s(t^{-1}k_H[t^{-1}]) \arrow[r, "\bar{\mathrm{F}}-1"] \arrow[d, "\mathrm{V}"'] \arrow[dr, phantom, "\odot"] & \mathbf{W}_s(t^{-1}k_H[t^{-1}]) \arrow[r, "\mathrm{M}(0,-)"] \arrow[d, "\mathrm{V}"'] & \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_{s+1}}) \\ \mathbf{W}_{s+1}(t^{-1}k_H[t^{-1}]) \arrow[r, "\bar{\mathrm{F}}-1"] & \mathbf{W}_{s+1}(t^{-1}k_H[t^{-1}]) \arrow[ur, "\mathrm{M}(0,-)"'] & \end{tikzcd} \tag{5.3.0.3}$$

This shows, by passing to the inductive limit, that we have again an exact sequence

$$\mathbf{CW}(t^{-1}k_H[t^{-1}]) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{CW}(t^{-1}k_H[t^{-1}]) \xrightarrow{\mathrm{M}(0,-)} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_\infty}) \ . \tag{5.3.0.4}$$

The group $\mathbb{Z}_p/\mathbb{Z}$ has no $p-$torsion element. On the other hand, every elements of $\mathbf{CW}(t^{-1}k_H[t^{-1}])$ is killed by a power of p. Since we are assuming that all solution are of the forme 5.2.2.1, this proves the following

Lemma 5.3.1. *Let H be an algebraic extension of K_∞. The image of* $\mathrm{M}(0,-)$ *is the sub-group of the $p-$torsion elements of* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_H)$*, and if H/K_∞ is Galois, then* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_H)$ *is isomorphic, as* $\mathrm{Gal}(H/K_\infty)-$*module, to the direct sum of $\mathbb{Z}_p/\mathbb{Z}$ with the image of* $\mathrm{M}(0,-)$. □

This consideration shows that we can descend by Galois. In other words

Corollary 5.3.2. *The map* $(a_0,\alpha) \mapsto \mathrm{M}(a_0,\alpha)$ *induces an isomorphism*

$$\mathbb{Z}_p/\mathbb{Z} \oplus \frac{\mathbf{CW}(t^{-1}k_\infty[t^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(t^{-1}k_\infty[t^{-1}])} \quad \xrightarrow[\sim]{\mathrm{M}(-,-)} \quad \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}) \ . \tag{5.3.2.1}$$

Proof : By Galois descent $\mathrm{M}(-,-)$ induce an isomorphism, with $k_\infty^{\mathrm{perf}} := (k^{\mathrm{alg}})^{\mathrm{Gal}(k^{\mathrm{alg}}/k_\infty)}$ instead of k_∞. But actually, the covector quotient is invariant under inseparable extension of k_∞ as explained in subsection 5.3.3.2 below. □

5.3.1 Characters of the abelianized wild inertia

On the other hand, it is well known that (cf. lemma 5.4.5 and 5.4.10)

$$\mathrm{H}^1\left(\mathrm{Gal}\big(\kappa(\!(t)\!)^{\mathrm{sep}}/\kappa(\!(t)\!)\big), \mathbb{Q}_p/\mathbb{Z}_p\right) = \mathbf{P}(\kappa) \oplus \mathrm{H}^1\left(\mathrm{Gal}\big(\kappa^{\mathrm{sep}}/\kappa\big), \mathbb{Q}_p/\mathbb{Z}_p\right) \ , \tag{5.3.2.2}$$

where $\mathbf{P}(k_\infty)$ is the character group of $\mathcal{P}_{\mathrm{E}_\infty}$, with $\mathrm{E}_\infty = k_\infty(\!(t)\!)$ (cf. 2.8.5). More precisely we have the following (for a more handy description of $\mathbf{P}(\kappa)$ see 6.1.1)

Lemma 5.3.3. *For all field κ of characteristic $p > 0$, we have*

$$\mathbf{P}(\kappa) = \frac{\mathbf{CW}(t^{-1}\kappa[t^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(t^{-1}\kappa[t^{-1}])} \ . \tag{5.3.3.1}$$

Proof : This will follow from 5.4.5 and 5.4.10. □

Furthermore we have

$$\mathbf{P}(k_\infty^{\mathrm{perf}}) = \mathbf{P}(k_\infty) \ , \tag{5.3.3.2}$$

because, by remark 2.7.6 (or 2.8.2.1), the Artin-Schreier complex is stable under purely inseparable extensions, that is $\mathrm{Gal}(k_\infty^{\mathrm{perf,sep}}/k_\infty^{\mathrm{perf}}) = \mathrm{Gal}(k_\infty^{\mathrm{sep}}/k_\infty)$. In other words, for all $r \geq 0$, the two co-vectors $\overline{\boldsymbol{f}^-}(t) = (\ldots, 0, \overline{f_0^-}(t), \ldots, \overline{f_s^-}(t))$ and $\bar{\mathrm{F}}^r(\overline{\boldsymbol{f}^-}(T)) = (\ldots, 0, \overline{f_0^-}(t)^{p^r}, \ldots, \overline{f_s^-}(t)^{p^r})$ have the same image in the right hand quotient of equation 5.3.3.1.

5.4 Proofs of the statements

We prove first the statements (3), (4), and (5) of the theorem 5.2.4. The idea is to express $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ as a product of $\boldsymbol{\pi}-$exponentials of the type $\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})$. The main tool will be the notion of $s-$*co-monomial* which reduce the study to $\boldsymbol{\pi}-$exponentials (see equation 5.4.11.1). The principal lemma will be 5.4.3.

Definition 5.4.1. Let H/K be an algebraic extension. Let $d = np^m > 0$, $(n,p) = 1$. Let $s \geq 0$. We will call $s-$*co-monomial* of degree $-d$ relative to $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$ the following Witt vector in $\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$

$$\begin{array}{lcll} \boldsymbol{\lambda}T^{-d} & := & (\overbrace{0, \ldots, 0}^{s-m}, \lambda_0 T^{-n}, \lambda_1 T^{-np}, \ldots, \lambda_m T^{-d}) & \text{if} \quad m \leq s \ , \\ \boldsymbol{\lambda}T^{-d} & := & (\lambda_r T^{-np^r}, \lambda_{r+1} T^{-np^{r+1}}, \ldots, \lambda_m T^{-d}) & \text{if} \quad m \geq s \ , \end{array} \tag{5.4.1.1}$$

where $r = m - s$. We denote by

$$\mathbf{W}_s^{(-d)}(\mathcal{O}_H) \tag{5.4.1.2}$$

the sub-group of $\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ formed by $s-$co-monomials of degree $-d$, and by

$$\mathbf{W}_s^{(-d)}(k_H) \tag{5.4.1.3}$$

its image in $\mathbf{W}_s(t^{-1}k_H[t^{-1}])$.

Remark 5.4.2. By looking at the phantom components we have an isomorphism of groups $\mathbf{W}_s^{(-d)}(\mathcal{O}_H) \xrightarrow{\sim} \mathbf{W}_{\min(s,m)}(\mathcal{O}_H)$, and hence $\mathbf{W}_s^{(-d)}(k_H) \xrightarrow{\sim} \mathbf{W}_{\min(s,m)}(k_H)$.

Lemma 5.4.3. *Let now H/K be an algebraic extension. Let $d = np^m > 0$, $(n,p) = 1$, let $s \geq 0$, and let $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$. If $m \leq s$, we have*

$$\mathrm{e}_{p^s}(\boldsymbol{\lambda}T^{-d}, 1) = \mathrm{e}_d(\boldsymbol{\lambda}, T^{-1}) \ . \tag{5.4.3.1}$$

Proof : The phantom vector of $(\underbrace{0,\ldots,0}_{s-m},\lambda_0T^{-n},\lambda_1T^{-np},\ldots,\lambda_mT^{-d})$, is

$$\langle 0,\ldots,0,p^{s-m}\phi_0T^{-n},p^{s-m}\phi_1T^{-np},\ldots,p^{s-m}\phi_mT^{-d}\rangle, \qquad (5.4.3.2)$$

where $\langle\phi_0,\cdots,\phi_m\rangle$ is the phantom vector of $(\lambda_0,\ldots,\lambda_m)$. The proof follows immediately from the definitions 5.1.1 and 3.3.1.1. $\Box$

Definition 5.4.4. For all algebraic extensions H/K we set $\mathrm{E}_H := k_H(\!(t)\!)$.

Lemma 5.4.5. *For all $s\geq 0$, we have a (functorial) decomposition*

1. $\mathbf{W}_s(\mathrm{E}_H)=\oplus_{d>0}\mathbf{W}_s^{(-d)}(k_H)\oplus\mathbf{W}_s(k_H)\oplus\mathbf{W}_s(tk_H[[t]])$;
2. $\mathbf{W}_s(\mathcal{O}_H[[T]][T^{-1}])=\oplus_{d>0}\mathbf{W}_s^{(-d)}(\mathcal{O}_H)\oplus\mathbf{W}_s(\mathcal{O}_H)\oplus\mathbf{W}_s(T\mathcal{O}_H[[T]])$.

Proof : Let $s=0$, then $k_H(\!(t)\!)=\oplus_{d>0}k_Ht^{-d}\oplus k_H\oplus tk_H[\![t]\!]$. The proof follows easily by induction from 4.2.7.1.$\Box$

Remark 5.4.6. Witt vectors in $\oplus_{d>0}\mathbf{W}_s^{(-d)}(\mathcal{O}_H)$ (resp. $\mathbf{W}_s(\mathcal{O}_H)$, $\mathbf{W}_s(T\mathcal{O}_K[[T]])$) have their phantom components in $T^{-1}\mathcal{O}_H[T^{-1}]$ (resp. $\mathcal{O}_H$, $T\mathcal{O}_K[[T]]$).

Corollary 5.4.7. *We have a (functorial) decomposition*

$$\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])=\oplus_{d>0}\mathbf{W}_s^{(-d)}(\mathcal{O}_H)\ ,\quad \mathbf{W}_s(t^{-1}k_H[t^{-1}])=\oplus_{d>0}\mathbf{W}_s^{(-d)}(k_H)\ .$$

Proof : The inclusion $\subseteq$ follows by the remark 5.4.6. The inclusion $\supseteq$ is evident since all monomials belong to $\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$. The right hand equality result by reduction. $\Box$

Definition 5.4.8. For all $\boldsymbol{f}(T)\in\mathbf{W}_s(\mathcal{O}_H[[T]][T^{-1}])$, we will denote by

$$\boldsymbol{f}(T)=\boldsymbol{f}^-(T)+\boldsymbol{f}_0+\boldsymbol{f}^+(T) \qquad (5.4.8.1)$$

the unique decomposition of $\boldsymbol{f}(T)$ satisfying $\boldsymbol{f}^-(T)\in\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$, $\boldsymbol{f}_0\in\mathbf{W}_s(\mathcal{O}_H)$, $\boldsymbol{f}^+(T)\in\mathbf{W}_s(T\mathcal{O}_H[[T]])$ (cf. 5.4.5). The same notation will be used for a Witt vector $\overline{\boldsymbol{f}}(t)\in\mathbf{W}_s(\mathrm{E}_H)$.

Remark 5.4.9. By 2.8.1.2, we have then a correspondent decomposition of $\alpha:=\delta(\overline{\boldsymbol{f}}(t))$, i.e. $\alpha=\alpha^-+\alpha_0$, ($\alpha^+=0$ by 5.4.10), with $\alpha^-=\delta(\overline{\boldsymbol{f}^-}(t))$, and $\alpha_0=\delta(\overline{\boldsymbol{f}}_0)$. This shows that $\mathrm{Gal}(\mathrm{E}_H^{\mathrm{sep}}/\mathrm{E}_H)^{\mathrm{ab}}\cong\mathrm{Gal}(\mathrm{E}_H^{\mathrm{sep}}/\mathrm{E}_H)^{\mathrm{ab}}\oplus\mathcal{I}^{\mathrm{ab}}_{\mathrm{E}_H}$, where $\mathrm{E}_H=k_H(\!(t)\!)$.

Proposition 5.4.10. $\mathbf{W}_s(tk_H[[t]])\subseteq(\bar{\mathrm{F}}-1)\mathbf{W}_s(tk_H[[t]])$*, for all $s\geq 0$.*

Proof : Since E_H is complete, then, by 2.7.5, $\mathbf{W}_s(\mathrm{E}_H)$ is complete. Let $\overline{\boldsymbol{f}}^+(t)\in\mathbf{W}_s(tk_H[[t]])$. Hence the series $\overline{\boldsymbol{g}}^+(t):=-\sum_{i\geq 0}\bar{\mathrm{F}}^i(\overline{\boldsymbol{f}}^+(t))$ is Cauchy for this topology, and hence converges in $\mathbf{W}_s(\mathrm{E}_H)$. Moreover $\overline{\boldsymbol{f}}^+(t)=\bar{\mathrm{F}}(\overline{\boldsymbol{g}}^+(t))-\overline{\boldsymbol{g}}^+(t)$.$\Box$

Remark 5.4.11. Let H/K be an algebraic extension and let $\boldsymbol{f}^-(T) \in \mathbf{W}_s(t^{-1}\mathcal{O}_H[t^{-1}])$. Let $v_p(-)$ be the $p-$adic valuation (namely $v_p(d) = m$ if $d = np^m$, $(n,p) = 1$). Let $\boldsymbol{f}^-(T) = \sum_{d>0} \boldsymbol{\lambda}_d T^{-d}$, with $\boldsymbol{\lambda}_d \in \mathbf{W}_{v_p(d)}(k_H)$ be its decomposition in $s-$co-monomials of degree $-d$. We can suppose $s \gg 0$ (cf. 5.3.0.2), then

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) = \mathrm{e}_{p^s}(\sum_{d>0} \boldsymbol{\lambda}_d T^{-d}, 1) = \prod_{d>0} \mathrm{e}_{p^s}(\boldsymbol{\lambda}_d T^{-d}, 1) \overset{5.4.3}{=} \prod_{d>0} \mathrm{e}_d(\boldsymbol{\lambda}_d, T^{-1}). \tag{5.4.11.1}$$

Then $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ is a (finite) product of elementary $\boldsymbol{\pi}-$exponentials. In terms of differential modules, we have

$$\mathrm{M}(0, \boldsymbol{f}^-(T)) = \otimes_{d>0} \mathrm{M}(0, \boldsymbol{\lambda}_d T^{-d}) \ . \tag{5.4.11.2}$$

Hence, by 2.2.0.4, the study can be reduced to $\boldsymbol{\pi}-$exponentials.

5.4.1 Proof of the statements $(3), (4), (5)$ of theorem 5.2.4 :

Notation 5.4.12. For all index $d > 0$, we set $d = np^m$, with $(n,p) = 1$, and $v_p(d) := m$. In the sequel the letters n and m will indicate always this decomposition of a given d.

First of all, lemma 5.4.3 shows that, for all d appearing in the (finite) product 5.4.11.1, we have (cf. definition 3.4.2)

$$\mathrm{L}_d(\boldsymbol{\lambda}_d) = \mathrm{L}(0, \boldsymbol{\lambda}_d T^{-d}) \qquad , \qquad \widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}_d) = \mathrm{M}(0, \boldsymbol{\lambda}_d T^{-d}) \ , \tag{5.4.12.1}$$

where $\boldsymbol{\lambda}_d T^{-d}$ is the $s-$co-monomial of degree $-d$ attached to $\boldsymbol{\lambda}_d \in \mathbf{W}_{v_p(d)}(\mathcal{O}_H)$ (cf. 5.4.1). Actually, by the rule 5.3.0.2, we can suppose $s \gg v_p(d) = m$, for all $d > 0$ appearing in the (finite) product 5.4.11.1.

The assertions (3) and (4) are consequences of the reduction theorem 3.4.6, and the Frobenius structure theorem for $\boldsymbol{\pi}-$exponentials 3.6.1, respectively. Let us show the assertion (3). We decompose $\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T)$ in $s-$co-monomials of degree $-d$, $\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T) = \sum_d \boldsymbol{\lambda}_d T^{-d}$, with $\boldsymbol{\lambda}_d \in \mathbf{W}_{v_p(d)}(\mathcal{O}_H)$ (cf. 5.4.5). Then

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T), 1) = \prod_{d>0} \mathrm{e}_{p^s}(\boldsymbol{\lambda}_d T^{-d}, 1) \overset{5.4.3}{=} \prod_{d>0} \mathrm{e}_d(\boldsymbol{\lambda}_d, T^{-1}) \ . \tag{5.4.12.2}$$

The over-convergence of $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T), 1)$ will result from the over-convergence of every $\mathrm{e}_d(\boldsymbol{\lambda}_d, T^{-1})$. In order to apply the reduction theorem 3.4.6, we shall show that the reduction of $\boldsymbol{\lambda}_d$ is 0, for all $d > 0$. Since the reduction of $\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T)$ is 0, hence, by lemma 5.4.5, the reduction of $\boldsymbol{\lambda}_d T^{-d}$ in $\mathbf{W}_s^{(-d)}(k_H)$ is 0, for all $d > 0$. By remark 5.4.2, for all $d > 0$, we have an isomorphism

$$\boldsymbol{\lambda}_d T^{-d} \mapsto \boldsymbol{\lambda}_d : \mathbf{W}_s^{(-d)}(\mathcal{O}_H) \xrightarrow{\sim} \mathbf{W}_{v_p(d)}(\mathcal{O}_H) \ . \tag{5.4.12.3}$$

Hence, for all $d > 0$, the reduction of $\boldsymbol{\lambda}_d$ in $\mathbf{W}_{v_p(d)}(k_H)$ is 0.

The assertion (4) follows the same line. Namely, by the assertion (3), the isomorphism class of $\mathrm{M}(0, \boldsymbol{f}^-(T))$ depends only on the reduction $\overline{\boldsymbol{f}^-}(t) \in \mathbf{W}_s(t^{-1}k_H[t^{-1}])$ of $\boldsymbol{f}^-(T)$. As usual, we decompose $\overline{\boldsymbol{f}^-}(t) = \sum_{d>0} \bar{\boldsymbol{\lambda}}_d t^{-d}$, with $\bar{\boldsymbol{\lambda}}_d t^{-d} \in \mathbf{W}_s^{(-d)}(k_H)$. The morphism $\bar{\mathrm{F}} : \mathbf{W}_s(\mathrm{E}_H) \to \mathbf{W}_s(\mathrm{E}_H)$ sends the monomial $\bar{\boldsymbol{\lambda}}_d t^{-d}$ into $\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd}$, Hence $\bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t)) = \sum_{d>0} \bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd}$. Then

$$\mathrm{M}(0, \overline{\boldsymbol{f}^-}(t)) \xrightarrow{\sim} \otimes_{d>0}\mathrm{M}_d(\bar{\boldsymbol{\lambda}}_d) \xrightarrow[3.6.1]{\sim} \otimes_{d>0}\mathrm{M}_{pd}(\mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)) \xrightarrow{\sim} \mathrm{M}(0, \bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t))) \;, \tag{5.4.12.4}$$

where the last isomorphism follows from the fact that $\mathrm{V}(\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd})$ and $\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd}$ define the same differential module (cf. 5.3.0.2).

The proof of the assertion (5) of theorem 5.2.4 follows from assertions (3) and (4) of theorem 5.2.4 in the following way. Suppose that $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ is over-convergent. We want to show that the equation $\bar{\mathrm{F}}(\bar{\boldsymbol{\nu}}) - \bar{\boldsymbol{\nu}} = \overline{\boldsymbol{f}^-}(t)$ has a solution $\bar{\boldsymbol{\nu}} \in \mathbf{W}_s(t^{-1}k_H[t^{-1}])$. In other words, we shall show that $\overline{\boldsymbol{f}^-}(t)$ belongs to $(\bar{\mathrm{F}} - 1)\mathbf{W}_s(t^{-1}k_H[t^{-1}])$. Let us write $\boldsymbol{f}^-(T) = \sum_{d>0} \boldsymbol{\lambda}_d T^{-d}$ as a (finite) sum of $s-$co-monomials. We need to replace $\boldsymbol{f}^-(T)$ by a more handy Witt vector.

Definition 5.4.13. A Witt vector $\boldsymbol{f}_p^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ is said *pure* if its decomposition in $s-$co-monomials is a (finite) sum of the type

$$\boldsymbol{f}_p^-(T) = \sum_{n \in \mathrm{J}_p} \boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}} \;, \tag{5.4.13.1}$$

where

$$\mathrm{J}_p := \{n \in \mathbb{Z} \mid (n,p) = 1 \;,\; n > 0\} \;, \tag{5.4.13.2}$$

and $\boldsymbol{\lambda}_{np^{m(n)}} \in \mathbf{W}_{m(n)}(\mathcal{O}_H)$.

Remark 5.4.14. We have

$$\partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}_p^-(T), 1)) = \sum_{n \in \mathrm{J}_p} -n \sum_{j=0}^{m(n)} \pi_{m(n)-j}\phi_{np^{m(n)},j} T^{-np^j} \;, \tag{5.4.14.1}$$

where $\langle \phi_{np^{m(n)},0}, \ldots, \phi_{np^{m(n)},m(n)} \rangle$ is the phantom vector of $\boldsymbol{\lambda}_{np^{m(n)}}$. In this case the coefficients of the differential equation are simpler and directly related to the Witt vector. This fact will be useful for explicit computations (cf. 6.3.18).

The interest of the notion of *pure* Witt vector is the following

Lemma 5.4.15. *Let $\boldsymbol{f}_p^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ be a pure Witt vector. The exponential $\mathrm{e}_{p^s}(\boldsymbol{f}_p^-(T), 1)$ is over-convergent if and only if $\overline{\boldsymbol{f}_p^-}(t) = 0$. Moreover,*

$$\mathrm{Irr}\left(\mathrm{M}(0, \boldsymbol{f}_p^-(T))\right) = \max_{n \in \mathrm{J}_p} \mathrm{Irr}\left(\mathrm{M}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})\right) . \tag{5.4.15.1}$$

Proof : Write $\mathrm{M}(0, \boldsymbol{f}_p^-(T)) = \otimes_{n\in \mathrm{J}_p}\mathrm{M}(0, \boldsymbol{\lambda}_{np^{m(n)}}T^{-np^{m(n)}})$. The irregularity of $\mathrm{M}(0, \boldsymbol{\lambda}_{np^{m(n)}}T^{-np^{m(n)}}) \xrightarrow[5.4.12.1]{\sim} \widetilde{\mathrm{M}}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})$ is, by theorem 3.4.6, a number belonging to the set

$$\{0\} \cup \{n \cdot p^m \mid m \geq 0\} \ . \tag{5.4.15.2}$$

Hence, for different values of n, we have different values of the $p-$adic slope of $\mathrm{M}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})$. Theorem 2.4.8 implies then the equation 5.4.15.1. Suppose now that $\mathrm{e}_{p^s}(\boldsymbol{f}_p^-(T), 1)$ is over-convergent, then this irregularity is equal to 0. Hence all $\mathrm{M}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})$ are trivial, and $\mathrm{e}_{p^s}(\boldsymbol{\lambda}_{np^{m(n)}}T^{-np^{m(n)}}, 1)$ is over-convergent (i.e. lies in $\mathcal{R}_H$), for all $n \in \mathrm{J}_p$. By theorem 3.4.6 this implies $\overline{\boldsymbol{\lambda}_{np^{m(n)}}} = 0$, for all $n \in \mathrm{J}_p$. $\square$

The assertion (5) of theorem 5.2.4 follows then by the point (1) of the following

Lemma 5.4.16. *Let $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$. Then*

1. *There exists a pure Witt vector $\boldsymbol{f}_p^-(T)$ such that*

$$\overline{\boldsymbol{f}^-(T) - \boldsymbol{f}_p^-(T)} \in (\bar{\mathrm{F}} - 1)\mathbf{W}_s(t^{-1}k_H[t^{-1}]) \ . \tag{5.4.16.1}$$

In particular, by assertion (4) *of theorem 5.2.4* , $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T) - \boldsymbol{f}_p^-(T), 1)$ *is over-convergent, and*

$$\mathrm{M}(0, \boldsymbol{f}^-(T)) \xrightarrow{\sim} \mathrm{M}(0, \boldsymbol{f}_p^-(T)) \ , \tag{5.4.16.2}$$

over $\mathcal{R}_{H_s}$;

2. *There exists a pure Witt vector $\boldsymbol{h}_p^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{H_\infty}[T^{-1}])$ such that*

$$[\pi_s]\boldsymbol{f}^-(T) = [\pi_s]\boldsymbol{h}_p^-(T) \ . \tag{5.4.16.3}$$

In particular

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) = \mathrm{e}_{p^s}(\boldsymbol{h}_p^-(T), 1) \ . \tag{5.4.16.4}$$

Proof : Let us write $\boldsymbol{f}^-(T) = \sum_{d>0} \boldsymbol{\lambda}_d T^{-d}$ as (finite) sum of $s-$co-monomials. Write

$$\boldsymbol{\lambda}_d T^{-d} = (0, \ldots, 0, \lambda_{d,0}T^{-n}, \ldots, \lambda_{d,m}T^{-np^m}) \in \mathbf{W}_{v_p(d)}(T^{-1}\mathcal{O}_H[T^{-1}]) \ , \tag{5.4.16.5}$$

where, for all $d > 0$, we set $d = np^m$, $m = v_p(d)$. Now set

$$\boldsymbol{\lambda}_{pd}^{(\bar{\mathrm{F}})}T^{-pd} := (0, \ldots, 0, \lambda_{d,0}^p T^{-np}, \ldots, \lambda_{d,m}^p T^{-np^{m+1}}) \ ,$$

then the reduction $\overline{\boldsymbol{\lambda}_{pd}^{(\bar{\mathrm{F}})}T^{-pd} - \boldsymbol{\lambda}_d T^{-d}}$ lies in $(\bar{\mathrm{F}} - 1)\mathbf{W}_s(k(\!(t)\!))$. Hence we can replace $\boldsymbol{\lambda}_d T^{-d}$ with $\boldsymbol{\lambda}_{pd}^{(\bar{\mathrm{F}})}T^{-pd}$. Replacing in this way $\boldsymbol{\lambda}_{np^m}T^{-np^m}$ with

$\boldsymbol{\lambda}_{np^m}^{(\bar{\mathrm{F}})} T^{-np^{m+1}}$, step by step, we obtain a pure Witt vector. In other words, we can suppose that, for all $n \in \mathrm{J}_p$, there exists a unique $m(n) \geq 0$ such that $\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}} \neq 0$. Now let us construct $\boldsymbol{h}_p^-(T)$. First we arrange the sum

$$\boldsymbol{f}^-(T) = \sum_{n \in \mathrm{J}_p} \sum_{m \geq 0} \boldsymbol{\lambda}_{np^m} T^{-np^m} . \tag{5.4.16.6}$$

Then we construct, for all $n \in \mathrm{J}_p$, a natural number $m(n) \geq 0$, and a Witt vector $\boldsymbol{\nu}_{np^{m(n)}} \in \mathbf{W}_s(\mathcal{O}_H)$, satisfying

$$\mathrm{e}_{p^s}(\boldsymbol{\nu}_{np^{m(n)}} T^{-np^{m(n)}}, 1) = \mathrm{e}_{p^s}(\sum_{m \geq 0} \boldsymbol{\lambda}_{np^m} T^{-np^m}, 1) . \tag{5.4.16.7}$$

Let $m(n) = \sup\{m \mid \boldsymbol{\lambda}_{np^m} \neq 0\}$. By 5.3.0.2, we can suppose $s \geq m(n)$. Let $\boldsymbol{\lambda}_{np^m} = (\lambda_{np^m,0}, \ldots, \lambda_{np^m,m})$, and let $\langle \phi_{np^m,0}, \ldots, \phi_{np^m,m} \rangle$ be its phantom vector. Then

$$\mathrm{e}_{p^s}(\sum_{m=0}^{m(n)} \boldsymbol{\lambda}_{np^m} T^{-np^m}, 1) = \exp\Big(\pi_{m(n)} a_0 T^{-n} + \cdots + \pi_0 a_{m(n)} \frac{T^{-np^{m(n)}}}{p^{m(n)}}\Big) , \tag{5.4.16.8}$$

where, for all $j = 0, \ldots, m(n)$, we have

$$a_j = \frac{\pi_0}{\pi_{m(n)-j}} \cdot \phi_{np^j,j} + \frac{\pi_1}{\pi_{m(n)-j}} \cdot \phi_{np^{j+1},j} + \cdots + \frac{\pi_{m(n)-j}}{\pi_{m(n)-j}} \cdot \phi_{np^{m(n)},j} . \tag{5.4.16.9}$$

Let $P(X)$ be the chosen Lubin-Tate series. Denote by $P^{(1)}(X) := P(X)$, $P^{(r)}(X) := P(P(\cdots P(X) \cdots))$, $r-$times. We set $h_0(X) := 1$, and $h_r(X) := P^{(r)}(X)/X$, for $r = 1, \ldots, m(n)$. The phantom vector of $[h_r(\pi_{m(n)})] \in \mathbf{W}_{m(n)}(\mathcal{O}_H)$ is as usual $\langle h_r(\pi_{m(n)}), h_r(\pi_{m(n)-1}), \ldots, h_r(\pi_0) \rangle$ and is then equal to

$$\langle \frac{\pi_{m(n)-r}}{\pi_{m(n)}}, \frac{\pi_{m(n)-r-1}}{\pi_{m(n)-1}}, \ldots, \frac{\pi_0}{\pi_r}, 0, \ldots, 0 \rangle \in \mathcal{O}_H^{m(n)+1} \text{ , if } r > 0 , \tag{5.4.16.10}$$

while $[h_0(\pi_{m(n)})] = 1$, and its phantom vector is $\langle 1, \ldots, 1 \rangle$. Hence we have

$$a_j = h_{m(n)}(\pi_{m(n)-j}) \phi^*_{n,j} + h_{m(n)-1}(\pi_{m(n)-j}) \phi^*_{np,j} + \cdots + h_0(\pi_{m(n)-j}) \phi^*_{np^{m(n)},j} ,$$

where, for all $k = 0, \ldots, m(n)$, $\langle \phi^*_{np^k,0}, \ldots, \phi^*_{np^k,m(n)} \rangle$ is the phantom vector of $\boldsymbol{\lambda}^*_{np^k} := (\lambda_{np^k,0}, \ldots, \lambda_{np^k,k}, *, \ldots, *) \in \mathbf{W}_{m(n)}(\mathcal{O}_H)$, where the last $m(n)-k$ components are arbitrarily chosen. Observe that $\phi^*_{np^k,j} = \phi_{np^k,j}$, for all $j = 0, \ldots, k$, while, if $j > k$ we have $h_{m(n)-k}(\pi_{m(n)-j}) = 0$. This shows that

$$\boldsymbol{\nu}_{np^{m(n)}} := [h_{m(n)}(\pi_{m(n)})] \boldsymbol{\lambda}^*_n + [h_{m(n)-1}(\pi_{m(n)})] \boldsymbol{\lambda}^*_{np} + \cdots + [h_0(\pi_{m(n)})] \boldsymbol{\lambda}^*_{np^{m(n)}} . \square$$

5.4.2 Proof of (1) and (2)

The assertions (1) and (2) of theorem 5.2.2 will be a direct consequence of the following theorem. The algorithm employed is due to Robba [Rob85, 10.10] (see also [CR94, 13.3]). We translate his techniques in terms of Witt vectors. Recall that, by 2.4.11, every rank one solvable equation has a basis in which the matrix is a polynomial in T^{-1} with coefficients in $\mathcal{O}_K$.

Theorem 5.4.17. *Let H/K be a finite extension. Let M be a solvable rank one differential module over $\mathcal{R}_H$, defined by an operator $\partial_T - g(T)$, $g(T) = \sum_{-d\leq i\leq -1} a_i T^i \in \mathcal{O}_H[T^{-1}]$. Then there exists a Witt vector $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{H'}[T^{-1}])$, whose coefficients lies in a finite extension H'/H, such that $\partial_T - g(T) = \mathrm{L}(0, \boldsymbol{f}^-(T))$. More explicitly we have (cf. 5.2.2.2)*

$$\sum_{-d\leq i\leq -1} a_i T^i = -\sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T)) \,. \qquad (5.4.17.1)$$

In particular

$$\exp\Big(\sum_{-d\leq i\leq -1} a_i T^i/i\Big) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) \,. \qquad (5.4.17.2)$$

Proof : We shall express $\exp(\sum_{-d\leq i\leq -1} a_i T^i/i)$ as a product of elementary $\boldsymbol{\pi}-$exponentials, with coefficients in H^{alg}. Observe that solvability does not changes by scalar extension of H. Let $d = np^m$, $(n,p) = 1$, and let $b_d \in H^{\mathrm{alg}}$ be such that $b_d^{p^m} = a_{-d}/(n\pi_0)$. By lemma 2.4.3, $|a_{-d}| \leq \omega < 1$, hence $|b_d| \leq 1$. We consider the Witt vector $\boldsymbol{\lambda}_d := (b_d, 0, \ldots, 0) \in \mathbf{W}_m(\mathcal{O}_{H^{\mathrm{alg}}})$, whose phantom vector is $\langle b_d, b_d^p, \ldots, b_d^{p^m}\rangle$. By construction, we have

$$L_d(\boldsymbol{\lambda}_d) = \partial_T + n\cdot(\pi_0 b_d^{p^m} T^{-d} + \pi_1 b_d^{p^{m-1}} T^{-d/p} + \cdots + \pi_m b_d T^{-n}). \quad (5.4.17.3)$$

Then $M \otimes \mathrm{M}_d(b_d, 0, \ldots, 0)$ is defined by an operator of the form $\partial_T - \sum_{-d+1\leq i\leq -1} \tilde{a}_i T^i$, $\exists\ \tilde{a}_i \in H^{\mathrm{alg}}$ (cf. 2.2.3). Moreover $M \otimes \mathrm{M}_d(b_d, 0, \ldots, 0)$ is again solvable, then, by 2.4.3, we have again $|\tilde{a}_{-d+1}| \leq \omega$. This show that we can iterate this process. More precisely there exist $\boldsymbol{\lambda}_i = (b_i, 0, \ldots, 0) \in \mathbf{W}_{v_p(i)}(\mathcal{O}_{H^{\mathrm{alg}}})$, $i = 1, \ldots, d$, such that the product

$$\epsilon(T) := \prod_{i=1,\ldots,d} \mathrm{e}_i(\boldsymbol{\lambda}_i, T^{-1}) = \mathrm{e}_{p^s}\Big(\sum_{i=1}^{d} \boldsymbol{\lambda}_i T^{-i}, 1\Big) \,, \quad s \gg 0, \qquad (5.4.17.4)$$

satisfy $\partial_{T,\log}(\epsilon(T)) = \sum_{-d\leq i\leq -1} a_i T^i$. Then $\boldsymbol{f}^-(T) := \sum_{1\leq i\leq d} \boldsymbol{\lambda}_i T^{-i}$ (cf. 5.4.11.1).$\square$

Chapitre 6

Applications

6.1 Description of the character group

Lemma 6.1.1. *For all fields κ of characteristic p, we the followings isomorphisms of additive groups (cf. 2.7.5) :*

$$\mathbf{CW}(t^{-1}\kappa[t^{-1}]) \cong \oplus_{d>0}\mathbf{W}_{v_p(d)}(\kappa) \qquad ; \qquad \mathbf{P}(\kappa) \cong \widetilde{\mathbf{CW}}(\kappa)^{(\mathrm{J}_p)} \ , \quad (6.1.1.1)$$

where $\mathrm{J}_p := \{n \in \mathbb{Z} \mid (n,p) = 1, n > 0\}$.

Proof : We have

$$\mathbf{CW}(t^{-1}\kappa[t^{-1}]) = \varinjlim_s \mathbf{W}_s(t^{-1}\kappa[t^{-1}]) \overset{5.4.5}{=} \varinjlim_s \oplus_{d>0}\mathbf{W}_s^{(-d)}(\kappa) \ . \quad (6.1.1.2)$$

Observe that $\mathbf{W}_s^{(-d)}(\kappa) = \mathbf{W}_{\min(s,v_p(d))}(\kappa)$ (cf. remark 5.4.2), hence

$$\mathbf{CW}(t^{-1}\kappa[t^{-1}]) = \oplus_{d>0} \varinjlim_s \mathbf{W}_{\min(s,v_p(d))}(\kappa) = \oplus_{d>0}\mathbf{W}_{v_p(d)}(\kappa) \ . \quad (6.1.1.3)$$

Now we write $d = np^m$, $n \in \mathrm{J}_p = \{n \in \mathbb{Z} \mid (n,p) = 1, n > 0\}$ and $m \geq 0$, then on the right hand side we have

$$\oplus_{d>0}\mathbf{W}_{v_p(d)}(\kappa) = \oplus_{n\in\mathrm{J}_p}(\oplus_m \mathbf{W}_{v_p(np^m)}(\kappa)) \ . \quad (6.1.1.4)$$

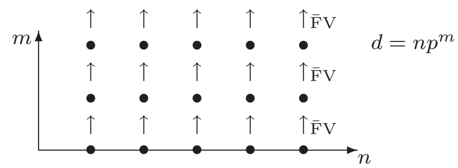
The Frobenius morphism $\bar{\mathrm{F}}$ sends $\mathbf{W}_s^{(-d)}(\kappa)$ into $\mathbf{W}_s^{(-pd)}(\kappa)$, and, under the isomorphism

$$\mathbf{W}_s^{(-d)}(\kappa) \xrightarrow{\sim} \mathbf{W}_{\min(s,v_p(d))}(\kappa) \quad (6.1.1.5)$$

(cf. remark 5.4.2), it becomes the morphism

$$\bar{\mathrm{F}}\mathrm{V} : \mathbf{W}_{v_p(np^m)}(\kappa) \to \mathbf{W}_{v_p(np^{m+1})}(\kappa) \quad (6.1.1.6)$$

as illustrated in the picture



Then

$$\begin{aligned}\mathbf{P}(\kappa) &\cong \oplus_{n\in \mathrm{J}_p}\left(\oplus_{m\geq 0}\mathbf{W}_{v_p(np^m)}(\kappa)/(\bar{\mathrm{F}}\mathrm{V}-1)(\oplus_{m\geq 0}\mathbf{W}_{v_p(np^m)}(\kappa))\right)\\ &\cong \left(\oplus_{m\geq 0}\mathbf{W}_m(\kappa)/(\bar{\mathrm{F}}\mathrm{V}-1)(\oplus_{m\geq 0}\mathbf{W}_m(\kappa))\right)^{(\mathrm{J}_p)} \ .\end{aligned}$$

One sees that $\oplus_{m\geq 0}\mathbf{W}_m(\kappa)/(\bar{\mathrm{F}}\mathrm{V}-1)(\oplus_{m\geq 0}\mathbf{W}_m(\kappa))$ is isomorphic to $\widetilde{\mathbf{CW}}(\kappa) = \varinjlim(\mathbf{W}_m(\kappa)\xrightarrow{\bar{\mathrm{F}}\mathrm{V}}\mathbf{W}_{m+1}(\kappa)\xrightarrow{\bar{\mathrm{F}}\mathrm{V}}\cdots)$. $\square$

6.2 Equations killed by an abelian extension

6.2.1 Extension of the field of constants

Corollary 6.2.1. *The natural morphism*

$$M\mapsto M\otimes K^{\mathrm{alg}} : \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)\to \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K^{\mathrm{alg}}})$$

is a monomorphism. In other words, two $\mathcal{R}_K-$differential modules are isomorphic if and only if they are isomorphic over $\mathcal{R}_{K^{\mathrm{alg}}}$ after scalar extension.

Proof : We show that the kernel of $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)\to \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K^{\mathrm{alg}}})$ is equal to 0. Let M be defined by the operator $L=\partial_T - g(T)$, $g(T):=\sum_i a_iT^i\in\mathcal{R}_K$, and suppose that $M\otimes K^{\mathrm{alg}}$ is trivial over $\mathcal{R}_{K^{\mathrm{alg}}}$. By 2.4.11, we can suppose $a_i=0$, for all $i\neq -d,\ldots,0$. We know that

$$M\otimes K^{\mathrm{alg}}\xrightarrow{\sim}\mathrm{M}(a_0,\boldsymbol{f}^-(T)) = \mathrm{M}(a_0,0)\otimes\mathrm{M}(0,\boldsymbol{f}^-(T))\ , \qquad (6.2.1.1)$$

for a convenable $\boldsymbol{f}^-(T)\in\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$. Then $M\otimes K^{\mathrm{alg}}$ is trivial only if both $\mathrm{M}(a_0,0)$ and $\mathrm{M}(0,\boldsymbol{f}^-(T))$ are trivial over K^{alg}. This implies that $a_0\in\mathbb{Z}$, and hence $\mathrm{M}(a_0,0)$ is trivial also over $\mathcal{R}_K$.

On the other hand, $\mathrm{M}(0,\boldsymbol{f}^-(T))$ is trivial if and only if $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ lies in $\mathcal{R}_{K^{\mathrm{alg}}}$. By 5.4.17, the series $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ has its coefficients in K, and $\mathrm{M}(0,\boldsymbol{f}^-(T))\in\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)$. Since the convergence does not change by scalar extension of the field K, hence $\mathrm{M}(0,\boldsymbol{f}^-(T))$ is trivial over $\mathcal{R}_K$. $\square$

Corollary 6.2.2. *We have*

$$\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)=\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})^{\mathrm{Gal}(K_\infty/K)}\ . \quad \square$$

6.2.2 Frobenius structure

Assume now that K has an absolute Frobenius $\sigma : K \to K$ (cf.2.5.1), and fix an absolute Frobenius $\varphi : \mathcal{R}_K \to \mathcal{R}_K$. By theorem 5.2.4−(5), for any Artin-Schreier character α, the module $\mathrm{M}(0, \alpha)$ has a Frobenius structure of order 1 over K_∞ (with respect to one, and hence all absolute Frobenius, cf.2.5.4). By lemma 6.2.1, this isomorphism descends on K.

Remark 6.2.3. Let $L = \partial_T + \sum_{i\in\mathbb{Z}} a_i T^i$, be an operator over $\mathcal{R}_K$ with Frobenius structure. The order h of the Frobenius structure depends only on the exponent $a_0 \in \mathbb{Z}_{(p)}$. Explicitly, if $a_0 = a/b$, $a, b \in \mathbb{Z}$, $(b, p) = 1$, and if $b = \prod_i q_i^{r_i} > 0$, $q_i > 0$, is a factorization of b in prime numbers, then, by 2.6.4, we have (cf. lemma 2.6.4)

$$h \leq \prod_i ([q_i]_{r_i} - 1). \tag{6.2.3.1}$$

Definition 6.2.4. Let us denote by $\mathrm{Pic}^{\mathrm{Frob}}(\mathcal{R}_{K_\infty}) \subseteq \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})$ the subgroup of differential modules having a Frobenius structure of some order h.

Corollary 6.2.5. $\mathrm{Pic}^{\mathrm{Frob}}(\mathcal{R}_{K_\infty}) \cong \mathbb{Z}_{(p)}/\mathbb{Z} \oplus \mathbf{P}(k_\infty)$. $\square$

6.2.3 Artin-Schreier extensions

In order to apply the theorems 4.2.3, and 6.2.6, in this section K will be discrete valued, and k will be perfect.

Proposition 6.2.6 ([Mat95, 3.4], [Tsu98a, 2.2.2])**.** *Let* $\mathrm{F}/k(\!(t)\!)$ *be a finite separable extension, and let* $\mathcal{F}^\dagger$ *be the corresponding unramified extension of* $\mathcal{E}^\dagger_{K,T}$. *We have the following statements.*

1. *There exist a finite unramified extension* $\widetilde{K}/K$, *a new variable* $\widetilde{T}$ *and an isometric isomorphism*

$$\tau : (\mathcal{F}^\dagger, |.|) \xrightarrow{\sim} (\mathcal{E}^\dagger_{\widetilde{K},\widetilde{T}}, |.|_{\widetilde{T},1}) \ , \tag{6.2.6.1}$$

where $|.|_{\widetilde{T},1}$ *is the Gauss norm with respect to* $\widetilde{T}$. *In particular, for all* $f(T) \in \mathcal{E}^\dagger_{K,T}$, *we have*

$$|f(T)|_{T,1} = |f(T)|_{\widetilde{T},1} \ . \tag{6.2.6.2}$$

2. *Let* $\tilde{t}$ *and* t *be the reductions of* $\widetilde{T}$ *and* T *respectively. Let* $F = \tilde{k}(\!(\tilde{t})\!)$. *Let* r *be the ramification index of* $\mathrm{F}/k(\!(t)\!)$. *Write*

$$t = \bar{a}_r \tilde{t}^r + \bar{a}_{r+1}\tilde{t}^{r+1} + \cdots \ , \tag{6.2.6.3}$$

with $\bar{a}_i \in \tilde{k}$. *Then* $\widetilde{T}$ *can be chosen such that*

$$\tau(T) = a_r \widetilde{T}^r + a_{r+1}\widetilde{T}^{r+1} + \cdots \ , \quad a_i \in \mathcal{O}_{\widetilde{K}} \ , \tag{6.2.6.4}$$

where the a_i*'s are lifting in* $\mathcal{O}_{\widetilde{K}}$ *of the* $\bar{a}_i$*'s.*

Proof : Let $Q(\widetilde{T}) := a_r\widetilde{T}^r + a_{r+1}\widetilde{T}^{r+1} + \cdots$. The proof consists in showing that $f(T) \mapsto \tau(f(T)) := f(Q(\widetilde{T})) : \mathcal{E}^{\dagger}_{K,T} \to \mathcal{E}^{\dagger}_{\widetilde{K},\widetilde{T}}$ is étale (cf. [Mat95, 3.4]).□

Notation 6.2.7. We denote by

$$\mathcal{R}_{\widetilde{K},\widetilde{T}} \tag{6.2.7.1}$$

the corresponding Robba ring.

Remark 6.2.8. We have

$$(\partial_{\widetilde{T}} \circ \tau)(f(T)) = \partial_{\widetilde{T},\log}(Q(\widetilde{T})) \cdot (\tau \circ \partial_T)(f(T)) \;, \tag{6.2.8.1}$$

where as usual $\partial_{\widetilde{T},\log}(Q(\widetilde{T})) = \frac{\partial_{\widetilde{T}}(Q(\widetilde{T}))}{Q(\widetilde{T})}$. Then, after scalar extension, a generic differential operator $\partial_T - g(T)$ becomes

$$\partial_{\widetilde{T}} - \partial_{\widetilde{T},\log}(Q(\widetilde{T})) \cdot g(Q(\widetilde{T})) \;. \tag{6.2.8.2}$$

Indeed the unique K_∞ derivation of the étale extension $\mathcal{R}_{K_\infty,\widetilde{T}}$ extending ∂_T is $\partial_{\widetilde{T},\log}(Q(\widetilde{T}))^{-1} \cdot \partial_{\widetilde{T}}$. The solutions of this operator are the same as $\partial_T - g(T)$.

Corollary 6.2.9. *Let* $\mathrm{E} = k(\!(t)\!)$. *Let* F/E *be the Artin-Schreier extension defined by the kernel of* $\alpha = \delta(\overline{\boldsymbol{f}}(t))$, *with* $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_s(\mathrm{E})$, *for some* $s \geq 0$. *Let* $\mathcal{R}_{K,T} \to \mathcal{R}_{\widetilde{K},\widetilde{T}}$ *be the corresponding étale extension. Then the kernel of the scalar extension map*

$$\mathrm{Res} : \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,T}) \longrightarrow \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{\widetilde{K}_\infty,\widetilde{T}}) \tag{6.2.9.1}$$

is the (finite and cyclic) sub-group of $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,T})$, *formed by (isomorphism classes of) modules of the type (cf.5.4.8)*

$$\mathrm{Ker(Res)} = \{\, \mathrm{M}(0, \boldsymbol{f}^-(T))^{\otimes k} \;, k \geq 0 \,\} \;. \tag{6.2.9.2}$$

where $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[[T]][T^{-1}])$ *is an arbitrary lifting of* $\overline{\boldsymbol{f}}(t)$. *This kernel has order* $[\mathrm{F} : \mathrm{E}]$.

Proof : By 6.2.1, we can suppose $K = K^{\mathrm{alg}}$. We decompose $\overline{\boldsymbol{f}}(t) = \overline{\boldsymbol{f}^-}(t) + \overline{\boldsymbol{f}}_0 + \overline{\boldsymbol{f}^+}(t)$ (cf. 5.4.8). Since $k = \tilde{k}$, we have $\delta(\overline{\boldsymbol{f}}_0) = 0$ (cf. 2.8.1.2). On the other hand, by 5.4.10, we have always $\delta(\overline{\boldsymbol{f}^+}(t)) = 0$. Hence we can suppose $\overline{\boldsymbol{f}}(t) = \overline{\boldsymbol{f}^-}(t) = (\overline{f_0^-}(t), \ldots, \overline{f_s^-}(t))$. Since the Artin-Schreier complex is invariant by V (cf. 2.8.1.2), we can suppose $\overline{f_0^-}(t) \neq 0$ (i.e. the degree $[\mathrm{F} : \mathrm{E}]$ is p^{s+1}). By corollary 5.3.2, the morphism 6.2.9.1 can be seen as a map

$$\mathbb{Z}_p/\mathbb{Z} \oplus \frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}} - 1)\mathbf{CW}(t^{-1}k[t^{-1}])} \xrightarrow{\mathrm{Res}} \mathbb{Z}_p/\mathbb{Z} \oplus \frac{\mathbf{CW}(\tilde{t}^{-1}k[\tilde{t}^{-1}])}{(\bar{\mathrm{F}} - 1)\mathbf{CW}(\tilde{t}^{-1}k[\tilde{t}^{-1}])} \;, \tag{6.2.9.3}$$

where $\tilde{t}$ is the reduction of $\widetilde{T}$. We start by studying the term $\mathbb{Z}_p/\mathbb{Z}$. By 6.2.6.4, $T = Q(\widetilde{T})$, with $Q(\widetilde{T}) = a_{p^{s+1}}\widetilde{T}^{p^{s+1}} + \cdots$, with $a_i \in \mathcal{O}_{K_\infty}$. The differential operator $\partial_T - a_0$, $a_0 \in \mathbb{Z}_p$ is sent in $\partial_{\widetilde{T}} - \partial_{\widetilde{T},\log}(Q(\widetilde{T})) \cdot a_0$. Observe that

$$\partial_{\widetilde{T},\log}(Q(T)) = p^{s+1} + Q_1(\widetilde{T}) \quad , \qquad Q_1(\widetilde{T}) \in \widetilde{T} \cdot \mathcal{O}_K[[\widetilde{T}]] \, , \tag{6.2.9.4}$$

hence the new operator is $\partial_{\widetilde{T}} - p^{s+1} \cdot a_0 - Q_1(\widetilde{T}) \cdot a_0$. By 2.4.10, this operator is isomorphic to $\partial_{\widetilde{T}} - p^{s+1}a_0$. Then the morphism 6.2.9.3 sends $\mathbb{Z}_p/\mathbb{Z}$ into itself by multiplication by $p^{s+1} = [\mathrm{F} : \mathrm{E}]$, and is then bijective on $\mathbb{Z}_p/\mathbb{Z}$.

On the co-vectors quotient, the morphism 6.2.9.3 is the usual functorial map corresponding to the inclusion $t^{-1}k[t^{-1}] \longrightarrow \tilde{t}^{-1}k[\tilde{t}^{-1}]$. The module $\mathrm{M}(0, \boldsymbol{f}(T)) \xrightarrow{\sim} \mathrm{M}(0, \boldsymbol{f}^-(T))$ lies then in the kernel. Indeed, by definition of F/E, there exists $\boldsymbol{\nu}(\tilde{t}) \in \mathbf{W}_s(\tilde{t}^{-1}k[\tilde{t}^{-1}])$ such that (cf. remark 2.8.3)

$$\bar{\mathrm{F}}(\overline{\boldsymbol{\nu}}(\tilde{t})) - \overline{\boldsymbol{\nu}}(\tilde{t}) = \overline{\boldsymbol{f}^-}(t), \tag{6.2.9.5}$$

hence, by theorem 5.2.4, $\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)$ lies in $\mathcal{R}_{K,\widetilde{T}}$. In other words, this exponential is over-convergent in the new variable $\widetilde{T}$. Conversely, a module $\mathrm{M}(0, \boldsymbol{g}^-(T))$ lies in the kernel, if and only if the exponential $\mathrm{e}_{p^s}(\boldsymbol{g}^-(T), 1)$ belongs to $\mathcal{R}_{K,\widetilde{T}}$. By theorem 5.2.4, this happens if and only if the equation $\bar{\mathrm{F}}(\boldsymbol{\nu}) - \boldsymbol{\nu} = \overline{\boldsymbol{g}^-}(t)$ has a solution $\boldsymbol{\nu} \in \mathbf{W}_s(k((\tilde{t})))$. This happens if and only if the kernel of $\delta(\overline{\boldsymbol{g}^-}(t))$ contains the kernel of $\alpha = \delta(\overline{\boldsymbol{f}^-}(t))$. Since the quotient $\mathrm{G}_\mathrm{E}/\mathrm{Ker}(\delta(\overline{\boldsymbol{f}^-}(t)))$ is cyclic, this implies that $\delta(\overline{\boldsymbol{g}^-}(t)) = m \cdot \delta(\overline{\boldsymbol{f}^-}(t))$, for some $m \geq 0$. Hence $\mathrm{M}(0, \boldsymbol{g}^-(T)) \xrightarrow{\sim} \mathrm{M}(0, \boldsymbol{f}^-(T))^{\otimes m}$. $\square$

6.2.4 Kummer extensions

Corollary 6.2.10. *Let* F/E *be an abelian totally ramified extension of degree* $[\mathrm{F} : \mathrm{E}] = n$, *with* $(n, p) = 1$. *Let* $\mathcal{R}_{K,T} \mapsto \mathcal{R}_{K,\widetilde{T}}$ *be the corresponding étale extension. Then the scalar extension morphism*

$$\mathrm{Res} : \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,T}) \longrightarrow \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,\widetilde{T}}) \tag{6.2.10.1}$$

is the multiplication by n, *and its kernel is then* $(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$.

Proof : Indeed, in this case we can choose $\tilde{t}$ satisfying $t = Q(\tilde{t}) = \tilde{t}^n$.$\square$

6.3 A criterion of solvability

This sub-section is devoted to prove the corollary 6.3.15. The aim of this result is to characterize the solvability of the differential equation $\partial_T - g(T)$, with $g(T) = \sum a_i T^i$ giving an explicit condition on the coefficients "a_i".

Roughly this theorem shows that every solvable differential equation over $\mathcal{E}_K$ has, without change of basis, a solution of the form

$$T^{a_0}\cdot E(\boldsymbol{f}^-(T),1)\cdot E(\boldsymbol{f}^+(T),1)\ , \tag{6.3.0.2}$$

where $\boldsymbol{f}^-(T)\in\mathbf{W}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ and $\boldsymbol{f}^-(T)\in\mathbf{W}(T\mathcal{O}_K[[T]])$ is a certain (*infinite*) Witt vector, satisfying some properties of convergence, in order that the series $E(\boldsymbol{f}^-(T),1)$ makes sense (cf. 6.3.1). Similarly to the precedent situation, this Witt vector will be a sum of *monomials* (dual notion of $s-$co-monomial, cf. 6.3.4). If a Lubin-Tate group $\mathfrak{G}_P$ is chosen, then this classification is a generalization of theorem 5.2.2, because $\mathbf{W}(T^{-1}\mathcal{O}_{K_\infty}[[T^{-1}]])$ contains $\mathbf{CW}(T^{-1}\mathcal{O}_{K_\infty}[[T^{-1}]])$, via the choice of a generator $\boldsymbol{\pi}\in\mathrm{T}(\mathfrak{G}_P)$ (cf. diagram 6.3.7.1), and the exponential $E(\boldsymbol{f}^-(T),1)$ becomes $\mathrm{e}_{p^s}(-,1)$ if applied to the image of a co-vector (cf. 6.3.6.5).

We preserve the notations of section 3.1. In the sequel we will work both with $T\mathcal{O}_K[[T]]$ and $T^{-1}\mathcal{O}_K[[T^{-1}]]$, almost all assertion has a dual meaning.

Lemma 6.3.1. *Let*

$$E(-,Y):\mathbf{W}(\mathcal{O}_K[[T]])\to 1+Y\mathcal{O}_K[[T]][[Y]] \tag{6.3.1.1}$$

be the Artin Hasse exponential (cf. 3.2.1). Let v_T be the $T-$adic valuation. Let $\boldsymbol{f}(T)=(f_0(T),f_1(T),\ldots)\in\mathbf{W}(T\mathcal{O}_K[[T]])$, and let $\phi_j(T)$ be its $j-$th phantom component. If

$$\lim_{j\to\infty}v_T(f_j(T))=+\infty\ , \tag{6.3.1.2}$$

then

$$\lim_{j\to\infty}v_T(\phi_j(T))=+\infty\ , \tag{6.3.1.3}$$

and then $E(\boldsymbol{f}(T),Y)$ converges T-adically at $Y=1$. $\square$

Definition 6.3.2. We denote by

$$\mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]]) \tag{6.3.2.1}$$

the ideal of $\mathbf{W}(\mathcal{O}_K[[T]])$ satisfying the condition of lemma 6.3.1.

Remark 6.3.3. For all $\boldsymbol{f}^+(T)=(f_0(T),f_1(T),\ldots)\in\mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$, we have

$$E(\boldsymbol{f}^+(T),1):=\prod_{j\geq 0}E(f_j(T))=\exp(\phi_0(T)+\frac{\phi_1(T)}{p}+\frac{\phi_2(T)}{p^2}+\cdots)\ , \tag{6.3.3.1}$$

where $\phi_j^+(T)$ is the $j-$th phantom component of $\boldsymbol{f}^+(T)$. The $T-$adic convergence of this product is guaranteed by lemma 6.3.1.

Definition 6.3.4 (Monomials)**.** Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathcal{O}_K)$, and let $d \geq 1$ be a positive integer. We will call

$$\boldsymbol{\lambda} T^d := (\ \lambda_0 T^d\ ,\ \lambda_1 T^{dp}\ ,\ \lambda_2 T^{dp^2}\ ,\ \ldots) \in \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]]) \qquad (6.3.4.1)$$

the *monomial* of degree d relative to the Witt vector $\boldsymbol{\lambda}$.[1] In analogy with 5.4.1, we call

$$\mathbf{W}^{(d)}(\mathcal{O}_K) \qquad (6.3.4.2)$$

the sub-group of $\mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$, formed by monomials of degree d.

Lemma 6.3.5. *Let* $\mathrm{J}_p := \{n \in \mathbb{Z} \mid (n,p) = 1, n > 0\}$. *We have an injection*

$$\prod_{n \in \mathrm{J}_p} \mathbf{W}^{(n)}(\mathcal{O}_K) \subset \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])\ , \qquad (6.3.5.1)$$

given by $(\boldsymbol{\lambda}_n T^n)_{n \in \mathrm{J}_p} \mapsto \sum_{n \in \mathrm{J}_p} \boldsymbol{\lambda}_n T^n$.

Proof : If $\phi_n = (\phi_{n,0}, \phi_{n,1}, \ldots)$ is the phantom vector of $\boldsymbol{\lambda}_n$, then the phantom vector of $\boldsymbol{\lambda}_n T^n$ is $(\phi_{n,0}T^n, \phi_{n,1}T^{np}, \phi_{n,2}T^{np^2}, \cdots)$. Hence all terms have different degree and they not "blend" when we sum the phantom components.$\square$

Remark 6.3.6. **1.** Let $\boldsymbol{f}(T) \in \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$ (resp. $\boldsymbol{f}(T) \in \mathbf{W}^{\downarrow}(\widetilde{T}^{-1}\mathcal{O}_K[[T^{-1}]])$), let $\boldsymbol{\lambda}, \boldsymbol{\lambda}_d \in \mathbf{W}(\mathcal{O}_K)$, $d > 0$. Then we have

$$\begin{aligned} E(\mathrm{V}(\boldsymbol{f}(T)), 1) &= E(\boldsymbol{f}(T), 1)\ , && (6.3.6.1)\\ E(\boldsymbol{\lambda}, T^d) &= E(\boldsymbol{\lambda} T^d, 1)\ , && (6.3.6.2)\\ \prod_{d \geq 1} E(\boldsymbol{\lambda}_d, T^d) &= E\Big(\sum_{d \geq 1} \boldsymbol{\lambda}_d T^d, 1\Big)\ . && (6.3.6.3) \end{aligned}$$

2. If $\phi_n = (\phi_{n,0}, \phi_{n,1}, \cdots)$ is the phantom vector of $\boldsymbol{\lambda}_{-n}$, then we have

$$E(\sum_{n \in \mathrm{J}_p} \boldsymbol{\lambda}_{-n} T^{-n}, 1) = \exp(\sum_{n \in \mathrm{J}_p} \sum_{m \geq 0} \phi_{-n,m} \frac{T^{-np^m}}{p^m}) \qquad (6.3.6.4)$$

3. If $\boldsymbol{f}^-(T) = (f_0^-(T), f_1^-(T), \ldots) \in \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ and if $\mathrm{pr}_m(\boldsymbol{f}^-(T))$ is the image of $\boldsymbol{f}^-(T)$ in $\mathbf{W}_m(T^{-1}\mathcal{O}_K[[T^{-1}]])$, then (cf. 3.3.1.1)

$$E([\pi_m] \cdot \boldsymbol{f}^-(T), 1) = \mathrm{e}_{p^m}\big(\mathrm{pr}_m(\boldsymbol{f}^-(T)), 1\big) = \prod_{j \geq 0}^{m} E_{m-j}(f_j^-(T)). \qquad (6.3.6.5)$$

The exponentials used in the precedent section are then a particular case of $E(-,1)$.

[1]Observe that if $\boldsymbol{\lambda} T^{-d}$ is a monomial in $\mathbf{W}(\mathcal{O}_K[[T^{-1}]])$, its reduction in $\mathbf{W}_m(\mathcal{O}_K[[T^{-1}]])$ is NOT a co-monomial of degree $-d$, but it is a co-monomial of degree $-dp^m$.

Remark 6.3.7. Recall that $\mathbf{W}_m(T^{-1}\mathcal{O}_{K_m}[T^{-1}]) \xrightarrow{\sim} [\pi_m]\mathbf{W}(T^{-1}\mathcal{O}_{K_m}[T^{-1}]) \subset \mathbf{W}(T^{-1}\mathcal{O}_{K_m}[[T^{-1}]])$ (see 3.1.8.3). We have the following commutative diagram

$$\begin{tikzcd}
\mathbf{W}_{m+1}(T^{-1}\mathcal{O}_{K_{m+1}}[T^{-1}]) \arrow[r, hook, "{[\pi_{m+1}]\cdot}"'] & \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_{K_{m+1}}[[T^{-1}]]) \arrow[r, "{E(-,1)}"] & 1+T^{-1}\mathcal{O}_{K_\infty}[[T^{-1}]] \\
\mathbf{W}_m(T^{-1}\mathcal{O}_{K_m}[T^{-1}]) \arrow[r, hook, "{[\pi_m]\cdot}"'] \arrow[u, "\mathrm{V}"] & \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_{K_m}[[T^{-1}]]) \arrow[u, "\mathrm{V}"] \arrow[ur, "{E(-,1)}"'] &
\end{tikzcd} . \tag{6.3.7.1}$$

Indeed, we see, looking at the phantom components, that

$$[\pi_m](f_0^-, \ldots, f_m^-, f_{m+1}^-, \cdots) = [\pi_m](f_0^-, \ldots, f_m^-, 0, 0, \cdots) \;, \tag{6.3.7.2}$$

for all $\boldsymbol{f}^-(T) = (f_0^-, \ldots, f_m^-, f_{m+1}^-, \cdots) \in \mathbf{W}(T^{-1}\mathcal{O}_K[T^{-1}])$. Hence $[\pi_m]\boldsymbol{f}^-(T)$ lies in $\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[T^{-1}])$, for all $\boldsymbol{f}^-(T) \in \mathbf{W}(T^{-1}\mathcal{O}_K[T^{-1}])$.

Remark 6.3.8. By definition one has

$$\mathbf{W}^{(d)}(\mathcal{O}_K) \subset \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]]) \;, \tag{6.3.8.1}$$

for all $d \geq 1$. The group $\mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$ is not generated by the family

$$\{\mathbf{W}^{(d)}(\mathcal{O}_K)\}_{d\geq 0} \tag{6.3.8.2}$$

of sub-groups. Indeed, for example, the m-th phantom component $\phi_m(T)$ of a Witt vector of the form $\sum_{d>0} \boldsymbol{\lambda}_d T^d$ is always of the type $\phi_m(T) = h(T^{p^m})$, for some $h(T) \in \mathcal{O}_K[[T]]$.

However, the basic fact is that, for all $\boldsymbol{f}^+(T) \in \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$ (resp. $\boldsymbol{f}^-(T) \in \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$), there exists an (infinite) family of monomials

$$\{\boldsymbol{\lambda}_n T^n\}_{n\in \mathrm{J}_p} \in \prod_{n\in\mathrm{J}_p} \mathbf{W}^{(n)}(\mathcal{O}_K) \tag{6.3.8.3}$$

(resp. $\{\boldsymbol{\lambda}_{-n} T^{-n}\}_{n\in\mathrm{J}_p} \in \prod_{n\in\mathrm{J}_p} \mathbf{W}^{(-n)}(\mathcal{O}_K)$) satisfying

$$E(\boldsymbol{f}^+(T),1) = E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1) \quad ; \quad E(\boldsymbol{f}^-(T),1) = E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-n} T^{-n}, 1).$$

In other words, a general Witt vector is not an infinite sum of monomials, but the Artin-Hasse exponential of this Witt vector is always equal to the Artin-Hasse exponential of an infinite sum of monomials with support in J_p.

Lemma 6.3.9. *The differential equation $\partial_T - g^+(T)$, $g^+(T) \in \mathcal{R}_K$, with*

$$g^+(T) = \sum_{i\geq 1} a_i T^i \tag{6.3.9.1}$$

is solvable if and only if there exists a family $\{\boldsymbol{\lambda}_n\}_{n\in \mathrm{J}_p}$, $\boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$, *with phantom components* $\phi_n = (\phi_{n,0}, \phi_{n,1}, \ldots)$ *satisfying*

$$a_{np^m} = n\phi_{n,m} \ , \qquad \textit{for all } n \in \mathrm{J}_p,\ m \geq 0 \ . \tag{6.3.9.2}$$

In other words we have

$$\exp(\sum_{i\geq 1} a_i \frac{T^i}{i}) = E(\sum_{n\in \mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1) \ . \tag{6.3.9.3}$$

Proof : The formal series $E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1) \in 1+T\mathcal{O}_K[[T]]$ is a solution of the equation

$$L := \partial_T - \sum_{n\in \mathrm{J}_p}\sum_{m\geq 0} n\cdot \phi_{n,m}\cdot T^{np^m} \ . \tag{6.3.9.4}$$

Since this exponential converge in the unit disk, then $Ray(L,\rho) = \rho$, for all $\rho < 1$ and L is solvable. Conversely if $\partial_T - g^+(T)$ is solvable, then the Witt vectors $\boldsymbol{\lambda}_n = (\lambda_{n,0}, \lambda_{n,1}, \ldots)$ is defined by the relation A.1.8.1 (cf. 2.7.1). For example for all $n \in \mathrm{J}_p$ we have

$$\lambda_{n,0} = \frac{a_n}{n} \quad , \qquad \lambda_{n,1} = \frac{1}{p}\Big(\frac{a_{np}}{n} - (\frac{a_n}{n})^p\Big) \ . \tag{6.3.9.5}$$

We must show that $|\lambda_{n,m}| \leq 1$ for all $n \in \mathrm{J}_p$, $m \geq 0$.

Step 1 : By the small radius lemma 2.3.3, we have $|a_i| \leq 1$, for all $i \geq 1$. Hence, for all $n \in \mathrm{J}_p$, we have $|\lambda_{n,0}| \leq 1$. Then the exponential

$$E(\sum_{n\in\mathrm{J}_p}(\lambda_{n,0}, 0, 0, \ldots)T^n, 1) = \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}\frac{T^{p^m}}{p^m})$$

converge in the unit disk and is solution of the operator $Q^{(0)} := \partial_T - h^{(0)}(T)$, with

$$h^{(0)}(T) = \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}T^{p^m} \ , \tag{6.3.9.6}$$

which is then solvable.

Step 2 : The tensor product operator

$$\partial_T - (g^+(T) - h^{(0)}(T)) \tag{6.3.9.7}$$

is again solvable and satisfy

$$g^+(T) - h^{(0)}(T) = p\cdot g^{(1)}(T^p) \ , \tag{6.3.9.8}$$

for some $g^{(1)}(T) \in TK[[T]]$. In other words the "antecedent by ramification" φ_p^* (cf. 2.5.4.1) of the equation $\partial_T - (g^+(T) - h^{(0)}(T))$ is given by $\partial_T - g^{(1)}(T)$, which is then solvable.

Step 3 : We observe that

$$g^{(1)}(T) = \frac{1}{p}\sum_{n\in \mathrm{J}_p}\sum_{m\geq 0}\Big(a_{np^{m+1}} - n(\frac{a_n}{n})^{p^{m+1}}\Big)\,T^{np^m}\ , \tag{6.3.9.9}$$

and again by the small radius lemma we have

$$|a_{np} - n(\frac{a_n}{n})^p| \leq 1\ , \tag{6.3.9.10}$$

which implies $|\lambda_{n,1}| \leq 1$. The process can be restarted indefinitely. $\square$

Remark 6.3.10. We shall now consider the general case of an equation $\partial_T - g(T)$, with $g(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K$, and get a criteria of solvability.

Suppose that $\partial_T - g(T)$ is solvable. We know that $\partial_T - g^-(T)$, $\partial_T - a_0$ and $\partial_T - g^+(T)$ are all solvable (cf. 2.4.9).

We can then consider $\partial_T - g^-(T)$ as an operator on $]1,\infty]$ (instead of $]1-\varepsilon,\infty]$) and the lemma 6.3.9 gives us the existence of a family of Witt vector $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ satisfying

$$a_{-np^m} = -n\phi_{-n,m}\ ,$$

for all $n \in \mathrm{J}_p$, and all $m \geq 0$.

Conversely suppose that we are given two families $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_{n}\}_{n\in\mathrm{J}_p}$, with $\boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$. Since the phantom components of $\boldsymbol{\lambda}_n$ are bounded by 1, then $|a_i|$ is bounded by 1 and then $g^+(T)$ belong to $\mathcal{R}_K$. But in general the series

$$g^-(T) := \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} -n\phi_{-n,m}T^{-np^m}$$

do not belongs to $\mathcal{R}_K$.

We shall now find a necessary and sufficient condition on the family $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ in order that the series $g^-(T)$ belongs to $\mathcal{R}_K$.

Lemma 6.3.11. *Let*

$$c \leq \omega = |p|^{\frac{1}{p-1}}\ , \quad n \in \mathrm{J}_p\ , \quad \rho \leq 1 \tag{6.3.11.1}$$

be fixed. Let $(\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathcal{O}_K)$ *and let* $\phi = (\phi_0, \phi_1, \ldots)$ *be its phantom vector. Then*

$$|\phi_i/p^i| \leq c\rho^{np^i}\ , \quad \textit{for all } i \geq 0 \tag{6.3.11.2}$$

if and only if

$$|\lambda_i| \leq c\rho^{np^i}\ , \quad \textit{for all } i \geq 0\ . \tag{6.3.11.3}$$

Proof : Recall that $c^{p^i} \leq |p|^i c$ for all $i \geq 0$. Suppose that $|\phi_i/p^i| \leq c\rho^{np^i}$ for all $i \geq 0$. Then $|\lambda_0| = |\phi_0| \leq c\rho^n$. By induction suppose that

$$|\lambda_j| \leq c\rho^{np^j}\ , \quad \text{for all } j = 0, \ldots, i-1\ , \tag{6.3.11.4}$$

then
$$|\lambda_i| = |\frac{1}{p^i}(\phi_i - \lambda_0^{p^i} - p\lambda_1^{p^{i-1}} - \ldots - p^{i-1}\lambda_{i-1}^p)| \;. \tag{6.3.11.5}$$
By induction $|\phi_i| \leq |p|^i c\rho^{np^i}$ and $|p^k\lambda_k^{p^{i-k}}| \leq |p|^k(c\rho^{np^k})^{p^{i-k}} = |p|^k c^{p^{i-k}}\rho^{np^i} \leq |p^i|c\rho^{np^i}$, hence
$$|\lambda_i| \leq c\rho^{np^i} \;. \tag{6.3.11.6}$$
Conversely suppose that $|\lambda_i| \leq c\rho^{np^i}$ for all $i \geq 0$. Then
$$\begin{aligned}
|\phi_i| &= |\lambda_0^{p^i} + p\lambda_1^{p^{i-1}} + \cdots + p^i\lambda_i| && (6.3.11.7)\\
&\leq \sup((c\rho^n)^{p^i}, |p|(c\rho^{np})^{p^{i-1}}, \cdots, |p|^i(c\rho^{np^i})) && (6.3.11.8)\\
&\leq |p|^i c\rho^{np^i} \;. \quad \square && (6.3.11.9)
\end{aligned}$$

Definition 6.3.12. Let $c \leq \omega$ and $\rho \leq 1$. We denote by
$$\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]]) \subset \prod_{n\in \mathrm{J}_p} \mathbf{W}^{(-n)}(\mathcal{O}_K) \overset{6.3.5}{\subset} \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]]) \tag{6.3.12.1}$$
the sub-group formed by the sums $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ such that every vector $\boldsymbol{\lambda}_{-n} = (\lambda_{-n,0},\lambda_{-n,1},\ldots) \in \mathbf{W}(\mathcal{O}_K)$ verifies the condition of lemma 6.3.11. In other words
$$|\lambda_{-n,m}| \leq c\rho^{np^m}. \tag{6.3.12.2}$$
Remark 6.3.13. Observe that, by lemma 6.3.11, a Witt vector $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ belongs to the subgroup $\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ if and only if the argument of the exponential
$$E(\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n},1) = \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}\frac{T^{-np^m}}{p^m}) \tag{6.3.13.1}$$
verifies
$$\left|\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}\frac{T^{-np^m}}{p^m}\right|_\rho := \sup_{n\in\mathrm{J}_p,m\geq 0}\Big(\frac{|\phi_{-n,m}|}{|p|^m}\rho^{-np^m}\Big) \leq c \leq \omega \;. \tag{6.3.13.2}$$

Definition 6.3.14. We denote by
$$\mathbf{W}^{\dagger}(T^{-1}\mathcal{O}_K[[T^{-1}]]) \tag{6.3.14.1}$$
the subgroup of $\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ defined as the *sum* of the sub-group
$$\bigcup_{c<\omega,\rho<1}\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]]) \tag{6.3.14.2}$$
with the sub-group
$$\Big(\bigcup_{j\geq 0}[\pi_j]\cdot\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}])\Big)\bigcap\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[T^{-1}]) \;. \tag{6.3.14.3}$$

Corollary 6.3.15 (Criterion of solvability)**.** *Let $\partial_T - g(T)$, $g(T) \in \mathcal{R}_K$*

$$g(T) := \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K \tag{6.3.15.1}$$

be a solvable equation. Then $a_0 \in \mathbb{Z}_p$ and there exist two families $\{\boldsymbol{\lambda}_{-n}\}_{n\in \mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$ such that for all $n \in \mathrm{J}_p$, and all $m \geq 0$ we have

$$a_{-np^m} = -n\phi_{-n,m} \quad ; \qquad a_{np^m} = n\phi_{n,m} \ ,$$

where $(\phi_{-n,0}, \phi_{-n,1}, \ldots)$ (resp. $(\phi_{n,0}, \phi_{n,1}, \ldots)$) is the phantom vector of $\boldsymbol{\lambda}_{-n}$ (resp. $\boldsymbol{\lambda}_n$). Moreover $\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-n} T^{-n}$ belongs to $\mathbf{W}^\dagger(T^{-1}\mathcal{O}_K[[T^{-1}]])$.

Conversely given a triplet

$$\Big(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-n} T^{-n} \ , \ a_0 \ , \ \sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n \Big) \ , \tag{6.3.15.2}$$

with

- $$\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-n} T^{-n} \in \mathbf{W}^\dagger(T^{-1}\mathcal{O}_K[[T^{-1}]]), \tag{6.3.15.3}$$
- $$a_0 \in \mathbb{Z}_p, \tag{6.3.15.4}$$
- $$\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n \in \mathbf{W}(T\mathcal{O}_K[[T]]), \tag{6.3.15.5}$$

then

$$g(T) := \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} -n\phi_{-n,m}T^{-np^m} + a_0 + \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} n\phi_{n,m}T^{np^m} \tag{6.3.15.6}$$

belongs to $\mathcal{R}_K$, and the equation $\partial_T - g(T)$ is solvable.

Remark 6.3.16. This corollary asserts that $\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-n}T^{-n}$ is a sum of a "small" vector i.e. verifying the relation 6.3.13.2 and a vector of "type Robba" i.e. of the type $[\pi_j]\boldsymbol{f}^-(T)$, $\boldsymbol{f}^-(T) \in \mathbf{W}(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[[T^{-1}]])$, for some $j \geq 0$ and such that the product $[\pi_j]\boldsymbol{f}^-(T)$ lies in $\mathbf{W}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ i.e. has its coefficients in K. Actually the proof will show that $\boldsymbol{f}^-(T)$ can be chosen pure.

Proof of 6.3.15 : Let $\partial_T - g(T)$ be solvable. By 6.3.9, we know the existence of $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$ (cf. A.1.9). We must show that $\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-n}T^{-n}$ lies in $\mathbf{W}^\dagger(T^{-1}\mathcal{O}_K[[T^{-1}]])$.

Let $d > 0$ be such that

$$\Big|\sum_{i<-d} a_i T^i / i\Big|_\rho < \omega \ , \ \text{for some } \rho < 1 \tag{6.3.16.1}$$

(cf. 2.4.1). Write

$$g^-(T) = \sum_{i<-d} a_i T^i + \sum_{-d\le i\le -1} a_i T^i \ . \tag{6.3.16.2}$$

By 2.4.1 we know that

$$\exp(\sum_{i<-d} a_i T^i/i) \in \mathcal{R}_K \ , \tag{6.3.16.3}$$

hence the equation $\partial_T - \sum_{i<-d} a_i T^i$ is solvable (and actually trivial). In particular

$$\partial_T - \sum_{-d\le i\le -1} a_i T^i \tag{6.3.16.4}$$

is solvable and hence, again by A.1.9, there exists a family $\{\boldsymbol{\lambda}'_{-n}\}_{n\in \mathrm{J}_p}$, such that

$$-n\phi'_{-n,m} = \begin{cases} a_{-np^m} & \text{if} \quad\quad -np^m < -d \\ 0 & \text{if} \quad -d \le -np^m \le -1 \end{cases}$$

where $(\phi_{-n,0}, \phi_{-n,1}, \cdots)$ is the phantom vector of $\boldsymbol{\lambda}_{-n}$. Since, by construction

$$|\sum_{i<-d} a_i T^i/i|_\rho < \omega \ , \tag{6.3.16.5}$$

this implies

$$|\sum_{n\in \mathrm{J}_p}\sum_{m\ge 0} \phi_{-n,m} T^{-np^m}/p^m|_\rho < \omega \ , \tag{6.3.16.6}$$

hence $\sum_{n\in \mathrm{J}_p} \boldsymbol{\lambda}'_{-n} T^{-n}$ lies in $\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ for some $c < \omega$.

Now we consider $\boldsymbol{\lambda}''_{-n} := \boldsymbol{\lambda}_n - \boldsymbol{\lambda}'_n$, the family $\{\boldsymbol{\lambda}''_{-n}\}_{n\in \mathrm{J}_p}$ verifies then

$$-n\phi''_{-n,m} = \begin{cases} 0 & \text{if} \quad\quad -np^m < -d \\ a_{-np^m} & \text{if} \quad -d \le -np^m \le -1 \end{cases}$$

By 5.4.17, and by 5.4.16 there exists a *pure* Witt vector

$$\boldsymbol{f}^-(T) = (f_0^-(T), \ldots, f_s^-(T)) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}]) \tag{6.3.16.7}$$

such that

$$\mathrm{L}(0, \boldsymbol{f}^-(T)) = \partial_T - \sum_{-d\le i\le -1} a_i T^i \ . \tag{6.3.16.8}$$

Hence $[\pi_s]\boldsymbol{f}^-(T)$ and $\sum_{n\in \mathrm{J}_p} \boldsymbol{\lambda}''_{-n} T^{-n}$ have the "same" phantom vector because $\boldsymbol{f}^-(T)$ is pure. Then $\sum_{n\in \mathrm{J}_p} \boldsymbol{\lambda}''_n T^{-n}$ lies in the image of the morphism

$$\mathbf{W}_s(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}]) \xrightarrow{\sim} [\pi_s]\cdot \mathbf{W}(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}]) \ . \quad \square \tag{6.3.16.9}$$

Remark 6.3.17. Let
$$L := \partial_T - \sum_{i\in\mathbb{Z}} a_i T^i \ , \tag{6.3.17.1}$$
be a given equation. Then L is solvable if and only if for all $n \in \mathrm{J}_p$
$$\partial_T - \sum_{m\geq 0} a_{np^m} T^{np^m} \tag{6.3.17.2}$$
and
$$\partial_T - \sum_{m\geq 0} a_{-np^m} T^{-np^m} \tag{6.3.17.3}$$
are both solvable.

Corollary 6.3.18. *If K is unramified over $\mathbb{Q}_p$, then every rank one solvable differential module over $\mathcal{R}_K$ is isomorphic to a moderate module (cf. 2.6). In other words,*
$$\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K) = \mathbb{Z}_p/\mathbb{Z} \tag{6.3.18.1}$$

Proof : We must show that all $\boldsymbol{\pi}-$exponential $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ whose logarithmic derivative has its coefficients on K is trivial. Actually we can suppose that the co-monomial $\boldsymbol{f}^-(T)$ is *pure* (cf. 5.4.16). Write
$$\partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)) = \sum_{-d\leq i\leq -1} a_i T^i \ , \tag{6.3.18.2}$$
with $a_i \in \mathcal{O}_K$, for all $i = -d, \ldots, -1$. Write $\boldsymbol{f}^-(T) = \sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-np^{m(n)}} T^{-np^{m(n)}}$, $\boldsymbol{\lambda}_{-np^{m(n)}} = (\lambda_{-np^{m(n)},0}, \ldots, \lambda_{-np^{m(n)},m(n)}) \in \mathbf{W}_{m(n)}(\mathcal{O}_{K^{\mathrm{alg}}})$. Since $\boldsymbol{f}^-(T)$ is pure, then (cf. 5.4.14)
$$a_{-np^j} = -n\pi_{m(n)-j}\phi_{-np^{m(n)},j} \ , \tag{6.3.18.3}$$
for all $j = 0, \ldots, m(n)$, where $\langle\phi_{-np^{m(n)},0}, \ldots, \phi_{-np^{m(n)},m(n)}\rangle$ is the phantom vector of $\boldsymbol{\lambda}_{-np^{m(n)}}$. On the other hand the criterion of solvability 6.3.15 asserts the existence of a family $\{\boldsymbol{\lambda}'_{-n}\}_{n\in\mathrm{J}_p}$ with phantom vector $\{\phi'_{-n}\}_{n\in\mathrm{J}_p}$, with $\phi'_{-n} := \langle\phi_{-n,0}, \phi_{-n,1}, \ldots\rangle$, such that
$$a_{-np^m} = -n\phi'_{-n,m} \ , \quad \text{for all } n \in \mathrm{J}_p \ , \ \text{ and } m \geq 0 \ . \tag{6.3.18.4}$$
Observe that $\phi'_{-n,m} \in \mathcal{O}_K$. Since K is unramified over $\mathbb{Q}_p$, then we can employ the lemma 2.7.2. Then $\phi'_{-n,m} \equiv \phi'_{-n,m-1} \mod p^m\mathcal{O}_K$ for all $n \in \mathrm{J}_p$, $m \geq 0$, that is
$$a_{-np^j} \equiv a_{-np^{j-1}} \mod p^j\mathcal{O}_K \quad \text{for all } j \geq 0. \tag{6.3.18.5}$$
Since $a_{-np^{m(n)+1}} = 0$ we obtain, by 6.3.18.3, the estimation $|\pi_{m(n)-j}\phi_{-np^{m(n)},j}| \leq |p|^{j+1}$, for all $j = 0, \ldots, m(n)$. Then we have a system of conditions
$$|\lambda_{-np^{m(n)},0}^{p^j} + p\lambda_{-np^{m(n)},1}^{p^{j-1}} + \cdots + p^j\lambda_{-np^{m(n)},j}| \leq |p|^{j+1}|\pi_{m(n)-j}|^{-1} \ ,$$
which gives easily $|\lambda_{-np^{m(n)},j}| \leq |p|^j|\pi_{m-j}|^{-1} < 1$. Then by 3.4.6, and 6.2.1, $\mathrm{e}_d(\boldsymbol{\lambda}_{-np^{m(n)}}, T^{-1})$ lies in $\mathcal{R}_K$, for all $n \in \mathrm{J}_p$, and $\mathrm{L}_d(0, \boldsymbol{f}^-(T))$ is trivial.□

6.4 Explicit computation of the Irregularity in some cases

Let v_t be the $t-$adic valuation of $k(\!(t)\!)$.

Lemma 6.4.1. *Let H/K be an algebraic extension. Let $f(T) \in T^{-1}\mathcal{O}_K[T^{-1}]$ be a polynomial in T^{-1}, and let $\overline{f}(t) \in t^{-1}k[t^{-1}]$ be the reduction of $f(T)$. Let $n := -v_t(\overline{f}(t)) > 0$. If $(n,p) = 1$, then*

$$\mathrm{Irr}\Big(\mathrm{M}\big(0, (\underbrace{0, \ldots, 0, f(T), 0, \ldots, 0}_{\ell+1})\big)\Big) = n \cdot p^{\ell} \ , \tag{6.4.1.1}$$

where $\ell = \ell(0, \ldots, 0, f(T), 0, \ldots, 0)$ (cf. 2.7.3).

Proof : We have $\mathrm{M}\big(0, (0, \ldots, 0, f(T), 0, \ldots, 0)\big) = \mathrm{M}(0, (f(T), 0, \ldots, 0))$. (cf. 5.3.0.2). Moreover, the isomorphism class of this module depends only on $\overline{f}(t)$, hence we can suppose that $f(T) = a_{-n}T^{-n} + \cdots + a_{-1}T^{-1}$, with $|a_{-n}| = 1$. We have

$$\mathrm{L}(0, (f(T), 0, \ldots, 0)) = \partial_T + \partial_{T,\log}(f(T)) \cdot \big[\pi_s f(T) + \pi_{s-1} f(T)^p + \cdots + \pi_0 f(T)^{p^s}\big] \ . \tag{6.4.1.2}$$

We have $\partial_{T,\log}(f(T)) = -n + TQ(T)$, with $Q(T) \in \mathcal{O}_K[[T]]$, then

$$g(T) = -\pi_0 \cdot n \cdot a_{-n}^{p^\ell} \cdot T^{-n \cdot p^\ell} + (\text{terms of degree} > -np^\ell) \ . \tag{6.4.1.3}$$

Since $(n,p) = 1$, we can apply 2.4.3 and $\mathrm{Irr}(\partial_T + g(T)) = \mathrm{Irr}_F(\partial_T + g(T)) = np^\ell$.$\square$

Corollary 6.4.2. *Let $\overline{\boldsymbol{f}^-}(t) = (\overline{f_0^-}, \ldots, \overline{f_s^-}) \in \mathbf{W}_s(t^{-1}k[t^{-1}])$. Let*

$$n_j := -v_t\left(\overline{f_j^-}\right) \ . \tag{6.4.2.1}$$

If $(n_j, p) = 1$, or $n_j = 0$, for all $j = 0, \ldots, s$ (cf. 6.1.1), then

$$\mathrm{Irr}\left(\mathrm{M}(0, \overline{\boldsymbol{f}^-}(t))\right) = \max_{0 \le j \le s} \left(n_j \cdot p^{s-j}\right) . \tag{6.4.2.2}$$

Proof : Let M_j be the differential module defined by $(0, \ldots, 0, \overline{f_j^-}(t), 0, \ldots, 0)$. By 6.4.1, $\mathrm{Irr}(M_j) = n_j \cdot p^{s-j}$. Since $\mathrm{M}(0, \boldsymbol{f}^-(T)) = \otimes_j M_j$ (cf. 4.2.7.1), and since $n_j p^{s-j}$ are all different, then by 2.4.8 we conclude. $\square$

6.5 Tannakian group

In this section we study the category of solvable differential modules over $\mathcal{R}_K$ which are extension of rank one sub objects. We remove the hypothesis

"K is spherically complete", present in the literature. Let H/K be an arbitrary algebraic extension. We set

$$\mathcal{H}_H^\dagger := \cup_\varepsilon \mathcal{A}_H(]1-\varepsilon,\infty[). \tag{6.5.0.3}$$

Let S be a sub-group of $\mathbb{Z}_p$ without Liouville numbers and containing $\mathbb{Z}$.

Definition 6.5.1. Let $\mathcal{C}$ be an additive category. If there exists a function *rank* on $\mathcal{C}$, then we denote by $\mathcal{C}_{\oplus\text{-}1}$ (resp. $\mathcal{C}_{\text{ext-}1}$) the full sub-category of $\mathcal{C}$ whose objects are finite direct sum (resp. finite successive extension) of rank one objects.

Definition 6.5.2. An object is said *simple* if it has no non trivial sub-objects. It is said *indecomposable* if it is not a direct sum of non trivial objects.

Definition 6.5.3. Let $\mathrm{MLS}(\mathcal{H}_H^\dagger)$ be the category of (free) differential modules over $\mathcal{H}_H^\dagger$ solvable at 1 (i.e. $Ray(N,1)=1$, cf. 2.4). Recall that, by definition, such a module comes, by scalar extension, from a module over $\mathcal{H}_L^\dagger$, for some finite extension L/K (cf. 2.1.1.1). Let $N \in \mathrm{MLS}_{\text{ext-}1}(\mathcal{H}_H^\dagger)$ be extension of rank one modules, say $\{N_i\}_{i=1,\ldots,k}$. We will say that N is regular at ∞, write $N \in \mathrm{MLS}^{\text{reg}}_{\text{ext-}1}(\mathcal{H}_H^\dagger)$, if, for all i, the module N_i is defined, in some basis, by an operator $\partial_T + g_i(T)$, satisfying

$$g_i(T) = \sum a_{i,j}T^j\ , \quad \text{with } a_{i,j}=0\ ,\ \text{for all } j \geq 1\ . \tag{6.5.3.1}$$

We will say that N belongs to $\mathrm{MLS}^{\text{reg}}_{\text{ext-}1}(\mathcal{H}_H^\dagger, S)$, if $N \in \mathrm{MLS}^{\text{reg}}_{\text{ext-}1}(\mathcal{H}_H^\dagger)$, and if $a_{i,0} \in S$, for all i.

Lemma 6.5.4 (Schur's Lemma)**.** *Let M_1, M_2 be two rank one objects in $\mathrm{MLS}(\mathcal{R}_H)$ (resp. $\mathrm{MLS}^{\text{reg}}_{\text{ext-}1}(\mathcal{H}_H^\dagger)$). Every non zero morphism $\varrho : M_1 \to M_2$ is an isomorphism.*

Proof : Let $a_{0,i} \in \mathbb{Z}_p$ be the exponent of M_i. In the theorem 5.4.17 we have seen that M_i has a basis $\mathbf{e}_i \in M_i$ in which the solution is of the type $T^{a_{0,i}}\epsilon_i(T)$, where $\epsilon_i(T) \in \mathcal{A}_H(]1,\infty])$ is a series with coefficients in H. We have then $\varrho(\mathbf{e}_1) = h(T)\mathbf{e}_2$, with $h(T) = T^{a_{0,2}-a_{0,1}}\epsilon_2(T)\epsilon_1(T)^{-1} \in \mathcal{R}_H$. Then $a_{0,2}-a_{0,1} \in \mathbb{Z}$, and $\epsilon_2(T)\epsilon_1(T)^{-1} \in \mathcal{R}_H$. Since $\epsilon_i(T)$ is a product of $\boldsymbol{\pi}-$exponentials, hence both $h(T)$ and its inverse lie in $\mathcal{R}_H$. If $M_1, M_2 \in \mathrm{MLS}^{\text{reg}}_{\oplus\text{-}1}(\mathcal{H}_H^\dagger)$, then, by the proof of 2.4.1, the base change necessary to obtain $\mathbf{e}_i$ lies in $(\mathcal{H}_H^\dagger)^\times$.$\square$

Remark 6.5.5. By 6.5.4, every rank one object of $\mathrm{MLS}(\mathcal{R}_H)$ is simple in Mod-$\mathcal{R}_H[\partial_T]$. Then, by the Jordan-Hölder theorem in Mod-$\mathcal{R}_H[\partial_T]$, the categories $\mathrm{MLS}_{\text{ext-}1}(\mathcal{R}_H)$, and $\mathrm{MLS}_{\text{ext-}1}(\mathcal{R}_H, S)$ are abelian, and, for all object M, the set of rank one objects appearing in a decomposition series of M

does not depend, up to the order, on the chosen decomposition. Moreover the sub-categories $\mathrm{MLS}_{\oplus\text{-}1}(\mathcal{R}_H)$ and $\mathrm{MLS}_{\oplus\text{-}1}(\mathcal{R}_H, S)$ are abelian and semi-simple. The same facts are true for $\mathrm{MLS}^{\mathrm{reg}}_{\mathrm{ext}\text{-}1}(\mathcal{H}^{\dagger}_H)$, $\mathrm{MLS}^{\mathrm{reg}}_{\mathrm{ext}\text{-}1}(\mathcal{H}^{\dagger}_H)$ and $\mathrm{MLS}^{\mathrm{reg}}_{\oplus\text{-}1}(\mathcal{H}^{\dagger}_H, S)$.

Theorem 6.5.6. *Let $N \in \mathrm{MLS}_{\mathrm{ext}\text{-}1}(\mathcal{R}_H, S)$ (resp. $N \in \mathrm{MLS}^{\mathrm{reg}}_{\mathrm{ext}\text{-}1}(\mathcal{H}^{\dagger}_H, S)$). There exists a basis of N in which the matrix of the derivation is in the Jordan canonical form. In other words, N is a direct sum of objects of the form*

$$M \otimes U_m, \tag{6.5.6.1}$$

where M is a rank one object and U_m is defined by the operator ∂^m_T.

Proof : Let $M_1, M_2 \in \mathrm{MLS}_{\mathrm{ext}\text{-}1}(\mathcal{R}_H)$. By the Robba's index theorem for rank one operators whose matrix is a rational fraction [Rob84], we have $\dim \mathrm{Hom}(M_1, M_2) = \dim \mathrm{Ext}^1(M_1, M_2)$. Observe that this fact does not need the "spherically complete" hypothesis on the field K. The theorem results then by classical considerations (see for example [vdPS03]).$\square$

Theorem 6.5.7 (Canonical extension)**.** *The canonical restriction functor* $\mathrm{Res} : \mathrm{MLS}^{\mathrm{reg}}_{\mathrm{ext}\text{-}1}(\mathcal{H}^{\dagger}_H, S) \to \mathrm{MLS}_{\mathrm{ext}\text{-}1}(\mathcal{R}_H, S)$ *is an equivalence.*

Proof : By the main theorem 5.2.2, $\mathrm{Res} : \mathrm{MLS}^{\mathrm{reg}}_{\mathrm{ext}\text{-}1}(\mathcal{H}^{\dagger}_H, S) \to \mathrm{MLS}_{\mathrm{ext}\text{-}1}(\mathcal{R}_H, S)$ is essentially surjective. Indeed $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$ has its coefficients in $\mathcal{H}^{\dagger}_H$. By 6.5.4, two rank one modules in $\mathrm{MLS}^{\mathrm{reg}}_{\mathrm{ext}\text{-}1}(\mathcal{H}^{\dagger}_H, S)$ are isomorphic if and only if they are isomorphic over $\mathcal{R}_H$, because the base change is given by an overconvergent exponential in $1 + T^{-1}\mathcal{O}_H[[T^{-1}]]$. Hence, by the Schur lemma 6.5.4, Res is also fully-faithful.$\square$

Corollary 6.5.8. *The Tannakian category* $\mathrm{MLS}_{\mathrm{ext}\text{-}1}(\mathcal{R}_H, S)$ *is neutral.*

Proof : Let $\omega_S : \mathrm{MLS}^{\mathrm{reg}}_{\oplus\text{-}1}(\mathcal{H}^{\dagger}_H, S) \to \underline{\mathrm{Vect}}^{\mathrm{fin}}(H)$ be the fiber functor sending a rank one object in its Taylor solution at 1 (cf. 2.3.0.1). A $H-$linear fiber functor of $\mathrm{MLS}_{\oplus\text{-}1}(\mathcal{R}_H, S)$ is given by composing ω_S with a quasi-inverse of Res.$\square$

Definition 6.5.9. An affine group scheme $\mathcal{H}$ (over H) is *linear* if there exists a closed immersion $\mathcal{H} \to \mathrm{GL}_H(V)$, for some finite dimensional vector space V.

Definition 6.5.10. Let $\omega_S : \mathrm{MLS}_{\oplus\text{-}1}(\mathcal{R}_H, S) \to \underline{\mathrm{Vect}}^{\mathrm{fin}}(H)$ be a fiber functor. We denote by

$$\boldsymbol{\mathcal{G}}_H := \mathrm{Aut}^{\otimes}(\omega_S) \tag{6.5.10.1}$$

the tannakian group of $\mathrm{MLS}_{\oplus\text{-}1}(\mathcal{R}_H, S)$.

Remark 6.5.11. By 6.5.6, the tannakian group of $\mathrm{MLS}_{\mathrm{ext}\text{-}1}(\mathcal{R}_H, S)$ is

$$\boldsymbol{\mathcal{G}}_H \times \mathbb{G}_a. \tag{6.5.11.1}$$

6.5.1 Study of $\mathcal{G}_H$

For all (finite dimensional) representations $\rho_V : \mathcal{G}_H \to \mathrm{GL}_H(V)$, we set $\mathcal{G}_{H,V} := \rho_V(\mathcal{G}_H)$. The group $\mathcal{G}_{H,V}$ is then linear and affine. Moreover, $\mathcal{G}_{H,V}$ is diagonalizable (i.e. closed subgroup of the group of diagonal matrices). The group $\mathcal{G}_H$ is the inverse limit of its linear (compact) quotients $\mathcal{G}_{H,V}$, and is endowed with the limit topology. Hence $\mathcal{G}_H$ is abelian, because every V is a direct sum of rank one objects, and $\mathcal{G}_{H,V}$ is abelian.

Remark 6.5.12. Let I be a non empty directed set. The functor $\varinjlim_{i\in I}$ is exact if applied to exact sequences of compact algebraic groups (see for example [Bou83b, ch.3,§7,Cor.1]). For this reason, all exact sequences in the sequel will be studied at level $\mathcal{G}_{H,V}$.

Definition 6.5.13. We set $\mathbf{X}(\mathcal{G}_H) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{gr}}(\mathcal{G}_H, \mathbb{G}_m \otimes H)$, where the word "cont" means that such a morphism $\mathcal{G}_H \to \mathbb{G}_m \otimes H$ factors on a linear quotient $\mathcal{G}_{H,V}$.

Let $\mathrm{Pic}^{\mathrm{sol}}_S(\mathcal{R}_{H_\infty})$ be the sup-group of $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_\infty})$ formed by modules whose residue lies in S. By tannakian equivalence, we have an isomorphism of groups

$$\mathbf{X}(\mathcal{G}_{H_\infty}) \cong \mathrm{Pic}^{\mathrm{sol}}_S(\mathcal{R}_{H_\infty}) \xrightarrow{\sim} S/\mathbb{Z} \oplus \mathbf{P}(k_{H_\infty}). \tag{6.5.13.1}$$

This leads us to recover the group $\mathcal{G}_{H_\infty}$ itself (cf. [Spr98, 3.2.6]). Let us write

$$S/\mathbb{Z} = S/\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}/\mathbb{Z}\ . \tag{6.5.13.2}$$

Theorem 6.5.14. *$\mathcal{G}_{H_\infty}$ is the product of a torus $\boldsymbol{T}_{H_\infty}$ (dual of $S/\mathbb{Z}_{(p)}$) with a pro-finite group $\boldsymbol{\mathcal{I}}_{H_\infty}$ (dual of $\mathbb{Z}_{(p)}/\mathbb{Z} \oplus \mathbf{P}(k_{H_\infty})$). This last is isomorphic to the Galois group $\mathcal{I}^{\mathrm{ab}}_{E_{H_\infty}} := \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}_{H_\infty}/\mathrm{E}_{H_\infty})^{\mathrm{ab}}$, where $\mathrm{E}_{H_\infty} = k_{H_\infty}((t))$ (cf. 2.8.5)*

$$\mathcal{I}^{\mathrm{ab}}_{\mathrm{E}_{H_\infty}} \xrightarrow{\sim} \boldsymbol{\mathcal{I}}_{H_\infty}\ . \tag{6.5.14.1}$$

Proof : The proof is standard. These two groups have the same character groups. Namely, by tannakian equivalence and by corollary 5.3.2, the character group of $\boldsymbol{\mathcal{I}}_{H_\infty}$ is $\mathbb{Z}_{(p)}/\mathbb{Z} \oplus \mathbf{P}(k_{H_\infty})$. By the Artin-Schreier theory, and the Kummer theory, this last is also the character group of $\mathcal{I}_{H_\infty}$. □

Remark 6.5.15. We will see in the next section that this isomorphism is induced by the Fontaine-Katz functor $\mathbf{M}^\dagger$. Actually, this isomorphism exists even without the hypothesis required in the definition of this functor.

6.6 Differential equations and $\varphi-$modules over $\mathcal{E}^\dagger_K$ in the abelian case

In this section the notations, and hypotheses, will follow [Tsu98a]. We recall that

$$\mathrm{w} = p \tag{6.6.0.1}$$

(cf. 4.0.5). We suppose k perfect (used in 6.2.6). Let $\Lambda/\mathbb{Q}_p$ be a finite extension containing $\mathbb{Q}_p(\xi_s)$. Let $\mathbb{F}_q$, with

$$q := p^r \ , \tag{6.6.0.2}$$

be the residue field of Λ.

Hypothesis 6.6.1. We assume the existence of an *absolute* Frobenius

$$\sigma_0 : \Lambda \to \Lambda \tag{6.6.1.1}$$

(i.e. lifting of the $p-$th power map $x \mapsto x^p$ of $\mathbb{F}_q$), satisfying

$$\sigma_0^r = \mathrm{Id}_\Lambda \tag{6.6.1.2}$$

and

$$\sigma_0(\pi_s) = \pi_s \ . \tag{6.6.1.3}$$

This is always possible if $\Lambda/\mathbb{Q}_p$ is Galois.

We let

$$K := \Lambda \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathbf{W}(k) \tag{6.6.1.4}$$

and

$$\sigma := \mathrm{Id}_\Lambda \otimes \mathrm{F}^r \ . \tag{6.6.1.5}$$

We denote again by σ_0 the morphism $(\sigma_0 \otimes \mathrm{F})$ on K, then $\sigma = \sigma_0^r$. We fix a continue absolute Frobenius φ_0 on $\mathcal{O}_K^\dagger$, by setting

$$\varphi_0(\sum a_i T^i) := \sum \sigma_0(a_i)\varphi_0(T)^i \ , \tag{6.6.1.6}$$

where $\varphi_0(T) \in \mathcal{O}_\Lambda^\dagger$, is a lifting of $t^p \in k(\!(t)\!)$ (see definition 2.5.1). Then φ_0 verifies $\varphi_0(\pi_s) = \pi_s$, and $\varphi_0(\mathcal{E}_\Lambda^\dagger) \subseteq \mathcal{E}_\Lambda^\dagger$. We set

$$\varphi := \varphi_0^r \ . \tag{6.6.1.7}$$

Both φ and φ_0 extend uniquely to all unramified extensions of $\mathcal{E}_K^\dagger$, hence they commute with the action of $\mathrm{G_E} := \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$.

Definition 6.6.2. Let $\underline{\mathrm{Rep}}_\Lambda^{\mathrm{fin}}(\mathrm{G_E})$ be the category of continue (finite dimensional) representations $\alpha : \mathrm{G_E} \to \mathrm{GL}_\Lambda(\mathrm{V})$, such that $\alpha(\mathcal{I}_\mathrm{E})$ is finite.

Definition 6.6.3. Let $\alpha : \mathrm{G_E} \to \Lambda^\times$ be a character such that $\alpha(\mathcal{I}_\mathrm{E})$ is finite. Then we denote by $\mathrm{V}_\alpha \in \underline{\mathrm{Rep}}_\Lambda^{\mathrm{fin}}(\mathrm{G_E})$ the rank one representation of $\mathrm{G_E}$ given by

$$\gamma(\mathbf{e}) := \alpha(\gamma) \cdot \mathbf{e} \ , \quad \text{for all } \gamma \in \mathrm{G_E},$$

where $\mathbf{e} \in \mathrm{V}_\alpha$ is a basis. We denote by $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ (resp. $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$) the $\varphi-\nabla-$module over $\mathcal{E}_K^\dagger$ (resp. differential module over $\mathcal{R}_K$) attached to V_α. Namely

$$\mathbf{D}^\dagger(\mathrm{V}_\alpha) = (\mathrm{V}_\alpha \otimes_\Lambda \mathcal{E}_K^{\dagger,\mathrm{unr}})^{\mathrm{G_E}} \quad , \quad \mathbf{M}^\dagger(\mathrm{V}_\alpha) = \mathbf{D}^\dagger(\mathrm{V}_\alpha) \otimes_{\mathcal{E}_K^\dagger} \mathcal{R}_K \ . \tag{6.6.3.1}$$

We recall that $\mathbf{M}^{\dagger}(\mathrm{V}_{\alpha})$ is endowed with the unique derivation "commuting" with φ (cf. [Fon90, 2.2.4]). This derivation is $\nabla = 1 \otimes \partial_T$. By 2.5.9, one has

$$\mathbf{M}^{\dagger}(\mathrm{V}_{\alpha}) \in \mathrm{MLS}(\mathcal{R}_K)\ . \tag{6.6.3.2}$$

Definition 6.6.4. We will identify $\mathbb{Z}/p^{s+1}\mathbb{Z}$ with $\boldsymbol{\mu}_{p^{s+1}}$, by sending

$$1 \mapsto \xi_s\ :\ \mathbb{Z}/p^{s+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{s+1}},$$

where ξ_s is the unique $p^{s+1}-$th root of 1 verifying (cf. 2.9.9)

$$|(\xi_s - 1) - \pi_s| < |\pi_s|\ . \tag{6.6.4.1}$$

If $\alpha \in \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Z}/p^{s+1}\mathbb{Z})$, we denote again by V_{α} the representation given by

$$\gamma(\mathbf{e}) := \xi_s^{\alpha(\gamma)} \cdot \mathbf{e}\ , \quad \text{for all } \gamma \in \mathrm{G_E}.$$

Remark 6.6.5. This definition is chosen "ad hoc" to be the inverse of the action of $\mathrm{G_E}$ described in 4.1.9.2.

Remark 6.6.6. Let $\alpha : \mathrm{G_E} \to \Lambda^{\times}$ be a continue character, then α factors on the abelianized $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}$. Let $\mathcal{I}_{\mathrm{E}}^{\mathrm{ab}}$ be the inertia of $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}$, and $\mathrm{G}_k^{\mathrm{ab}}$ be the abelianized of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. Since k is perfect, then, by 6.2.6, the exact sequence $1 \to \mathcal{I}_{\mathrm{E}}^{\mathrm{ab}} \to \mathrm{G}_{\mathrm{E}}^{\mathrm{ab}} \to \mathrm{G}_k^{\mathrm{ab}} \to 1$ is split, hence

$$\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}} = \mathcal{I}_{\mathrm{E}}^{\mathrm{ab}} \oplus \mathrm{G}_k^{\mathrm{ab}}\ , \tag{6.6.6.1}$$

and $\alpha = \alpha^{-} \cdot \alpha_0$, where $\alpha^{-} : \mathcal{I}_{\mathrm{E}}^{\mathrm{ab}} \to \Lambda^{\times}$ and $\alpha_0 : \mathrm{G}_k^{\mathrm{ab}} \to \Lambda^{\times}$. Then

$$\mathrm{V}_{\alpha} = \mathrm{V}_{\alpha^{-}} \otimes \mathrm{V}_{\alpha_0}.$$

We observe that

$$\mathbf{M}^{\dagger}(\mathrm{V}_{\alpha_0}) \xrightarrow{\sim} \mathcal{R}_K. \tag{6.6.6.2}$$

is trivial because its solution is a constant. Indeed, the extension of $\mathcal{O}_K^{\dagger}$ defined by α_0 is $\mathcal{O}_K^{\dagger} \otimes_K H$, for some unramified extensions H/K. In the sequel we will treat only characters $\alpha : \mathrm{G_E} \to \Lambda^{\times}$ with *finite image*, this will be restrictive in terms of $\varphi-$modules but not in terms of differential modules (cf. 6.6.6). Indeed,

$$\mathbf{D}^{\dagger}(\mathrm{V}_{\alpha}) = \mathbf{D}^{\dagger}(\mathrm{V}_{\alpha^{-}}) \otimes \mathbf{D}^{\dagger}(\mathrm{V}_{\alpha_0}) \qquad ; \qquad \mathbf{M}^{\dagger}(\mathrm{V}_{\alpha}) = \mathbf{M}^{\dagger}(\mathrm{V}_{\alpha^{-}}). \tag{6.6.6.3}$$

Remark 6.6.7. Points (4) and (5) of the following theorem have been already proved in [Mat95] in the case $p \neq 2$ and rank one, and in [Cre00], [Mat02], [Tsu98b] in the general case. Moreover it seems that the explicit form of the differential operator (answer to the question (5) of the introduction) was written in the proof of Lemma 5.2 of [Mat95], in the case $p \neq 2$.

Theorem 6.6.8. *Let $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_s(\mathrm{E})$ and let $\alpha = \delta(\overline{\boldsymbol{f}}(t))$ be the Artin-Schreier character defined by $\overline{\boldsymbol{f}}(t)$ (cf. (2.8.1.2)). Let $(\mathcal{E}_K^\dagger)'$ be the unramified extension of $\mathcal{E}_K^\dagger$ corresponding, by henselianity, to the separable extension of $k(\!(t)\!)$ defined by α. Then*

1. *a basis of $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ is given by*

$$y := \mathbf{e} \otimes \theta_{p^s}(\boldsymbol{\nu}, 1) \;, \tag{6.6.8.1}$$

where $\mathbf{e} \in \mathrm{V}_\alpha$ is the basis of 6.6.4 and $\boldsymbol{\nu} \in \mathbf{W}_s(\widehat{\mathcal{E}}_K^{\mathrm{unr}})$ is a solution of

$$\varphi_0(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{f}(T) \;, \tag{6.6.8.2}$$

where $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[[T]][T^{-1}])$ is an arbitrary lifting of $\overline{\boldsymbol{f}}(t)$;

2. *the Frobenius φ_0 acts on V_α, moreover $\varphi_0(y) = \theta_{p^s}(\boldsymbol{f}(T), 1) \cdot y$. Hence, if $\mathrm{Tr}(\boldsymbol{f}(T)) := \boldsymbol{f}(T) + \varphi_0(\boldsymbol{f}(T)) + \cdots + \varphi_0^{r-1}(\boldsymbol{f}(T))$, we have*

$$\varphi(y) = \theta_{p^s}(\mathrm{Tr}(\boldsymbol{f}(T)), 1) \cdot y; \tag{6.6.8.3}$$

3. *By 4.2.5, one has $(\mathcal{E}_K^\dagger)'(\pi_s) = \mathcal{E}_{K_s}^\dagger[\theta_{p^s}(\boldsymbol{\nu}^-, 1)]$ (cf. definition 5.4.8). This extension can be identified with the extension*

$$\mathcal{E}_{K_s}^\dagger[\theta_{p^s}(\boldsymbol{\nu}^-, 1)] \xrightarrow{\sim} \mathcal{E}_{K_s}^\dagger[\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)] \tag{6.6.8.4}$$

by sending $\theta_{p^s}(\boldsymbol{\nu}, 1)$ into $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$. In particular, if $\pi_s \in K$, one has

$$\widetilde{y} = \mathbf{e} \otimes \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) \;. \tag{6.6.8.5}$$

Moreover $\varphi_0(\widetilde{y}) = \theta_{p^s}(\boldsymbol{f}^-(T), 1) \cdot \widetilde{y}$, and $\varphi(\widetilde{y}) = \theta_{p^s}(\mathrm{Tr}(\boldsymbol{f}^-(T)), 1) \cdot \widetilde{y}$.

4. *The isomorphism class of $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ depends only on α^- and*

$$\mathbf{M}^\dagger(\mathrm{V}_\alpha) \xrightarrow{\sim} \mathrm{M}(0, \alpha^-) \;; \tag{6.6.8.6}$$

5. *the irregularity of $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ is equal to the Swan conductor of V_α.*

Proof : Let $\mathrm{E} = k(\!(t)\!)$. For all $\gamma \in \mathrm{G}_\mathrm{E} = \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$, we have (cf. (4.1.11.2))

$$\gamma(\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{\nu}, 1)) = (\xi_s^{\alpha(\gamma)} \cdot \mathbf{e}) \otimes (\xi_s^{-\alpha(\gamma)} \cdot \theta_{p^s}(\boldsymbol{\nu}, 1)) = \mathbf{e} \otimes \theta_{p^s}(\boldsymbol{\nu}, 1), \tag{6.6.8.7}$$

hence $\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{\nu}, 1) \in \mathbf{D}^\dagger(\mathrm{V}_\alpha)$. Moreover, $\varphi_0(\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{\nu}, 1)) = \mathbf{e} \otimes \varphi_0(\theta_{p^s}(\boldsymbol{\nu}, 1))$ and

$$\varphi_0(\theta_{p^s}(\boldsymbol{\nu}, 1)) \overset{(*)}{=} \theta_{p^s}(\varphi_0(\boldsymbol{\nu}), 1) = \theta_{p^s}(\varphi_0(\boldsymbol{\nu}) - \boldsymbol{\nu}, 1) \cdot \theta_{p^s}(\boldsymbol{\nu}, 1) \;. \tag{6.6.8.8}$$

The equality $(*)$ is true because $\varphi_0(\pi_s) = \sigma_0(\pi_s) = \pi_s$, for all $s \geq 0$. The derivation on $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ arises from the derivation on $\mathcal{E}_K^{\dagger,\mathrm{unr}}$ (cf. (6.6.3.1)), hence the operator attached to the basis $\mathbf{e}\otimes\theta_{p^s}(\boldsymbol{\nu},1)\in\mathbf{M}^{\dagger}_K(\mathrm{V}_\alpha)$ is (cf. 2.2.2) $\partial_T-\partial_{T,\log}(\theta_{p^s}(\boldsymbol{\nu},1))$. As explained in 6.6.6, the isomorphism class of $\mathbf{M}^{\dagger}(\mathrm{V}_\alpha)$ depends only on $\alpha^-=\delta(\overline{\boldsymbol{f}^-}(t))$. Hence, we can suppose $\alpha=\alpha^-$ and $\boldsymbol{f}(T)=\boldsymbol{f}^-(T)\in\mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$ (cf. 5.4.8). Let us write $\boldsymbol{\nu}^-$ instead of $\boldsymbol{\nu}$. By equation (5.1.0.1), we have

$$\theta_{p^s}(\boldsymbol{\nu}^-,1)^{p^{s+1}}=\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)^{p^{s+1}}, \tag{6.6.8.9}$$

hence $\partial_{T,\log}(\theta_{p^s}(\boldsymbol{\nu}^-,1))=\partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1))$. This establishes point (3) and (4).

Both the Swan conductor and the irregularity are stable under extension of the constant field K, hence we can suppose $K=K^{\mathrm{alg}}$. We can suppose that $\boldsymbol{f}^-(T)$ is pure (cf. 5.4.13), because both the Swan conductor and the irregularity depend only on α^-. Write $\boldsymbol{f}^-(T)=\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{np^{m(n)}}T^{-np^{m(n)}}$. Since the irregularities of the $\boldsymbol{\lambda}_{np^{m(n)}}T^{-np^{m(n)}}$'s are all different we can suppose $\boldsymbol{f}^-(T)=\boldsymbol{\lambda}_{np^{m(n)}}T^{-np^{m(n)}}$. Now write explicitly (cf. definition 5.4.1)

$$\begin{aligned}\boldsymbol{\lambda}_{np^{m(n)}}T^{-np^{m(n)}} &= (\lambda_rT^{-np^r},\lambda_{r+1}T^{-np^{r+1}},\ldots,\lambda_mT^{-np^{m(n)}})\\ &= (\lambda_rT^{-np^r},0\ldots,0)+\cdots+(0,\ldots,0,\lambda_mT^{-np^{m(n)}})\ .\end{aligned}$$

Since, by Reduction Theorem 3.4.6, the irregularities of the these vectors are all different, and since both the irregularity and the Swan conductor are invariant by V (cf. (6.4.1.1)), we can suppose $\boldsymbol{f}^-(T)=(\lambda T^{-n},0,\ldots,0)$, with $|\lambda|=1$. Moreover, since $K=K^{\mathrm{alg}}$, the residue field is perfect and, replacing λT^{-n} with $\lambda^{1/p^k}T^{-n/p^k}$, we can suppose $(n,p)=1$ (cf. 2.8.2.1). The irregularity is then np^s (cf. 6.4.1), and it is equal to the Swan conductor (see for example [Bry83]). This Theorem is the analogue of 6.4.2 for Artin-Schreier characters of $\mathrm{G_E}$). □

Remark 6.6.9. Suppose that the character is totally ramified, and choose $\boldsymbol{f}^-(T)$ in $\mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$, then $\theta_{p^s}(\boldsymbol{\nu},1)=\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$.

Remark 6.6.10. Let $a_0=\frac{m}{n}\in\mathbb{Z}_p\cap\mathbb{Q}$. Suppose that $\boldsymbol{\mu}_n\subset k$. Let

$$\beta_{a_0}:\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))\to\boldsymbol{\mu}_n\subset\Lambda^\times \tag{6.6.10.1}$$

be the Kummer character defined by t^{a_0}. We have $\beta_{a_0}(\gamma)=\gamma(t^{a_0})/t^{a_0}$. As before, a basis of $\mathbf{D}^{\dagger}(\mathrm{V}_{\beta_{a_0}})$ is given by $\mathbf{e}\otimes T^{-a_0}\in\mathrm{V}_{\beta_{a_0}}\otimes\mathcal{E}_K^{\dagger,\mathrm{unr}}$, because $\gamma(\mathbf{e}):=\beta_{a_0}(\gamma)\mathbf{e}$. Then

$$\varphi(\mathbf{e}\otimes T^{-a_0})=T^{a_0}\varphi(T^{-a_0})\cdot(\mathbf{e}\otimes T^{-a_0})\ , \tag{6.6.10.2}$$

and $\mathbf{M}^{\dagger}(\mathrm{V}_{\beta_0})=\mathrm{M}(a_0,0)$. We do not have necessary an action of φ_0, because σ_0 does not fix the $n-$th root of 1.

Troisième partie

Strong $p-$adic Confluence and $p-$adic local monodromy of $q-$Difference Equations

Abstract

In this paper we study a particular kind of confluence of $p-$adic $q-$difference equations to differential equations. Our main goal is to show that, in the $p-$adic framework, the solutions of $q-$difference equations are not only an approximation of the solutions of the differential equations, but are actually *equal* to these solutions.

We develop a language which describes this situation and we deduce the quasi-unipotence of $q-$difference equations, from the confluence and from the quasi unipotence of $p-$adic differential equations with Frobenius structure.

Chapitre 1

Introduction

1.1 Heuristic

Let us consider a family of $q-$difference equations

$$\{ \sigma_q(Y) = A_q(T) \cdot Y \}_{q \in D} , \tag{1.1.0.1}$$

where D is a domain admitting 1 as limit point, and such that, for all $q \in D$, the matrix $A_q(T)$ is an invertible $n \times n$ matrix, with coefficients in a ring of functions B, endowed with an action of the operator

$$\sigma_q : \mathrm{B} \xrightarrow[f(T) \mapsto f(qT)]{} \mathrm{B} . \tag{1.1.0.2}$$

Denote by

$$\Delta_q := \frac{\sigma_q - 1}{q - 1} , \qquad \delta_1 := T \frac{d}{dT} , \tag{1.1.0.3}$$

the $q-$derivation and the derivation ($\lim_{q \to 1} \Delta_q = \delta_1$). The family 1.1.0.1 is equivalent to the family of equations

$$\Delta_q(Y) = G_q \cdot Y , \quad \text{with} \quad G_q := \frac{A_q - I_n}{q - 1} . \tag{1.1.0.4}$$

The phenomena of *confluence* and of *deformation* describe heuristically the fact that in some cases, if $q \to 1$, the equation 1.1.0.4 “tends” (in a suitable meaning) to a differential equation

$$\delta_1(Y) = G_1 \cdot Y , \quad \text{with} \quad G_1 := \lim_{q \to 1} G_q , \tag{1.1.0.5}$$

and that the eventual solution matrix Y_q of 1.1.0.4 “tends” to a solution Y_1 of 1.1.0.5 :

$$\lim_{q \to 1} Y_q \ = \ Y_1 . \tag{1.1.0.6}$$

The object of this paper is to show that, in the $p-$adic framework, the dream of the lazy mathematician is realized : it is possible to choose the family of equations 1.1.0.1 in such a way that the solution Y_q is not only a "discretization" of the solution Y_1, but actually one has

$$Y_q = Y_1 \ , \tag{1.1.0.7}$$

for $|q-1|$ sufficiently small. We show that the class of $q-$difference equations, for which this principle holds, contains the family of equations, that we call *Taylor admissible*, which, roughly speaking, are defined by the fact that they admit a "$q-$Taylor solution" at every point (cf. 7.1.1). Observe that, under the condition 1.1.0.7, the family 1.1.0.1 is uniquely determined by the matrix Y_1, by the relation $A_q(T) = \sigma_q(Y_1) \cdot Y_1^{-1}$.

This shows that, given the (admissible) differential equation $\delta_1 - G(T)$, there exists a canonical family of $q-$difference equations attached to it, and it is characterized by the fact that the Taylor solution of the differential equation is simultaneously solution of every equation of this family of $q-$difference equations.

Example 1.1.1. We consider the rank one differential equation $\delta_1 + \pi T^{-1}$, where π is "the π of Dwork" (i.e. is a solution of $\pi^{p-1} = -p$). Its Taylor solution at infinity is

$$y(T) := \exp(\pi T^{-1}). \tag{1.1.1.1}$$

We consider this equation over the ring of analytic function $\mathrm{B} := \mathcal{A}_K(I)$ on an annulus $\mathcal{C}(I_\varepsilon)$, where I_ε is the interval $]1-\varepsilon, 1[$, with $0 < \varepsilon < 1$. Then the canonical family of $q-$difference equations attached to $\delta_1 + \pi T^{-1}$ is given by

$$\{ \ \sigma_q - A(q,T) \ \}_{q \in \mathrm{D}^+(1,\tau)} \ , \tag{1.1.1.2}$$

where $\tau = 1 - \varepsilon$, and

$$A(q,T) = \exp(\pi(q^{-1} - 1)T^{-1}) \ . \tag{1.1.1.3}$$

Indeed the solution of $\sigma_q - \exp(\pi(q^{-1} - 1)T^{-1})$ is always $\exp(\pi T^{-1})$, and $\exp(\pi(q^{-1} - 1)T^{-1}) \in \mathcal{A}_K(I)^\times$, for all $q \in \mathrm{D}^+(1, \tau)$.

1.1.1 Admissible $\sigma-$modules

Let $(K, |.|)$ be a spherically closed field of characteristic 0, with residual field of characteristic $p > 0$. Let A be a bounded affinoïd (cf. 2.1.2.1). Denote by $\mathcal{H}(A)$ the ring of analytic elements on A (cf. 2.1.3.1). Let $\mathcal{Q}$ be the subgroup of K formed by numbers $q \in K$ such that σ_q acts on $\mathcal{H}_K(A)$. We put on $\mathcal{Q}$ the topology induced by the ultrametric value of K. In the most important examples $\mathcal{Q}$ is an open Lie subgroup of K.

Now fix a q_0 in the Lie group $\mathcal{Q}$. Suppose that q_0 is not a root of unity, and consider a q_0-difference equation :

$$\sigma_{q_0}(Y) = A_{q_0}(T)Y \ , \tag{1.1.1.4}$$

with $A_{q_0}(T) \in GL_n(\mathcal{H}(A))$. Denote by $Y(T,c)$ the Taylor solution of that equation, at $c \in A(K)$. We show that, under certain conditions of convergence of the Taylor solution $Y(T,c)$, there exist an open subgroup $U \subseteq \mathcal{Q}$ containing q_0, and a canonical family of equations

$$\{\sigma_q(Y) = A(q,T)Y\}_{q \in U} \ , \tag{1.1.1.5}$$

characterized by the fact that $Y(T,c)$ is solution of every equation of this family (in particular $A(q_0,T) = A_{q_0}(T) = Y(q_0 T) \cdot Y(T)^{-1}$). The strong deformation consists actually in pass from an equation to another of this canonical family.

Starting from this fact we introduce a category of "sheaves", called *admissible $\sigma-$modules*, on the metric space $\mathcal{Q}$. We denote this category by

$$\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}}_{\mathcal{Q}}. \tag{1.1.1.6}$$

An admissible $\sigma-$module on an open subset $U \subseteq \mathcal{Q}$ is a free $\mathcal{H}_K(A)-$module, together with an action of σ_q, for all $q \in U$, depending "*analytically*" on q, and admitting a "solution" which is simultaneously solution of every $q-$difference equation defined by M.

Admissible $\sigma-$modules form actually a full sub-category of the category of "sheaves" of modules over the sheaf of rings $\mathcal{O}_{\mathcal{Q}}$ on $\mathcal{Q}$, defined as

$$\mathcal{O}_{\mathcal{Q}}(U) := \mathrm{B}[\{\sigma_q, \sigma_q^{-1}\}_{q \in U}] \ , \quad U \subseteq \mathcal{Q} \ . \tag{1.1.1.7}$$

with the evident relations $\sigma_q \circ f(T) = \sigma_q(f(T)) \circ \sigma_q$, for all $q \in U$. Observe that to the inclusion $U \subset V$ corresponds the inclusion $\mathcal{O}_{\mathcal{Q}}(U) \subset \mathcal{O}_{\mathcal{Q}}(V)$, hence the $\mathcal{O}_{\mathcal{Q}}$ is a *covariant* functor on the category of open subsets of $\mathcal{Q}$. If one shall works with "real" sheaves one must consider the opposite category.

Admissible $\sigma-$modules have a "mixed" nature, in the sense that :

- if q is not a root of unity, then the stalk at a $q \in \mathcal{Q}$ of such an object is a *$q-$difference equation* ;
- the stalk at 1 is a *differential equation* ;
- the stalk at a root of unity $\xi \in \mathcal{Q}$ is a "B*$-$module together with the action of the finite order operator σ_ξ and the action of $\delta_1 := T\frac{d}{dT}$*".

It happens that every admissible $\sigma-$module is a "constant" sheaf in the sense that such a sheaf is completely given by its stalk at some q. This fact implies that two stalks of the same $\sigma-$module are canonically related. The language is then particularly simple.

We are leaded then to define the *strong deformation* functor $\mathrm{Def}_{q,q'}$ as the functor which sends the stalk at q into the stalk at q' of the same sheaf. Roughly speaking the strong deformation functor sends a $q-$equation into the $q'-$equation having the same solutions.

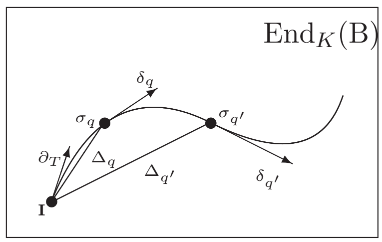
1.2 $q-$tangent operators

The "singularities" of these sheaves are the roots of 1. In the particular case in which the root is *equal* to 1, then $\sigma_1 = \mathrm{Id}$ is the identity, and the formula $\Delta_1 := \frac{\sigma_1 - \mathrm{Id}}{1-1}$ has no meaning.

If $q = 1$, the expected object is a differential equation : the operator Δ_q must be replaced by δ_1. We must look at a neighborhood of $q = 1$ and not only at the point $q = 1$. In other words the derivative δ_1 can be *defined* by using Δ_q as follows

$$\delta_1 := \lim_{q \to 1} \Delta_q = \lim_{q \to 1} \frac{\sigma_q - \mathrm{Id}}{q-1} \ . \tag{1.2.0.8}$$

The idea is now that the operator Δ_q fails if q is a p^n-th root of 1, and it must be replaced by another operator. The main idea borns from the following picture :



Definition 1.2.1. For all $q \in K$, we define the *$q-$tangent operator* δ_q : $\mathcal{H}_K(A) \to \mathcal{H}_K(A)$ as the derivative at q of the map $\sigma : K \to \mathrm{End}_K^{\mathrm{cont}}(\mathcal{H}_K(A))$, $q \mapsto \sigma_q$. More precisely

$$\delta_q := q \cdot \lim_{q' \to q} \frac{\sigma_{q'} - \sigma_q}{q' - q} \ , \tag{1.2.1.1}$$

where the limit is taken with respect to the simple convergence topology.

One find that

$$\delta_q = \sigma_q \circ \delta_1 = \delta_1 \circ \sigma_q \ . \tag{1.2.1.2}$$

Then δ_q verifies

$$\delta_q(fg) = \delta_q(f) \cdot \sigma_q(g) + \sigma_q(f) \cdot \delta_q(g) \ . \tag{1.2.1.3}$$

If $q = \xi$ is a root of 1, then the category of $\sigma_\xi-$modules (i.e. $\xi-$differences equations) is completely different from the case in which q is not a root of 1. The main problem is that this category is not $K-$linear, since the ring of endomorphisms of the unit object is not reduced to K. In other words, the ring of constants of σ_q is not reduced to K. To solve this problem we will consider a new definition of "constants" :

Definition 1.2.2. An element $b \in \mathrm{B}$ is called a $(\sigma_q, \delta_q)-$constant over U if both

$$\begin{cases} \sigma_q(b) & = & b \\ \delta_q(b) & = & 0 \end{cases} \tag{1.2.2.1}$$

are verified.

In the most important cases, the ring of $(\sigma_q, \delta_q)-$constants of B is reduced to K, even if q is a root of 1.

These notions lead to define a new kind of mixed objects called *admissible* $(\sigma, \delta)-$*modules on* U. An admissible $(\sigma, \delta)-$module on U is an admissible $\sigma-$module, together with an action of δ_1 (and hence of $\delta_q = \sigma_q \circ \delta_1$). The Taylor solution of the $\sigma-$module is then automatically also solution of the differential equation defined by the operator δ_1 (cf. 6.1.10).

Analytic $(\sigma, \delta)-$modules coincide actually with analytic $\sigma-$modules, since the operator δ_q can be defined from σ_q by the formula 1.2.1.1. They are introduced since they describe better the situation on the root of unity.

Then the "stalk" of a $\sigma-$module at a root of unity $q = \xi$ is a module on $\mathcal{H}_K(A)$, together with the actions of the two operators σ_ξ and δ_ξ.

Example 1.2.3. We give now an example of confluence which is not of *strong* type. Consider the family

$$\{\sigma_q - \widetilde{A}(q, T)\}_{q \in \mathrm{D}^-(1,\tau)} , \tag{1.2.3.1}$$

where $\tau \leq 1 - \varepsilon$, and

$$\widetilde{A}(q, T) := \exp((1 - q)\pi T^{-1}) . \tag{1.2.3.2}$$

We consider this family over the ring $\mathcal{A}_K(I_\varepsilon)$, as in the example 1.1.1.

The family 1.2.3.1 defines a so called *analytic* $\sigma-$module on $\mathrm{D}^-(1, 1)$, which is not admissible in this case. Indeed the solution of $\sigma_q - \widetilde{A}(q, T)$ is

$$\widetilde{y}_q(T) := \exp(q\pi T^{-1}) , \tag{1.2.3.3}$$

and is not "constant with q". Observe that, for all $q \in \mathrm{D}^-(1, \tau)$, the $q-$difference equation $\sigma_q - \widetilde{A}(q, T)$ is isomorphic to the $q-$difference equation $\sigma_q - A(q, T)$ defined in the example 1.1.1. Indeed, the function

$$\frac{\exp(q\pi T^{-1})}{\exp(\pi T^{-1})} = \exp((q - 1)\pi T^{-1}) \tag{1.2.3.4}$$

belongs to $\mathcal{A}_K(I_\varepsilon)^\times$. But the two $\sigma-$modules are not isomorphic since there is no analytic function $\theta(T)$ in $\mathcal{A}_K(I_\varepsilon)^\times$ which verifies simultaneously

$$\theta(T) \cdot \widetilde{y}_q(T) = \exp(\pi T^{-1}) , \tag{1.2.3.5}$$

for all $q \in \mathrm{D}(1, \tau)$.

The stalk at $q = 1$ of this $\sigma-$module is the differential equation $\delta_1 - G(1,T)$, where $G(1,T)$ is defined by

$$G(1,T) := \lim_{q\to 1} \frac{\widetilde{A}(q,T) - 1}{q - 1} = -\pi T^{-1} \ . \tag{1.2.3.6}$$

If q goes to 1, then $\lim_{q\to 1} y_q(T) = \exp(\pi T^{-1})$.

Remark 1.2.4. We discuss in 8.4 the relation between our strong confluence and the confluence defined in [ADV04].

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Chapitre 2

Notations

We set

$$\mathbb{R}_{\geq} := \{r \in \mathbb{R} \mid r \geq 0\} \ . \tag{2.0.4.1}$$

2.1 Rings of function

Let $R > 0$ be a real number. We define the ring of analytic functions on the disk $\mathrm{D}^{-}(c, R)$, $c \in K$, as

$$\mathcal{A}_K(c, R) := \{\sum_{n \geq 0} a_n (T - c)^n \mid a_n \in K, \liminf_n |a_n|^{-1/n} \geq R\} \ . \tag{2.1.0.2}$$

Its topology is given by the family of norms $|\sum a_i (T - c)^i|_{(c,\rho)} := \sup |a_i| \rho^i$, for all $\rho < R$. Let $\emptyset \neq I \subseteq \mathbb{R}_{\geq 0}$ be some interval. We denote the annulus relative to I by

$$\mathcal{C}_K(I) := \{x \in K \mid |x| \in I\} \ . \tag{2.1.0.3}$$

Remark 2.1.1. By $\mathcal{C}(I)$, without the index K, we mean the annulus itself and not its $K-$valued points.

The ring of analytic power series on $\mathcal{C}(I)$ is

$$\mathcal{A}_K(I) := \{\sum_{i \in \mathbb{Z}} a_i T^i \mid a_i \in K, \lim_{i \to \pm\infty} |a_i| \rho^i = 0, \text{ for all } \rho \in I\} \ . \tag{2.1.1.1}$$

We have $|\sum_i a_i T^i|_\rho := \sup_i |a_i| \rho^i < +\infty$, for all $\rho \in I$. The ring $\mathcal{A}_K(I)$ is complete with respect to the topology given by the family of norms $\{|.|_\rho\}_{\rho \in I}$. Set

$$I_\varepsilon :=]1 - \varepsilon, 1[\ , \quad 0 < \varepsilon < 1 \ . \tag{2.1.1.2}$$

The Robba ring is defined as

$$\mathcal{R}_K := \bigcup_{\varepsilon > 0} \mathcal{A}_K(I_\varepsilon) \ , \tag{2.1.1.3}$$

and is endowed with the limit Frechet topology, and is complete.

2.1.1 Affinoïds

Definition 2.1.2. A $K-$*affinoïd* is an analytic subset of $\mathbb{P}^1$ defined by

$$A := \mathbb{P}^1 - \bigcup_{i=1}^{n} \mathrm{D}^-(c_i, R_i) , \tag{2.1.2.1}$$

for some $c_1, \dots, c_n \in K \cup \{\infty\}$, $R_1, \dots, R_n > 0$. For all ultrametric valued $K-$algebras $(L, |.|)$, one has

$$A(L) := \mathbb{P}^1(L) - \bigcup_{i=1}^{n} \mathrm{D}_L^-(c_i, R_i) , \tag{2.1.2.2}$$

where, if $c = \infty$, we set $\mathrm{D}_L^-(\infty, R) := \{x \in L \mid |x| > R^{-1}\}$.

Remark 2.1.3. We denote by A the $K-$affinoïd itself, and by $A(L)$ its $L-$rational points.

Let A be an affinoïd. We denote by

$$\mathcal{H}_K(A) \tag{2.1.3.1}$$

the completion of the ring $H^{\mathrm{rat}}(A)$ of rational fractions $f(T)$ in $K(T)$, without poles in $A(K^{\mathrm{alg}})$, with respect to the norm

$$\|f(T)\|_A := \sup_{x \in A(K^{\mathrm{alg}})} |f(x)| . \tag{2.1.3.2}$$

If $A = \mathbb{P}^1 - \big(\mathrm{D}^-(0, \rho_1) \cup \mathrm{D}^-(\infty, \rho_2^{-1})\big)$, then $\mathcal{H}_K(A) = \mathcal{A}_K(I)$, where $I = [\rho_1, \rho_2]$.

Let $\varepsilon > 0$ be a "small" real number. If $A = \mathbb{P}^1 - \bigcup_{i=1}^{n} \mathrm{D}^-(c_i, R_i)$, then we set

$$A_\varepsilon := \mathbb{P}^1 - \bigcup_{i=1}^{n} \mathrm{D}^-(c_i, R_i - \varepsilon) . \tag{2.1.3.3}$$

We set then

$$\mathcal{H}_K^\dagger(A) := \bigcup_{\varepsilon > 0} \mathcal{H}_K(A_\varepsilon) . \tag{2.1.3.4}$$

The ring $\mathcal{H}_K^\dagger(A)$ is complete with respect to the limit topology. We set

$$\mathcal{H}_K := \mathcal{H}_K(A_1) \quad , \qquad \mathcal{H}_K^\dagger := \mathcal{H}_K^\dagger(A_1) , \tag{2.1.3.5}$$

where $A_1 := \{x \mid |x| = 1\}$.

2.2 Norms and radii of convergence

2.2.1 Logarithmic properties

We recall that if $r \mapsto N(r) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a function, then the log $-$function attached to N is the function $\widetilde{N} : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$, defined by $\widetilde{N}(t) := \log(N(\exp(t)))$:

$$\begin{tikzcd} \mathbb{R}_{\geq 0} \arrow[r, "N"] & \mathbb{R}_{\geq 0} \arrow[d, "\wr" ', "\log"] \\ \mathbb{R} \cup \{-\infty\} \arrow[u, "\exp", "\wr" '] \arrow[r, dotted, "\widetilde{N}"] & \mathbb{R} \cup \{-\infty\} . \end{tikzcd} \tag{2.2.0.6}$$

We will say that N has logarithmically a given property if $\widetilde{N}$ has that property.

2.2.2 Norms

The absolute value on K, or, more generally, on every extension field Ω/K, will be extended to a norm on $M_n(\Omega)$, by setting $\|(a_{i,j})\|_\Omega := \max_{i,j} |a_{i,j}|_\Omega$.

Let $f(T) = \sum_{i \in \mathrm{Z}} a_i (T-c)^i$, $a_i \in K$, be a formal series. We set

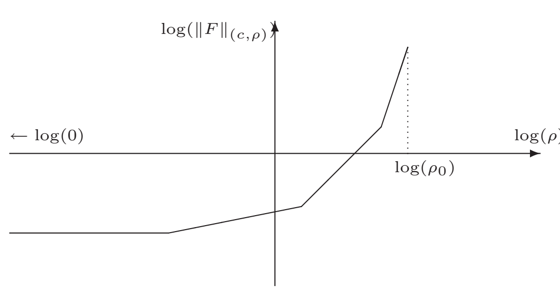
$$|f|_{(c,\rho)} := \sup_i |a_i| \rho^i \ , \tag{2.2.0.7}$$

this number can be equal to $+\infty$. If $F(T) = (f_{h,k}(T))_{h,k}$, is a matrix, we set

$$\|F\|_{(c,\rho)} := \max_{h,k} |f_{h,k}|_{(c,\rho)} \quad . \tag{2.2.0.8}$$

Lemma 2.2.1 ([CR94, ch.II]). *Let $F(T) \in M_n(K[[T-c]])$. Suppose that $\|F\|_{(c,\rho)} < \infty$, for all $\rho < \rho_0$. Then the function $\rho \mapsto \|F\|_{(c,\rho)} : [0, \rho_0[\longrightarrow \mathbb{R}_{\geq 0}$ is log-convex, piecewise log-affine, and log-increasing. Moreover*

$$\|F(T)\|_{(c,\rho)} = \sup_{|x-c| \leq \rho, x \in K^{\mathrm{alg}}} \|F(x)\|_{K^{\mathrm{alg}}} \ . \tag{2.2.1.1}$$



2.2.3 Radii

Let $f(T) = \sum_{i\geq 0} a_i(T-c)^i$, $a_i \in K$ be a formal series. The radius of convergence of $f(T)$ at c is

$$Ray_c(\ f(T)\ ) := \liminf_{i\geq 0} |a_i|^{-1/i}\ . \qquad (2.2.1.2)$$

If $F(T) = (f_{h,k}(T))_{h,k}$, is a matrix, then we set

$$Ray_c(F(T)) := \min_{h,k}\ Ray_c(\ f_{h,k}(T)\ )\ . \qquad (2.2.1.3)$$

Chapitre 3

Discrete or analytic $\sigma-$modules and $(\sigma, \delta)-$modules

Definition 3.0.2. Let B be one of the rings of 2.1. We denote by

$$\mathcal{Q}(\mathrm{B}) \tag{3.0.2.1}$$

the set of $q \in K$ such that $\sigma_q : f(T) \mapsto f(qT)$ is an automorphism of B, and that $|q-1| < 1$. We will write $\mathcal{Q}$ if no confusion is possible.

Remark 3.0.3. Clearly $\mathcal{Q}$ is a topological group and contains always a disk $\mathrm{D}^-(1, \tau_0)$, for some $\tau_0 > 0$. One has

$$\mathcal{Q}(\mathcal{A}_K(I)) = \mathcal{Q}(\mathcal{R}_K) = \mathcal{Q}(\mathcal{H}_K^\dagger) = \{q \in K \mid |q| = 1\} . \tag{3.0.3.1}$$

If A is a *bounded* affinoïd (i.e. contained in some disk $\mathrm{D}^-(0, r)$, $r > 0$), and if $\mathrm{B} = \mathcal{H}_K^\dagger(A)$, then

$$\mathcal{Q}(\mathcal{H}_K(A)) \subseteq \{q \in K \mid |q| = 1\} . \tag{3.0.3.2}$$

Observe that if there exists a $q \in \mathcal{Q}(\mathcal{H}_K(A))$ such that $|q| \neq 1$, then $A = \mathbb{P}^1_K$.

Definition 3.0.4. Let $S \subseteq \mathcal{Q}$ be a subset. We denote by

$$\langle S \rangle \tag{3.0.4.1}$$

the subgroup of $\mathcal{Q}$ generated by S. Let $\boldsymbol{\mu}(\mathcal{Q})$ be the set of all roots of 1 belonging to $\mathcal{Q}$. Then we set

$$S^\circ := S - \boldsymbol{\mu}(Q) . \tag{3.0.4.2}$$

Let $U \subset \mathcal{Q}$ be an open subset with respect to the "*naïve*" $p-$adic topology of $\mathcal{Q}$, induced by the absolute value of K. Let $S \subset \mathcal{Q}$ be an arbitrary sub-set.

In this section we define the following categories

$$\begin{array}{rcl}\sigma-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}} & = & \text{discrete } \sigma-\text{modules over } S\ ,\\ \sigma-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} & = & \text{analytic } \sigma-\text{modules over } U\ ,\\ (\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}} & = & \text{discrete } (\sigma,\delta)-\text{modules over } S\ ,\\ (\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} & = & \text{analytic } (\sigma,\delta)-\text{modules over } U\ .\end{array}$$

Remark 3.0.5. Recall that, by assumption, K is spherically closed, and that every finite dimensional free $\mathrm{B}-$module M has the product topology.

3.1 Discrete $\sigma-$modules

Definition 3.1.1 (definition of the category of discrete $\sigma-$modules)**.**

1. Objects : Let $S\subset\mathcal{Q}$ be an arbitrary subset. An object of σ–$\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is a finite dimensional free $\mathrm{B}-$module M, together with a group morphism

$$\sigma^{\mathrm{M}}:\langle S\rangle\xrightarrow[q\mapsto\sigma_q^{\mathrm{M}}]{}\mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})\ , \tag{3.1.1.1}$$

such that, for all $q\in S$, the operator σ_q^{M} is σ_q-semi-linear, that is

$$\sigma_q^{\mathrm{M}}(fm)=\sigma_q(f)\cdot\sigma_q^{\mathrm{M}}(m)\ , \tag{3.1.1.2}$$

for all $f\in\mathrm{B}$, and all $m\in\mathrm{M}$. Objects (M,σ^{M}) in $\sigma-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ will be called *discrete $\sigma-$modules over S*.

2. Morphisms : A morphism between $(\mathrm{M},\sigma^{\mathrm{M}})$ and $(\mathrm{N},\sigma^{\mathrm{N}})$ is a $\mathrm{B}-$linear map $\alpha:\mathrm{M}\to\mathrm{N}$ such that

$$\alpha\circ\sigma_q^{\mathrm{M}}=\sigma_q^{\mathrm{N}}\circ\alpha\ , \tag{3.1.1.3}$$

for all $q\in S$. We will denote the $K-$vector space of morphisms by

$$\mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{N})\ . \tag{3.1.1.4}$$

Notation 3.1.2. If $S=\{q\}$ is reduced to a point, then the category of discrete $\sigma-$modules over $\{q\}$ is the usual category of $q-$difference modules. We will use then a simplified notation :

$$\sigma_q-\mathrm{Mod}(\mathrm{B})\ \ =\ \ \sigma-\mathrm{Mod}(\mathrm{B})_{\{q\}}^{\mathrm{disc}}\ . \tag{3.1.2.1}$$

Remark 3.1.3. 1.— Conditions 3.1.1.2 and 3.1.1.3 for $q\in S$ implies the same conditions for all $q\in\langle S\rangle$.

2.— If $\mathrm{M}\neq 0$, then σ^{M} is automatically injective. Indeed, the equality $\sigma_q^{\mathrm{M}}(fm)=\sigma_{q'}^{\mathrm{M}}(fm)$, for all $f\in\mathrm{B}$ and all $m\in\mathrm{M}$, implies $\sigma_q(f)\sigma_q^{\mathrm{M}}(m)=$

$\sigma_{q'}(f)\sigma_{q'}^{\mathrm{M}}(m)$, and hence the contradiction $\sigma_q(f) = \sigma_{q'}(f)$, because B is integral and M is free.

3.— The morphism σ^{M} is determined by its restriction to the set S. Reciprocally, if a map $S \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ is given, then this map extends to a group morphism $\langle S\rangle \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ if and only if the following conditions are verified

i. $\sigma_q^{\mathrm{M}} \circ \sigma_{q'}^{\mathrm{M}} = \sigma_{q'}^{\mathrm{M}} \circ \sigma_q^{\mathrm{M}}$, for all $q, q' \in S$;

ii. If $\exists\, n, m \in \mathbb{Z}$, $\exists\, q_1, q_2 \in S$, such that $q_1^n = q_2^m$, then $(\sigma_{q_1}^{\mathrm{M}})^n = (\sigma_{q_2}^{\mathrm{M}})^m$;

iii. If $1 \in S$, then $\sigma_1^{\mathrm{M}} = \mathrm{Id}$.

3.1.1 Matrices of σ^{M}

Let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. In this basis σ_q^{M} acts as

$$\sigma_q^{\mathrm{M}}(f_1, \ldots, f_n) = (f_1, \ldots, f_n) \cdot A(q, T) , \tag{3.1.3.1}$$

where the matrix $A(q,T) := (a_{i,j}(q,T))_{i,j}$ is defined by the relation $\sigma_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j a_{i,j}(q,T) \cdot \mathrm{e}_j$. By definition, if $q = 1$, we have $A(1,T) = \mathrm{Id}$. One has

$$A(qq', T) = A(q', qT) \cdot A(q, T) . \tag{3.1.3.2}$$

3.1.2 Internal Hom and $\otimes$

Let $(\mathrm{M}, \sigma^{\mathrm{M}})$, $(\mathrm{N}, \sigma^{\mathrm{N}})$ be two discrete $\sigma-$modules over S.

Definition 3.1.4 ($\sigma - \mathrm{Hom}$)**.** We define a structure of discrete $\sigma-$module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N})$ by setting

$$\sigma_q^{\mathrm{Hom}(\mathrm{M},\mathrm{N})}(\alpha) := \sigma_q^{\mathrm{N}} \circ \alpha \circ (\sigma_q^{\mathrm{M}})^{-1} , \tag{3.1.4.1}$$

for all $q \in S$, and all $\alpha \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N})$.

Definition 3.1.5 ($\sigma - \otimes$)**.** We define on $\mathrm{M} \otimes_{\mathrm{B}} \mathrm{N}$ a structure of discrete $\sigma-$module over S by setting

$$\sigma_q^{\mathrm{M}\otimes\mathrm{N}}(m \otimes n) := \sigma_q^{\mathrm{M}}(m) \otimes \sigma_q^{\mathrm{N}}(n) , \tag{3.1.5.1}$$

for all $q \in S$, and all $m \in \mathrm{M}$, $n \in \mathrm{N}$.

Remark 3.1.6. If $S^\circ \neq \emptyset$ (cf. 3.0.4.2), then the category $\sigma - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is $K-$linear. Moreover, since B is a Bezout ring (i.e. every finitely generated ideal of B is principal), then the category is Tannakian.

3.2 Discrete $(\sigma, \delta)-$modules

Let $S \subset \mathcal{Q}(\mathrm{B})$ be an arbitrary subset.

Definition 3.2.1 (discrete $(\sigma,\delta)-$modules)**.**
1. Objects : An object of

$$(\sigma,\delta)\text{–Mod}(\mathrm{B})_S^{\mathrm{disc}} \tag{3.2.1.1}$$

is a $\sigma-$module over S, together with a connection[1]

$$\delta_1^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}\ . \tag{3.2.1.2}$$

Objects $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}})$ of $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ will be called discrete $(\sigma,\delta)-$*modules over* S.

2. Morphisms : A morphism between $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}})$ and $(\mathrm{N}, \sigma^{\mathrm{N}}, \delta_1^{\mathrm{N}})$ is a morphism $\alpha : (\mathrm{M}, \sigma^{\mathrm{M}}) \to (\mathrm{N}, \sigma^{\mathrm{N}})$ of discrete $\sigma-$modules satisfying also

$$\alpha \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{N}} \circ \alpha\ . \tag{3.2.1.3}$$

We will denote the $K-$vector space of morphisms by

$$\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})\ . \tag{3.2.1.4}$$

Remark 3.2.2. If $S = \{q\}$ is reduced to a point, then, by analogy with 3.1.2.1 we will use the simplified notation :

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}) \ := \ (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_{\{q\}}^{\mathrm{disc}}\ . \tag{3.2.2.1}$$

Remark 3.2.3. Suggested by the definition of 1.2.1.1, we introduce the operator

$$\delta_q^{\mathrm{M}} := \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}\ . \tag{3.2.3.1}$$

Then one has

$$\delta_q^{\mathrm{M}}(f \cdot m) = \sigma_q(f) \cdot \delta_q^{\mathrm{M}}(m) + \delta_q(f) \cdot \sigma_q^{\mathrm{M}}(m)\ , \tag{3.2.3.2}$$

for all $f \in \mathrm{B}$, all $m \in \mathrm{M}$, and all $q \in \langle S \rangle$. Moreover, for all $\alpha \in \mathrm{Hom}^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$, and all $q \in \langle S \rangle$, one has

$$\alpha \circ \delta_q^{\mathrm{M}} = \delta_q^{\mathrm{N}} \circ \alpha\ . \tag{3.2.3.3}$$

We heuristically imagine M as endowed with the map $q \mapsto \delta_q^{\mathrm{M}} : \langle S \rangle \to \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$. This is the reason for the notation 3.2.1.1, and 3.2.2.1.

[1] By definition δ_1^{M} is a connection if and only if $\delta_1^{\mathrm{M}}(fm) = \delta_1(f) \cdot m + f \cdot \delta_1^{\mathrm{M}}(m)$, for all $f \in \mathrm{B}$, and all $m \in \mathrm{M}$. Recall that $\delta_1 := T\frac{d}{dT}$.

3.2.1 Matrices of δ_q^{M}

Let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. Let $A(q,T) \in GL_n(\mathrm{B})$ be the matrix of σ_q^{M} in the basis $\mathbf{e}$ (cf. 3.1.3.1). Then δ_q^{M} acts in the basis $\mathbf{e}$ as :

$$\delta_q^{\mathrm{M}}(f_1, \ldots, f_n) = (\delta_q(f_1), \ldots, \delta_q(f_n)) \cdot A(q,T) + (\sigma_q(f_1), \ldots, \sigma_q(f_n)) \cdot G(q,T) \ , \tag{3.2.3.4}$$

where the matrix $G(q,T) = (g_{i,j}(q,T))_{i,j}$ is defined by $\delta_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j g_{i,j}(q,T) \cdot \mathrm{e}_j$. One has

$$G(q' \cdot q, T) = G(q', qT) \cdot A(q,T) \ . \tag{3.2.3.5}$$

3.2.2 Internal Hom and $\otimes$

Let $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}})$, $(\mathrm{N}, \sigma^{\mathrm{N}}, \delta^{\mathrm{N}})$ be two discrete $(\sigma, \delta)-$modules over S.

Definition 3.2.4 ($(\sigma,\delta)-$Hom)**.** We define a structure of discrete $(\sigma,\delta)-$module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N})$ by setting

$$\delta_q^{\mathrm{Hom(M,N)}}(\alpha) := \left(\delta_q^{\mathrm{N}} \circ \alpha - \sigma_q^{\mathrm{Hom(M,N)}}(\alpha) \circ \delta_q^{\mathrm{M}}\right) \circ (\sigma_q^{\mathrm{M}})^{-1} \ . \tag{3.2.4.1}$$

This definition gives the following relation in which $\mathrm{H} := \mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N})$

$$\delta_q^{\mathrm{N}}(\alpha \circ m) = \sigma_q^{\mathrm{H}}(\alpha) \circ \delta_q^{\mathrm{M}}(m) + \delta_q^{\mathrm{H}}(\alpha) \circ \sigma_q^{\mathrm{M}}(m) \ , \tag{3.2.4.2}$$

for all $\alpha \in \mathrm{Hom}(\mathrm{M}, \mathrm{N})$, and all $m \in \mathrm{M}$.

Definition 3.2.5 ($(\sigma,\delta)-\otimes$)**.** We define on $\mathrm{M} \otimes_{\mathrm{B}} \mathrm{N}$ a structure of discrete $(\sigma,\delta)-$module over S by setting

$$\delta_q^{\mathrm{M}\otimes\mathrm{N}}(m \otimes n) := \delta_q^{\mathrm{M}}(m) \otimes \sigma_q^{\mathrm{N}}(n) + \sigma_q^{\mathrm{M}}(m) \otimes \delta_q^{\mathrm{N}}(n) \ , \tag{3.2.5.1}$$

for all $q \in S$, and all $m \in \mathrm{M}$, $n \in \mathrm{N}$.

Remark 3.2.6. The category $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is $K-$linear and Tannakian.

3.3 Analytic $\sigma-$modules

The definition of analytic $\sigma-$modules is given only for the rings $\mathrm{B} := \mathcal{A}_K(I)$, $\mathcal{H}_K(A)$, $\mathcal{H}_K$, $\mathcal{H}_K^{\dagger}$, $\mathcal{R}_K$.

Let $I \subset \mathbb{R}_{\geq 0}$ be some interval and let A be an affinoïd. Let $\mathrm{B} = \mathcal{A}_K(I)$ (resp. $\mathrm{B} = \mathcal{H}_K(A)$), and let $U \subset \mathcal{Q}(\mathrm{B})$ be an open subset. Then the subgroup $\langle U \rangle \subseteq \mathcal{Q}(\mathrm{B})$ generated by U is open, i.e. $\langle U \rangle$ contains a disk $\mathrm{D}_K^-(1, \tau)$, for some $\tau > 0$.

Definition 3.3.1. Let $\mathrm{B} := \mathcal{A}_K(I)$ (resp. $\mathrm{B} = \mathcal{H}_K(A)$). Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be a discrete $\sigma-$module over U. Let

$$A(q,T) \in GL_n(\mathrm{B}) \tag{3.3.1.1}$$

be the matrix of σ_q^{M} in a fixed basis. We will say that $(\mathrm{M}, \sigma^{\mathrm{M}})$ is an *analytic* $\sigma-$*module* if, for all $q \in U$, there exist a disk $\mathrm{D}^-(q, \tau_q) = \{q' \in K \mid |q'-q| < \tau_q\}$, with $\tau_q > 0$, and a matrix

$$A_q(Q,T) \tag{3.3.1.2}$$

such that :

1. $A_q(Q,T)$ is an analytic element in the domain

$$\begin{aligned} (Q,T) \ &\in\ \mathrm{D}^-(q,\tau_q) \times \mathcal{C}(I) & (3.3.1.3)\\ (\text{resp.}\quad (Q,T) \ &\in\ \mathrm{D}^-(q,\tau_q) \times A\ )\ ; & (3.3.1.4)\end{aligned}$$

2. For all $q' \in \mathrm{D}^-(q,r)$, one has

$$A_q(q',T) = A(q',T)\ . \tag{3.3.1.5}$$

This definition does not depend on the chosen basis $\mathbf{e}$. We define

$$\sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \tag{3.3.1.6}$$

as the full sub-category of $\sigma-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{disc}}$, whose objects are analytic $\sigma-$modules.

Remark 3.3.2. If $(\mathrm{M}, \sigma^{\mathrm{M}})$ and $(\mathrm{N}, \sigma^{\mathrm{N}})$ are two analytic $\sigma-$modules over U, then $(\mathrm{Hom}(\mathrm{M},\mathrm{N}), \sigma^{\mathrm{Hom}(\mathrm{M},\mathrm{N})})$ and $(\mathrm{M}\otimes\mathrm{N}, \sigma^{\mathrm{M}\otimes\mathrm{N}})$ are analytic. This follows from the explicit dependence of the matrix of $\sigma^{\mathrm{Hom}(\mathrm{M},\mathrm{N})}$ and $\sigma^{\mathrm{M}\otimes\mathrm{N}}$ on terms of the matrix of σ^{M} and σ^{N}.

3.3.1 Discrete and analytic $\sigma-$modules over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$

Definition 3.3.3. Let $S \subseteq \mathcal{Q}$ be a subset, and let $U \subseteq \mathcal{Q}$ be an open subset. We set

$$\begin{aligned} \sigma - \mathrm{Mod}(\mathcal{R}_K)_U^{\mathrm{an}} \ &:=\ \bigcup_{\varepsilon>0} \sigma - \mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_U^{\mathrm{an}}\ ; & (3.3.3.1)\\ \sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{disc}} \ &:=\ \bigcup_{\varepsilon>0} \sigma - \mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_S^{\mathrm{disc}}\ . & (3.3.3.2)\end{aligned}$$

We set also

$$\begin{aligned} \sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)_U^{\mathrm{an}} \ &:=\ \bigcup_{\varepsilon,\varepsilon'>0} \sigma - \mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1+\varepsilon'[))_U^{\mathrm{an}}\ , & (3.3.3.3)\\ \sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{\mathrm{disc}} \ &:=\ \bigcup_{\varepsilon,\varepsilon'>0} \sigma - \mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1+\varepsilon'[))_S^{\mathrm{disc}}\ . & (3.3.3.4)\end{aligned}$$

Remark 3.3.4. Since U is open, one has $U^\circ \neq \emptyset$. If $\mathrm{B} = \mathcal{R}_K$, $\mathcal{H}_K^\dagger$, $\mathcal{A}_K(I)$, $\mathcal{H}_K(A)$, then by the point 5 of remark 3.1.3, the category $\sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}}$ is always $K-$linear and Tannakian.

3.4 Analytic $(\sigma, \delta)-$modules

We maintain the previous notations. In this sub-section we define a *fully faithful* functor (cf. 3.4.3)

$$\sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \longrightarrow (\sigma, \delta) - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{disc}} \ , \tag{3.4.0.1}$$

which is a "local" section of the functor which "forgets" the action of δ.

Definition 3.4.1. We will call

$$(\sigma, \delta) - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \tag{3.4.1.1}$$

the essential image of this functor.

By definition, the functor which "forgets" the action of δ is an equivalence

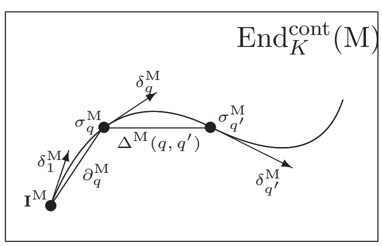
$$(\sigma, \delta) - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \xrightarrow[\sim]{\text{Forget } \delta} \sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \ . \tag{3.4.1.2}$$

3.4.1 Construction of δ

Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be an analytic $\sigma-$module. We shall define a $(\sigma, \delta)-$structure on M. The map $q \mapsto \sigma_q^{\mathrm{M}} : \langle U \rangle \to \mathrm{Aut}_K(\mathrm{M})$ is *derivable*, in the sense that, for all $q \in \langle U \rangle$, the limit

$$\delta_q^{\mathrm{M}} := q \cdot \lim_{q' \to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q' - q} \tag{3.4.1.3}$$

exists in $\mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$, with respect to the simple convergence topology. Moreover, for all $q \in \langle U \rangle$, the rule 3.2.3.2 holds, and $\delta_q^{\mathrm{M}} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}$.



Remark 3.4.2. A morphism between analytic $(\sigma, \delta)-$modules is, by definition, a morphism of *discrete* $(\sigma, \delta)-$modules.

Remark 3.4.3. Let $\varphi : (\mathrm{M}, \sigma^{\mathrm{M}}) \to (\mathrm{N}, \sigma^{\mathrm{N}})$ be a morphism of analytic $\sigma-$modules, that is $\varphi \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{N}} \circ \varphi$, for all $q \in U$. Passing to the limit in the definition 3.4.1.3, one has that φ commutes with δ_q^{M}, for all $q \in U$. Hence we obtain a bijective map

$$\mathrm{Hom}_U^{\sigma}(\mathrm{M}, \mathrm{N}) \xrightarrow[\varphi \mapsto \varphi]{\sim} \mathrm{Hom}_U^{(\sigma,\delta)}(\mathrm{M}, \mathrm{N}) \ . \tag{3.4.3.1}$$

Remark 3.4.4. If $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subset \mathrm{M}$ is a basis in which the matrix of σ_q^{M} is $A(q,T)$, then the matrix of δ_q^{M} is (cf. 3.2.3.4, 3.3.1.2)

$$G(q,T) \quad := \quad q \cdot \lim_{q' \to q} \frac{A(q',T) - A(q,T)}{q' - q} \quad = \quad \Big(\partial_Q(A_q(Q,T))\Big)_{|Q=q}, \tag{3.4.4.1}$$

where ∂_Q is the derivation $Q\frac{d}{dQ}$, and $A_q(Q,T)$ is the matrix 3.3.1.2.

Chapitre 4

Solutions (formal definition)

4.1 Discrete σ−algebras and (σ, δ)−algebras

Let again $S \subseteq \mathcal{Q}(\mathrm{B})$ be an arbitrary subset.

Definition 4.1.1 (Discrete σ−algebra over S)**.** A *discrete σ−algebra over S* is a B−algebra C satisfying :

1. C is an *integral domain*,
2. there exists a group morphism $\sigma^{\mathrm{C}} : \langle S \rangle \to \mathrm{Aut}_K(\mathrm{C})$ such that σ_q^{C} extends σ_q^{B}, for all $q \in \langle S \rangle$,
3. The following sub-ring of C

$$\mathrm{C}_S^{\sigma} := \{c \in \mathrm{C} \mid \sigma_q(c) = c\ ,\ \text{for all } q \in S\}\ , \tag{4.1.1.1}$$

   called *the ring of σ_q−constants*, must be equal to K.

In particular, if C is a free finite dimensional B−module, then it is a discrete σ−module. We will write σ_q instead of σ_q^{C}, if no confusion is possible.

Remark 4.1.2. Observe that no topology is required on C. The word *discrete* is employed, here and later on, to emphasize that we do not ask "continuity" with respect to q.

Remark 4.1.3. Suppose that $S = \{\xi\}$, with $\xi \in \boldsymbol{\mu}(\mathcal{Q})$. Since $\mathrm{B}_S^{\sigma} = \mathrm{B}^{\sigma_\xi} \neq K$, then B itself is not a discrete σ−algebra over S. Hence there is no discrete σ−algebra over S. On the other hand, if $S^{\circ} \neq \emptyset$ (cf. 3.0.4.2), then $\mathrm{B}_S^{\sigma} = K$, and B is a discrete σ−algebra over S.

Definition 4.1.4 (Discrete (σ, δ)−algebra over S)**.** A discrete (σ, δ)−algebra C over S is a B−algebra satisfying :

1. C verifies the properties 1. and 2. of the definition 4.1.1,
2. there exists a derivation δ_1^{C}, extending the derivation $\delta_1 = T\frac{d}{dT}$ on B,

3. the following sub-ring of C is equal to K :

$$\mathrm{C}_S^{(\sigma,\delta)} := \{f \in \mathrm{C} \mid f \in \mathrm{C}_S^{\sigma}\ ,\ \delta_1(f) = 0\}\ . \tag{4.1.4.1}$$

We will call $\mathrm{C}_S^{(\sigma,\delta)}$ the sub-ring of $(\sigma,\delta)-$constants of C on S. We will write δ_1 instead of δ_1^{C}, if no confusion is possible.

Remark 4.1.5. The operator $\delta_q^{\mathrm{C}} := \sigma_q^{\mathrm{C}} \circ \delta_1^{\mathrm{C}}$ satisfies analogous property of 3.2.3.2.
Moreover, since $\mathrm{B}_S^{(\sigma,\delta)} = K$, then B is always a $(\sigma,\delta)-$algebra over S, for all sub-set $S \subseteq \mathcal{Q}(\mathrm{B})$.

4.2 Discrete Solutions

Definition 4.2.1 (Discrete solutions on S)**.** Let $(\mathrm{M},\sigma^{\mathrm{M}})$ (resp.$(\mathrm{M},\sigma^{\mathrm{M}},\delta^{\mathrm{M}})$) be a *discrete* $\sigma-$module (resp. $(\sigma,\delta)-$module) over S, and let C be a discrete $\sigma-$algebra (resp. $(\sigma,\delta)-$algebra) over S. A *discrete solution* of M, with values in C, is a $B-$linear morphism

$$\alpha : \mathrm{M} \longrightarrow \mathrm{C} \tag{4.2.1.1}$$

such that $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$ (resp. α verifies simultaneously $\alpha \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{C}} \circ \alpha$, and $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$).

We denote by $\mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C})$ (resp. $\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C})$) the $K-$vector space of the solutions of M in C.

4.2.1 Matrices of solutions

Let M be a discrete $(\sigma,\delta)-$module over S (resp. let M be a discrete $\sigma-$module over S, with $S^{\circ} \neq \emptyset$). Let C be a discrete $(\sigma,\delta)-$algebra over S (resp. let C be a discrete $\sigma-$algebra over S).

Let $\mathbf{e} = \{\mathrm{e}_1,\ldots,\mathrm{e}_n\}$ be a basis of M, and let $A(q,T)$ (resp. $G(q,T)$) be the matrix of σ_q^{M} (resp. δ_q^{M}) in this basis (cf. 3.2.3.4). We identify a morphism $\alpha : \mathrm{M} \to \mathrm{C}$ with the vector $(y_i)_i \in \mathrm{C}^n$, given by $y_i := \alpha(\mathrm{e}_i)$. In this way discrete solutions become solutions in the usual *vector form.* Indeed

$$\begin{aligned} \begin{pmatrix} \sigma_q(y_1) \\ \vdots \\ \sigma_q(y_n) \end{pmatrix} &= A(q,T)\cdot\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\ , \quad \text{for all } q \in S\ , \\ \Bigg(\text{resp. } \begin{pmatrix} \delta_q(y_1) \\ \vdots \\ \delta_q(y_n) \end{pmatrix} &= G(q,T)\cdot\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\ , \quad \text{for all } q \in S\ .\Bigg) \end{aligned}$$

By a *fundamental matrix of solutions* of M (in the basis $\mathbf{e}$) we mean a matrix $Y \in GL_n(\mathrm{C})$ satisfying *simultaneously*

$$\sigma_q(Y) = A(q,T)\cdot Y\ , \quad \text{for all } q \in S\ , \tag{4.2.1.2}$$

(resp. satisfying *simultaneously*

$$\left\{\begin{array}{lcll}\sigma_q(Y) & = & A(q,T)\cdot Y\ , & \text{for all } q\in S\ , \\ \delta_1(Y) & = & G(1,T)\cdot Y\ . & \end{array}\right. \quad).\tag{4.2.1.3}$$

Following this identification, the sub-ring of $\sigma-$constants of B (resp. $(\sigma,\delta)-$constants of B) (cf 4.1.1.1 (resp. 4.1.4.1)) can be identified with the solutions of the unit object $\mathbb{I} = \mathrm{B}$, in the discrete $\sigma-$algebra $\mathrm{C} := \mathrm{B}$ (resp. $(\sigma,\delta)-$algebra $\mathrm{C} := \mathrm{B}$).

4.2.2 Dimension of the space of solutions

Remark 4.2.2. Let $F := \mathrm{Frac}(\mathrm{C})$ be the fraction field of C, then both σ_q and δ_1 extend to F (cf. [vdPS03, Ex.1.5]).

Lemma 4.2.3. *Let* M *be a $(\sigma,\delta)-$module (resp. $\sigma-$module) over S, and let* C *be a discrete $(\sigma,\delta)-$algebra (resp. $\sigma-$algebra) over S. One has*

$$\dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})\ .\tag{4.2.3.1}$$

(resp. if $S^\circ \neq \emptyset$ (cf. 3.0.4.2), then

$$\dim_K \mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})\ .\ )\tag{4.2.3.2}$$

Proof : The equation 4.2.3.1 is evident, since

$$\dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) \leq \dim_K \mathrm{Hom}^{\delta_1}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})\ .\tag{4.2.3.3}$$

On the other hand, if $q \in S^\circ$, then $\mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$. Hence

$$\dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) \leq \dim_K \mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})\ .\ \square\tag{4.2.3.4}$$

Chapitre 5

Discrete Strong Confluence

5.1 Constant $\sigma-$modules and $(\sigma,\delta)-$modules

Let B be one of the rings defined in 2.1 and let $S \subset \mathcal{Q}(\mathrm{B})$ be a subset.

Definition 5.1.1 (Constant $\sigma-$modules(resp. $(\sigma,\delta)-$modules)). Let M be a discrete $\sigma-$module over S (resp. $(\sigma,\delta)-$module). We will say that M is *constant* on S if there exists a discrete $\sigma-$algebra C over S (resp. a discrete $(\sigma,\delta)-$algebra) (cf. 4.1.1) such that

$$\dim_K \mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C}) = \mathrm{rk}_{\mathrm{B}}\mathrm{M} \tag{5.1.1.1}$$

$$(\text{resp. } \dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) = \mathrm{rk}_{\mathrm{B}}\mathrm{M}\ ) . \tag{5.1.1.2}$$

We will say that M is trivialized by C. The full sub-category of $\sigma-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$), whose objects are constant on S, and trivialized by C, will be denoted by

$$\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \tag{5.1.1.3}$$

$$(\text{resp.} \quad (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \quad) . \tag{5.1.1.4}$$

Remark 5.1.2. In other words, if $n = \mathrm{rk}_{\mathrm{B}}\mathrm{M}$, then the $\sigma-$module M (resp. $(\sigma,\delta)-$module M) is constant on S if there exists a matrix $Y \in GL_n(\mathrm{C})$ such that Y is *simultaneously* solution, for all $q \in S$, of the equations 4.2.1.2 (resp. both the equations 4.2.1.3). Roughly speaking M is constant on S if it admits a basis of $q-$solutions which "do not depend on $q \in S$".

Lemma 5.1.3. *Let* M, N *be two constant discrete $\sigma-$modules over S (resp. $(\sigma,\delta)-$modules over S). If* M,N *are constant, then* $\mathrm{M}\otimes\mathrm{N}$ *and* $\mathrm{Hom}(\mathrm{M},\mathrm{N})$ *are constant.*

Proof : The fundamental matrix solution of $\mathrm{M}\otimes\mathrm{N}$ (resp. $\mathrm{Hom}(\mathrm{M},\mathrm{N})$) is obtained by taking products of entries of the two matrices of solutions of M and N respectively. Hence "it does not depend on $q \in S$". □

Remark 5.1.4. Since B itself is constant, hence the dual of a constant module is constant.

Lemma 5.1.5. *Let $S' \subseteq S$ be a non empty subset. Let* C *be a discrete $(\sigma,\delta)-$algebra over S. Then the restriction functor, sending* $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ *into* $(\mathrm{M},\sigma^{\mathrm{M}}_{|_{\langle S'\rangle}},\delta_1^{\mathrm{M}})$ *:*

$$\mathrm{Res}^S_{S'} : (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow (\sigma,\delta)-\mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_{S'} \tag{5.1.5.1}$$

is fully faithful. In particular the restriction

$$\mathrm{Res}^S_{S'} : (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_{S'} \tag{5.1.5.2}$$

is fully faithful. The same facts are true for discrete $\sigma-$modules under the additional hypothesis : $(S')^\circ \neq \emptyset$.

Proof : The proof is formally the same in both cases, here we show the case of $(\sigma,\delta)-$modules. We must show that the morphism

$$\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{N}) \to \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M},\mathrm{N})$$

is an isomorphism, for all M, N in $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S$.

The $K-$vector space $\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{N})$ can be identified with the $K-$vector space of the solutions of $\mathrm{M}\otimes\mathrm{N}^\vee$ with values in B :

$$\begin{aligned}\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{N}) &= \mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ ; \\ \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M},\mathrm{N}) &= \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ .\end{aligned} \tag{5.1.5.3}$$

Observe that $\mathrm{M}\otimes\mathrm{N}^\vee$ is the dual of the "internal hom" $\mathrm{Hom}(\mathrm{M},\mathrm{N})$. By lemma 5.1.3, $\mathrm{M}\otimes\mathrm{N}^\vee$ is constant. The restriction of $\mathrm{M}\otimes\mathrm{N}^\vee$ to S' is again constant on S', and trivialized by C. This implies that

$$\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{C}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{C})\ . \tag{5.1.5.4}$$

This shows that a morphism with values in C commutes with all σ_q and δ_q, for all $q\in S$, if and only if it commutes with all σ_q and δ_q, for all $q\in S'$. Hence

$$\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ .\ \square \tag{5.1.5.5}$$

Remark 5.1.6. By the previous lemma one sees that if $\xi\in S\cap\boldsymbol{\mu}(\mathcal{Q})$, then

$$\mathrm{Res}^S_{\{\xi\}} : (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow (\sigma_\xi,\delta_\xi)-\mathrm{Mod}(\mathrm{B}) \tag{5.1.6.1}$$

is again fully faithful, while (if $S^\circ\neq\emptyset$) the restriction

$$\mathrm{Res}^S_{\{\xi\}} : \sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow \sigma_\xi-\mathrm{Mod}(\mathrm{B}) \tag{5.1.6.2}$$

is *not* fully faithful, since $\sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S$ is $K-$linear, while $\sigma_\xi-\mathrm{Mod}(\mathrm{B})$ is not $K-$linear.

5.2 Strong deformation and strong confluence

As usual $S \subseteq \mathcal{Q}(\mathrm{B})$ is a subset.

Definition 5.2.1 (Strongly confluent σ_q-modules (resp. $(\sigma_q, \delta_q)-$modules))**.** Let $q \in S$. Let C be a discrete $\sigma-$algebra over S (resp. $(\sigma, \delta)-$algebra over S). A $q-$difference module M (resp. a $(\sigma_q, \delta_q)-$module) is said *strongly confluent on* S if it belongs to the essential image of the restriction functor

$$\begin{array}{rrcl} & \mathrm{Res}^S_{\{q\}} : \quad \sigma - \mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_S & \longrightarrow & \sigma_q - \mathrm{Mod}(\mathrm{B}) \ , \\ (\text{resp.} & \mathrm{Res}^S_{\{q\}} : (\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_S & \longrightarrow & (\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}) \quad) \ . \end{array}$$

The *full sub-category* of σ_q-Mod(B) (resp. $(\sigma_q, \delta_q)-$Mod(B)), whose objects are strongly confluent over S, will be denoted by

$$\sigma_q - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \tag{5.2.1.1}$$

$$(\text{resp.} \quad (\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \quad) \ . \tag{5.2.1.2}$$

Lemma 5.1.5 and definition 5.2.1 give easily the following

Corollary 5.2.2. *Let $S \subseteq \mathcal{Q}(\mathrm{B})$ be a subset. Let* C *be a fixed discrete $(\sigma, \delta)-$algebra over S. The restriction functor 5.2.1.1*

$$(\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_S \xrightarrow[\sim]{\mathrm{Res}^S_{\{q\}}} (\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \tag{5.2.2.1}$$

is an equivalence. The analogous fact is true for discrete $\sigma-$modules over S, under the additional hypothesis : $q \in S^\circ$.

Proof : The proof is the same of 5.1.5. □

Definition 5.2.3. 1.– Let $S \subseteq \mathcal{Q}(\mathrm{B})$ be a subset and let $q, q' \in \langle S \rangle$. We will call the *strong deformation functor*, denoted by

$$\mathrm{Def}_{q,q'} : (\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \xrightarrow{\sim} (\sigma_{q'}, \delta_{q'}) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \ , \tag{5.2.3.1}$$

the equivalence obtained by composition with the restriction functor 5.2.2.1 :

$$\mathrm{Def}_{q,q'} := \mathrm{Res}^S_{\{q'\}} \circ (\mathrm{Res}^S_{\{q\}})^{-1} \ . \tag{5.2.3.2}$$

2.– We will call the *strong confluence functor*, the equivalence

$$\mathrm{Conf}_q := \mathrm{Def}_{q,1} : (\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \xrightarrow{\sim} (\sigma_1, \delta_1) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \ . \tag{5.2.3.3}$$

3.– Suppose that $q \in S^\circ$ and $q' \in S$, then we will call again the *strong deformation functor*, denoted by

$$\mathrm{Def}_{q,q'} : \sigma_q - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \longrightarrow \sigma_{q'} - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_S \ , \tag{5.2.3.4}$$

the functor obtained by composition with the restriction functor 5.2.2.1 : $\mathrm{Def}_{q,q'} := \mathrm{Res}^S_{\{q'\}} \circ (\mathrm{Res}^S_{\{q\}})^{-1}$. If $q' \in S^\circ$, then $\mathrm{Def}_{q,q'}$ is an equivalence.

5.3 Analytic $(\sigma, \delta)-$modules and strong confluence

In this section we shall describe the situation on the roots of 1. Let $U \subseteq \mathcal{Q}(\mathrm{B})$ be an open subset. Let C be a *fixed* discrete $(\sigma, \delta)-$algebra over U.

Remark 5.3.1. The condition $U^{\circ} \neq \emptyset$ is automatically verified since U is open.

Definition 5.3.2. We denote by

$$(\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an,const}} \tag{5.3.2.1}$$

the full subcategory of $(\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{const}}$ whose objects are analytic.

Lemma 5.3.3. *If $U' \subset U$, then the restriction functor*

$$\mathrm{Res}_{U'}^{U} : (\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an,const}} \longrightarrow (\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_{U'}^{\mathrm{an,const}} \tag{5.3.3.1}$$

is again fully faithful.

Proof : Same proof as 5.1.5. $\square$

Remark 5.3.4. By 5.1.5, for all (non empty) subsets $S \subset U$, the restriction

$$\mathrm{Res}_{S}^{U} : (\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an,const}} \longrightarrow (\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_{S}^{\mathrm{const}} \tag{5.3.4.1}$$

is fully faithful. On the other hand, under the assumption $S^{\circ} \neq \emptyset$, the restriction

$$\mathrm{Res}_{S}^{U} : \sigma - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an,const}} \longrightarrow \sigma - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_{S}^{\mathrm{const}} \tag{5.3.4.2}$$

is fully faithful.

Definition 5.3.5 (Strongly confluent objects)**.** Let $S = \{q\}$. We denote by

$$\sigma_q - \mathrm{Mod}(B, \mathrm{C})_U^{\mathrm{an}} \tag{5.3.5.1}$$

$$(\text{resp.} \quad (\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an}} \ ) \tag{5.3.5.2}$$

the full sub-category of $\sigma_q - \mathrm{Mod}(\mathrm{B})$ (resp. $(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B})$) whose objects belong to the essential image of $\sigma - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an,const}}$ (resp. $(\sigma, \delta) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an,const}}$).

Remark 5.3.6. One has then a family of equivalences, for all $q, q' \in U$:

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an}} \xrightarrow[\sim]{\mathrm{Def}_{q,q'}} (\sigma_{q'}, \delta_{q'}) - \mathrm{Mod}(\mathrm{B}, \mathrm{C})_U^{\mathrm{an}} \ . \tag{5.3.6.1}$$

The same fact is true for analytic $\sigma-$modules only under the condition $q, q' \notin \boldsymbol{\mu}(\mathcal{Q})$.

Remark 5.3.7. a.— Let $q \in U$. The advantage of the use of analytic objects is that we can work in a neighborhood of q, and not really on the point q. The situation is described in the following diagram (compare with 7.2.3) :

$$\begin{tikzcd}
\bigcup_U \ \sigma - \mathrm{Mod(B,C)}_U^{\mathrm{an,const}} \arrow[d, "\bigcup_U \ \mathrm{Res}^U_{\{q\}}"'] \arrow[r, equal, "\underline{3.4.1.2}"] \arrow[dr, phantom, "\odot"] & \bigcup_U \ (\sigma,\delta) - \mathrm{Mod(B,C)}_U^{\mathrm{an,const}} \arrow[d, "\wr"', "\bigcup_U \ \mathrm{Res}^U_{\{q\}}"] \\
\bigcup_U \ \sigma_q - \mathrm{Mod(B,C)}_U \arrow[d, "i_\sigma"'] \arrow[dr, phantom, "\odot"] & \bigcup_U \ (\sigma_q,\delta_q) - \mathrm{Mod(B,C)}_U \arrow[l, "\mathrm{Forget}\ \delta_q"] \arrow[d, "i_{(\sigma,\delta)}"] \\
\sigma_q - \mathrm{Mod(B)} & (\sigma_q,\delta_q) - \mathrm{Mod(B)} \arrow[l, "\mathrm{Forget}\ \delta_q"]
\end{tikzcd} \tag{5.3.7.1}$$

where U runs in the set of open neighborhoods of q, and where i_σ and $i_{(\sigma,\delta)}$ are the trivial inclusions of full sub-categories. The category we shall mainly study is the sub-category of $\sigma_q - \mathrm{Mod(B)}$, formed by equations which are Taylor admissible (cf. 7.1.1). Roughly speaking, these are equations admitting a Taylor solution at every point $c \in K$ (cf. section 6). Our main theorem (cf. 7.2.1) says that this category is contained in the essential image of $\bigcup_{q\in U} \sigma_q - \mathrm{Mod(B,C)}_U$ (resp. $\bigcup_{q\in U}(\sigma_q,\delta_q) - \mathrm{Mod(B,C)}_U$), where $\mathrm{C} := \bigcup_{R>|q-1||c|} \mathcal{A}_K(c,R)$, for some point c on which the equation is defined. In other words, every $(\sigma_q,\delta_q)-$module (resp. σ_q-module if q is not a root of unity) is the stalk at q of an analytic $\sigma-$module on a small neighborhood of q.

Hence, *all formal trivialities given until now will assume a serious meaning.*

b.— We know that, if $q \notin U \cap \boldsymbol{\mu}(\mathcal{Q})$, then the functor $\bigcup_{q\in U} \mathrm{Res}^U_{\{q\}}$ is an equivalence in the case of $\sigma-$modules. On the other hand, in the case of $(\sigma,\delta)-$modules, the functor $\bigcup_{q\in U} \mathrm{Res}^U_{\{q\}}$ is always an equivalence, for all values of $q \in \mathcal{Q}$. Hence, if $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then the functor "Forget δ_q" is an equivalence too. In this situation all rows of the diagram in the top are equivalences : the data of the $q-$tangent operator δ_q is not essential in this description.

On the other hand, if $q = \xi \in U \cap \boldsymbol{\mu}(\mathcal{Q})$, then $\bigcup_{q\in U} \mathrm{Res}^U_{\{q\}}$ is again an equivalence for $(\sigma,\delta)-$modules, but not for $\sigma-$modules. It happens that the category $\sigma_\xi - \mathrm{Mod(B)}$ has "*too many morphisms*", and then some object in $(\sigma_\xi,\delta_\xi) - \mathrm{Mod(B)}$ "*vanishes*" in $\sigma_\xi - \mathrm{Mod(B)}$ (i.e. becomes isomorphic to a direct sum of the unit object).

c.— The first impression we have is that all the information must be contained in the functor "Forget δ_q". But, actually, we will show for example (cf. 8.1.7) that, if ξ is a p^n-th root of 1, and if $\mathrm{B} = \mathcal{R}_K$, or $\mathrm{B} = \mathcal{H}_K^\dagger$, then every $\sigma_\xi-$difference equation over $\mathcal{R}_K$ (or $\mathcal{H}_K^\dagger$) with Frobenius structure is "*vanishing*" (i.e. is isomorphic to a direct sum of the unit object). In particular, every σ_ξ−difference equation coming, by deformation $\mathrm{Def}_{1,\xi}$, from a differential equation with Frobenius structure over $\mathcal{R}_K$ (or $\mathcal{H}_K^\dagger$), is "vanishing" at ξ. Then the category of equations which "vanish" at ξ is very large, hence the data of δ_ξ is necessary in this case, to preserve the information.

Chapitre 6

Taylor solutions

In this chapter $\mathrm{B} = \mathcal{H}_K(A)$, for some affinoïd A, and $S = \{q\} \in \mathcal{Q}(\mathcal{H}_K(A))$ is reduced to a point.

In this chapter we shall find convergent solutions of a $q-$difference equation. For this reason we will assume also that

$$q \in \mathrm{D}^-(1,1) \cap \mathcal{Q}(\mathcal{H}_K(A)) , \tag{6.0.7.1}$$

in order that the disk of convergence of Y is always $q-$invariant.

6.1 Existence of a Taylor solution

In this section we will discuss the existence of a Taylor solution at point c of a given σ_q-module or $(\sigma_q, \delta_q)-$module. In order to treat at the same time ordinary points and *generic points* (see def. 6.2.1), we will fix an arbitrary complete valued extension $(\Omega, |.|)/(K, |.|)$, and fix a point $c \in A(\Omega)$. Let

$$\mathrm{D}^-(c\ ,\ \rho_{c,A}) \subseteq A \tag{6.1.0.2}$$

be the biggest disk centered at c and contained in A. Suppose that this disk is $q-$invariant (i.e. q verifies $|q-1| < \rho_{c,A}/|c|$). Let us fix a $q-$difference equation

$$\sigma_q(Y) \;=\; A(q,T) \cdot Y\ , \qquad A(q,T) \in GL_n(\mathcal{H}_K(A))\ , \tag{6.1.0.3}$$

or, in the case of $(\sigma_q, \delta_q)-$modules, let us fix a system of equations

$$\left\{ \begin{array}{rcll} \sigma_q(Y) & = & A(q,T) \cdot Y\ , & A(q,T) \in GL_n(\mathcal{H}_K(A))\ , \\ \delta_q(Y) & = & G(q,T) \cdot Y\ , & G(q,T) \in M_n(\mathcal{H}_K(A))\ . \end{array} \right. \tag{6.1.0.4}$$

We shall find solutions $Y(T)$, convergent in a neighborhood of c, of the equation 6.1.0.3 (resp. of the system 6.1.0.4).

There is two kinds of obstructions to the existence of such a Taylor solution :

1. The first obstruction arises from the system 6.1.0.3, since, for a fixed $R > 0$, there exist $q-$difference equations without (non trivial) solutions in $\mathcal{A}_\Omega(c, R)$.
2. The second obstruction is a condition of compatibility between σ_q and δ_q. An example of this kind of obstruction is given at 6.1.1 below.

Example 6.1.1. Suppose that $q \in \mathrm{D}^-(1,1)$ is not a root of unity. Let

$$\begin{aligned} A &:= \mathrm{D}^+(0,1)\ , & (6.1.1.1)\\ A(q,T) &:= \exp((q-1)T) \in \mathcal{H}_K(A)^\times\ , & (6.1.1.2)\\ G(q,T) &:= 0\ . & (6.1.1.3) \end{aligned}$$

Let $c = 0$. Then every solution $y(T) \in K[[T]]$ of the operator $\sigma_q - A(q,T)$ is of the form $y(T) = \lambda \cdot \exp(T)$, with $\lambda \in K$. If now we ask, moreover, that $\delta_q(y) = 0$, then $y = 0$. Hence, the $(\sigma_q, \delta_q)-$module defined by $A(q,T)$ and $G(q,T)$ has no (non trivial) solutions in $K[[T]]$.

We will discuss these obstructions in the next section. The first obstruction is the "real" obstruction to the existence of a solution, while the second one will not play a role in the sequel since it will be automatically verified by *analytic objects* (cf. 6.1.2).

6.1.1 The first obstruction

Hypothesis 6.1.2. In the subsection 6.1.1 we suppose that

$$q \notin \boldsymbol{\mu}(\mathcal{Q})\ . \qquad (6.1.2.1)$$

Recall that we assumed that $q \in \mathrm{D}^-(1,1) \cap \mathcal{Q}$ (cf. 6.0.7.1). The first obstruction has been described in [DV04, Cor.3.3 and appendix A]. Consider the sequence

$$1\ ,\ (T-c)\ ,\ \frac{(T-c)_{q,2}}{[2]^!_q}\ ,\ \dots\ ,\ \frac{(T-c)_{q,n}}{[n]^!_q}\ ,\ \dots \qquad (6.1.2.2)$$

where

$$\begin{aligned} (T-c)_{q,n} &:= (T-c)(T-qc)(T-q^2c)\cdots(T-q^{n-1}c)\ , & (6.1.2.3)\\ [n]^!_q &:= \frac{(q-1)(q^2-1)(q^3-1)\cdots(q^n-1)}{(q-1)^n}\ . & (6.1.2.4) \end{aligned}$$

If $|q-1| < R/|c|$, then this family is a topological basis of $\mathcal{A}_\Omega(c,R)$ ([DV04, 14.1]). Moreover, if we set

$$d_q := \frac{1}{T}\Delta_q = \frac{\sigma_q - 1}{T(q-1)}\ , \qquad (6.1.2.5)$$

then the basis 6.1.2.2 is *adapted* to that $q-$derivation : for all $n \geq 1$ one has

$$d_q\left(\frac{(T-c)_{q,n}}{[n]_q^!}\right) = \frac{(T-c)_{q,n-1}}{[n-1]_q^!} \quad . \tag{6.1.2.6}$$

More generally, one introduces the $q-$difference algebras (cf. [DV04])

$$\begin{aligned} \Omega[[T-c]]_q &:= \{\sum_{n\geq 0} a_n(T-c)_{q,n} \ : \ a_n \in \Omega\} \ , & (6.1.2.7)\\ \Omega\{T-c\}_{q,R} &:= \{\sum_{n\geq 0} a_n(T-c)_{q,n} \ : \ a_n \in \Omega \ , \ \liminf_n |a_n|^{-1/n} \geq R\} \ , & (6.1.2.8)\end{aligned}$$

whose multiplicative laws are defined explicitly in [DV04, 1.3]. If $R > |q-1||c|$, then one has the identification

$$\mathcal{A}_\Omega(c,R) \xrightarrow{\sim} \Omega\{T-c\}_{q,R} \ , \tag{6.1.2.9}$$

which sends $f(T)$ into its $q-$Taylor expansion $\sum_{n\geq 0} d_q^n(f)(c)\frac{(T-c)_{q,n}}{[n]_q^!}$. The basis 6.1.2.2 allows us to define the analogous of the formal Taylor expansion :

Definition 6.1.3. We set

$$Y_{A(q,T)}(x,y) \ := \ \sum_{n\geq 0} H_n(y)\frac{(x-y)_{q,n}}{[n]_q^!} \ \in \ M_n(K[[x-y]]_q) \ , \tag{6.1.3.1}$$

where $H_n(T) \in M_n(\mathcal{A}_K(I))$ is the matrix of the operator d_q^n (i.e. $d_q^n(Y) = H_n(T)\cdot Y$, recall that by assumption $A(q,T) \in GL_n(\mathcal{H}_K(A))$).

We will omit the index $A(q,T)$ appearing in 6.1.3.1, if no confusion is possible.

Remark 6.1.4. One has the inductive relation :

$$\begin{aligned} H_1(T) &:= \frac{A(q,T)-\mathrm{I}}{(q-1)T} \ , & (6.1.4.1)\\ H_{n+1}(T) &= \sigma_q(H_n(T))\cdot H_1(T) + d_q(H_n(T)) \ . & (6.1.4.2)\end{aligned}$$

Remark 6.1.5. Note that, if q is a $m-$th root of 1, then $[n]_q^! = 0$, for all $n \geq m$. This is the reason for the hypothesis $q \notin \boldsymbol{\mu}(\mathcal{Q})$ given in 6.1.2.

Lemma 6.1.6. *Let*

$$\begin{aligned} \sigma_q^x \ &: \ f(x,y) \mapsto f(qx,y) \ , & \sigma_q^y \ &: \ f(x,y) \mapsto f(x,qy) \ ,\\ d_q^x \ &:= \ \tfrac{\sigma_q^x - 1}{(q-1)x} \ , & d_q^y \ &:= \ \tfrac{\sigma_q^y - 1}{(q-1)y} \ . \end{aligned} \tag{6.1.6.1}$$

Then :

$$\begin{aligned}
Y(x,x) &= \mathrm{Id}\ , && (6.1.6.2)\\
d_q^y\, Y(x,y) &= -\sigma_q^y(Y(x,y))\cdot H_1(y)\ , && (6.1.6.3)\\
\sigma_q^y\, Y(x,y) &= Y(x,y)\cdot A(y)^{-1}\ , && (6.1.6.4)\\
Y(x,y)\cdot Y(y,z) &= Y(x,z)\ , && (6.1.6.5)\\
Y(x,y)^{-1} &= Y(y,x)\ , && (6.1.6.6)\\
d_q^x\, Y(x,y) &= H_1(x)\cdot Y(x,y)\ . && (6.1.6.7)\\
\sigma_q^x\, Y(x,y) &= A(x)\cdot Y(x,y)\ . && (6.1.6.8)
\end{aligned}$$

Proof : 6.1.6.2 is evident and 6.1.6.3 is easy to compute explicitly. The relation 6.1.6.4 is equivalent to 6.1.6.3. Moreover, 6.1.6.3 implies that

$$d_q^y\ (\ Y(x,y)\cdot Y(y,z)\ ) = 0\ . \tag{6.1.6.9}$$

Since q is not a root of unity, hence $Y(x,y)\cdot Y(y,z)$ is not dependent on y. Then $Y(x,y)\cdot Y(y,z) = Y(x,z)\cdot Y(z,z) = Y(x,z)$, and hence 6.1.6.5 holds. If $x = z$ in the relation 6.1.6.5, then one gets 6.1.6.6. Then the relation 6.1.6.7 follows from 6.1.6.6 and 6.1.6.3.□

Lemma 6.1.7 ([DV04, Cor. 3.3]). *Let $q \in \mathcal{Q}$. Suppose that $|q-1||c| < \rho_{c,A}$ (cf. 6.1.0.2). We consider the σ_q-algebra*

$$\mathrm{C} := \bigcup_{R>|q-1||c|} \mathcal{A}_\Omega(c,R)\ . \tag{6.1.7.1}$$

The system 6.1.0.3 has a matrix solution in $GL_n(\mathrm{C})$ if and only if the number

$$R_c := \liminf_n \Big(\frac{\|H_n(c)\|_\Omega}{|[n]_q^!|}\Big)^{-1/n} \tag{6.1.7.2}$$

verifies $R_c > |q-1||c|$. In this case, the solution of 6.1.0.3 is $Y_{A(q,T)}(T,c)$, and it lies in

$$GL_n(\mathcal{A}_\Omega(c,\widetilde{R}_c))\ , \tag{6.1.7.3}$$

where $\widetilde{R}_c := \min(\ R_c\ ,\ \rho_{c,A}\ )$.

Proof : □

Remark 6.1.8. Suppose that $q \notin \boldsymbol{\mu}(\mathcal{Q})$, and that there exists a solution matrix $Y(T,c)$ of 6.1.0.3 in the algebra C (cf. 6.1.7.1). Then the set of all solutions in $GL_n(\mathcal{A}_\Omega(c,R))$ is given by

$$Y(T,c)\cdot GL_n(\Omega)\ . \tag{6.1.8.1}$$

Suppose for a moment that $q \in \boldsymbol{\mu}(\mathcal{Q})$. If a solution $Y_c(T) \in GL_n(\mathcal{A}_\Omega(c,R_c))$ exists, then the set of all solutions in the disk $\mathcal{A}_\Omega(c,R_c)$ is given by (cf. 4.1.1.1)

$$Y_c(T)\cdot GL_n(\mathcal{A}_\Omega(c,R_c)^{\sigma_q})\ . \tag{6.1.8.2}$$

For this reason, the radius of convergence of the solution $Y_c(T)$ can be different from the radius of $Y_c(T) \cdot H(T)$, if $H(T) \in GL_m(\mathcal{A}_\Omega(c,R)^{\sigma_q})$ (cf. example 6.2.7). For this reason, the radius of convergence of the system 6.1.0.3 will be not defined for $q \in \boldsymbol{\mu}(\mathcal{Q})$.

6.1.2 The second obstruction

Remark 6.1.9. In this subsection and in the sequel the number q can be equal to a root of unity.

The second obstruction, illustrated in the example 6.1.1, will not appear in the sequel of the paper, since we will work only with analytic $\sigma-$modules and $(\sigma,\delta)-$modules. Indeed the compatibility relation we need between σ_q^{M} and δ_q^{M} will be automatically verified for $(\sigma_q,\delta_q)-$modules in $(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}_K(A))_U^{\mathrm{an}}$, for some open subset U. More precisely, one has the following lemma :

Lemma 6.1.10. *Let $U \subseteq \mathcal{Q}(\mathcal{H}_K(A)) \cap \mathrm{D}^-(1,1)$ be an open subset. Let $R \leq \rho_{c,A}$ be a real number such that*

$$R > |q-1||c| \ , \quad \textit{for all } \ q \in U. \tag{6.1.10.1}$$

Let M *be an analytic $(\sigma,\delta)-$module on U, representing the family of equations*

$$\{ \ \sigma_q(Y) = A(q,T) \cdot Y \ \}_{q \in U} \ , \tag{6.1.10.2}$$

with $A(q,T) \in GL_n(\mathcal{H}_K(A))$, for all $q \in U$. Let $Y_c(T) \in GL_n(\mathcal{A}_\Omega(c,R))$ be a simultaneously solution of every equation of this family. Then $Y_c(T)$ is also solution of the equation

$$\delta_q(Y) = G(q,T) \cdot Y \ , \tag{6.1.10.3}$$

where $G(q,T) := q\frac{d}{dq}(A(q,T))$ (cf. 3.4.4.1), and hence $Y_c(T)$ is solution of the differential equation defined in section 3.4.1 :

$$\delta_1(Y_c(T)) = G(1,T) \cdot Y_c(T) \ , \tag{6.1.10.4}$$

where $G(1,T) = G(q,q^{-1}T) \cdot A(q,q^{-1}T)^{-1} \in M_n(\mathcal{H}_K(A))$ (cf. 3.2.3.5).

Proof : In terms of modules, the columns of the matrix $Y_c(T)$ correspond to $\mathcal{H}_K(A)-$linear maps $\alpha : \mathrm{M} \to \mathcal{A}_\Omega(c,R)$, verifying $\sigma_q \circ \alpha = \alpha \circ \sigma_q^{\mathrm{M}}$, for all $q \in U$ (cf. 4.2.1). We must show that such an α commutes also with δ_q. This follows immediately by the continuity of α. Indeed, the inclusion $\mathcal{H}_K(A) \to \mathcal{A}_\Omega(c,R)$ is continuous, and hence every $\mathcal{H}_K(A)-$linear map $\mathcal{H}_K(A)^n \to \mathcal{A}_\Omega(c,R)$ is continuous. $\square$

Remark 6.1.11. Observe that Lemma 6.1.10 is not a formal consequence of the previous theory. Indeed, by definition 4.1.4, the general $(\sigma,\delta)-$algebra C used in 4.1.4 has the discrete topology, hence the morphism $\alpha : \mathrm{M} \to \mathrm{C}$ defining the solution is not continuous in general.

We recall the definition of the classical Taylor solution of a differential equation

Definition 6.1.12. Let $\delta_1 - G(1,T)$, be a differential equation. Let $G_{[n]}(T)$ be the matrix of $(d/dx)^n$. We set

$$Y_{G(1,T)}(x,y) := \sum_{n\geq 0} G_{[n]}(y)\frac{(x-y)^n}{n!}\ . \tag{6.1.12.1}$$

One has (cf. for instance [CM02, p.137]) :

$$\begin{aligned}
Y_G(x,x) &= \mathrm{Id}\ , && (6.1.12.2)\\
d/dy\ (Y_G(x,y)) &= -Y_G(x,y)\cdot G_{[1]}(y)\ , && (6.1.12.3)\\
Y_G(x,y)\cdot Y_G(y,z) &= Y_G(x,z)\ , && (6.1.12.4)\\
Y_G(x,y)^{-1} &= Y_G(y,x)\ , && (6.1.12.5)\\
d/dx\ (Y_G(x,y)) &= G_{[1]}(x)\cdot Y_G(x,y)\ . && (6.1.12.6)
\end{aligned}$$

6.1.3 Taylor development of $Y_c(T)$

We give now the first rough estimate of the radius of convergence of $Y_c(T)$. This is the analogous of the same classical rough estimate for differential and $q-$difference equations (cf. [DV04, 4.3], [Chr83, 4.1.2]).

We recall that we have fixed a real number $\rho_{c,A}$, such that $\mathrm{D}^-(c,\rho_{c,A})$ is the biggest disk contained in A, centered at c. We supposed moreover that this disk is $q-$invariant (i.e. q verifies $|q-1| < \rho_{c,A}/|c|$). We recall also that R_c is the formal radius of convergence of $Y_c(T)$ at c (cf. 6.1.7.2).

Proposition 6.1.13. *Let* M *be the* $(\sigma_q,\delta_q)-$*module defined by the system 6.1.0.4. Suppose that* M *admits, in some basis, a solution* $Y_c(T)\in M_n(\mathcal{A}_\Omega(c,R))$, *with* $R > |q-1||c|$. *Then*

$$Y_c(T)\in GL_n(\mathcal{A}_\Omega(c,\widetilde{R}_c))\ , \tag{6.1.13.1}$$

with $\widetilde{R}_c := \min(R_c\,,\,\rho_{c,A})$. *Moreover, for all* ρ *satisfying*

$$|q-1||c| < \rho \leq \rho_{c,A},$$

one has (where $\omega = |p|^{\frac{1}{p-1}}$*)*

$$R_c \geq \frac{\omega\cdot\rho}{\max(\ \|A(q,T)\|_{(c,\rho)}\ ,\ \frac{\|G(q,T)\|_{(c,\rho)}}{\max(1,|c|/\rho)}\ )}\ . \tag{6.1.13.2}$$

In particular, if $\rho\leq|c|$, *then*

$$R_c \geq \frac{\omega\cdot\rho}{\max(\ \|A(q,T)\|_{(c,\rho)}\ ,\ \frac{\rho}{|c|}\|G(q,T)\|_{(c,\rho)}\ )}\ . \tag{6.1.13.3}$$

Proof : By assumption one has $R_c := Ray_c(Y_c(T)) > |q-1||c|$. Let us show that $Y_c(T)$ lies in $\mathcal{A}_\Omega(c,\widetilde{R}_c)$. By assumption, $Y_c(T)$ is solution of both equations 6.1.0.4. This implies that $Y_c(T)$ verifies also the differential equation

$$\delta_1(Y_c(T)) = G(1,T)\cdot Y_c(T)\ , \tag{6.1.13.4}$$

where $G(1,T) = G(q,q^{-1}T)\cdot A(q,q^{-1}T)^{-1}$. Then $w(T) := \det(Y_c(T))$ verifies $\delta_1(w(T)) = \mathrm{Tr}(G(1,T))\cdot w(T)$. Since $G(1,T)\in M_n(\mathcal{H}_K(A))\subset M_n(\mathcal{A}_\Omega(c,\widetilde{R}_c))$, hence $w(T)$ is invertible in $\mathcal{A}_\Omega(c,\widetilde{R}_c)$.

It remains to show the estimate 6.1.13.2. To do this we need the operator

$$D_q := \sigma_q\circ\frac{d}{dT} = \lim_{q'\to q}\frac{\sigma_{q'}-\sigma_q}{T(q'-q)} = \frac{1}{qT}\cdot\delta_q\ . \tag{6.1.13.5}$$

Remark 6.1.14. For all $q\in\mathcal{Q}(A)\cap\mathrm{D}^-(1,1)$, for all $f(T)\in\mathcal{H}_K(A)$, and all $|q-1||c|<\rho<\rho_{c,A}$, one has

$$\begin{aligned}
\mathrm{D}_q(f\cdot g) &= \sigma_q(f)\cdot\mathrm{D}_q(g)+\mathrm{D}_q(f)\cdot\sigma_q(g)\ , && (6.1.14.1)\\
(d/dT\circ\sigma_q) &= q\cdot(\sigma_q\circ d/dT)\ , && (6.1.14.2)\\
\mathrm{D}_q^n &= q^{n(n-1)/2}\cdot\sigma_q^n\circ(d/dT)^n\ , && (6.1.14.3)\\
|\mathrm{D}_q^n(f(T))|_{(c,\rho)} &\leq \frac{|n!|}{\rho^n}\cdot|f(T)|_{(c,\rho)}\ . && (6.1.14.4)
\end{aligned}$$

Hence, for all $c\in K$, one has

$$\mathrm{D}_q^n(T-c)^i = \begin{cases}\frac{i!}{(i-n)!}\cdot q^{n(n-1)/2}\cdot(q^nT-c)^{i-n} & \text{if}\quad n\leq i\ ,\\[2ex] 0 & \text{if}\quad n>i\ .\end{cases} \tag{6.1.14.5}$$

This shows that if $R>|q-1||c|$, and if

$$f(T) := \sum_{i\geq 0} a_i\cdot\frac{(T-c)^i}{(i!)\cdot q^{i(i-1)/2}}\in\mathcal{A}_\Omega(c,R) \tag{6.1.14.6}$$

is a formal series, then $a_n = \mathrm{D}_q^n(f)(c/q^n)$, and the usual Taylor formula can be written as

$$f(T) = \sum_{n\geq 0}\mathrm{D}_q^n(f)(c/q^n)\cdot\frac{(T-c)^n}{(n!)\cdot q^{n(n-1)/2}}\quad . \tag{6.1.14.7}$$

The matrix $Y_c(T)$ verifies

$$\begin{aligned}
\sigma_q^n(Y_c(T)) &= A_{[n]}(q,T)\cdot Y_c(T)\ , && (6.1.14.8)\\
D_q^n(Y_c(T)) &= F_{[n]}(q,T)\cdot Y_c(T)\ , && (6.1.14.9)
\end{aligned}$$

where $F_{[0]} = A_{[0]} = \mathrm{Id}$, $A_{[1]} := A(q,T)$, $F_{[1]} := \frac{1}{qT}G(q,T)$, and

$$\begin{aligned} A_{[n]} &:= \sigma_q^{n-1}(A_{[1]}) \cdots \sigma_q(A_{[2]}) \cdot A_{[1]} \;, & (6.1.14.10) \\ F_{[n+1]} &:= \sigma_q(F_{[n]}) \cdot F_{[1]} + D_q(F_{[n]}) \cdot A_{[1]} \;. & (6.1.14.11) \end{aligned}$$

Hence one has

$$Y_c(T) := \sum_{i \geq 0} F_{[n]}(c/q^n) \frac{(T-c)^n}{(n!) \cdot q^{n(n-1)/2}} \;, \qquad (6.1.14.12)$$

which is an hybrid between the usual Taylor formula and the Taylor formula for $q-$difference equations. The proposition 6.1.13 follows then from 6.1.14.12 and from the inequality

$$\begin{aligned} \|F_{[n]}\|_{(c,\rho)} &\leq \max\left(\|F_{[1]}\|_{(c,\rho)}, \frac{1}{\rho} \cdot \|A\|_{(c,\rho)} \right)^n & (6.1.14.13) \\ &= \frac{1}{\rho^n} \cdot \max\left(\frac{\|G(q,T)\|_{(c,\rho)}}{\max(1, |c|/\rho)} \;,\; \|A(q,T)\|_{(c,\rho)} \right)^n . & (6.1.14.14) \end{aligned}$$

Indeed $F_{[1]} = \frac{1}{qT}G(q,T)$, and $|T|_{(c,\rho)} = |(T-c)+c|_{(c,\rho)} = \max(\rho, |c|)$, hence $\|F_{[1]}\|_{(c,\rho)} = \frac{1}{|q| \max(|c|,\rho)} \cdot \|G(q,T)\|_{(c,\rho)}$ (cf. 6.1.14.11). $\Box$

6.2 Generic radius of convergence and solvability

In this chapter we will introduce the notion of generic points and of generic radius of convergence. We will not use the language of Berkovich analytic spaces (cf. [Ber90]) since, in the case of affinoïd spaces, a Berkovich point is always a semi-norm $|.|_{(c,\rho)}$ attached to a generic point $t_{c,\rho}$ (see below), for some $\rho > 0$, and some $c \in A(L)$, where L/K is some spherically closed extension (cf. [Ber90, 1.4.4]). The reader who knows the language of Berkovich will not have difficulties to translate the contents of this paper in the language of Berkovich.

We recall that $\rho_{c,A}$ is the biggest real number such that $\mathrm{D}^+(c\,,\,\rho_{c,A}) \subset A$. Let now $(\Omega, |.|)/(K, |.|)$ be a complete extension of valued field such that $|\Omega| = \mathbb{R}_{\geq}$, and such that k_Ω/k is not algebraic.

Proposition 6.2.1. *For all $c \in A(K)$ and all $0 < \rho \leq \rho_{c,A}$, there exists a point $t_{c,\rho} \in A(\Omega)$ such that $|\, t_{c,\rho} - c\,|_\Omega = \rho$, and that*

$$\mathrm{D}^-_\Omega(\, t_{c,\rho}\;,\;\rho\,) \cap K^{\mathrm{alg}} = \emptyset \;. \qquad (6.2.1.1)$$

Proof : [CR94, 9.1.2]. $\Box$

Remark 6.2.2. For all $f(T) \in \mathcal{H}_K(A)$, one has

$$|f(t_{c,\rho})|_\Omega = |f(T)|_{(c,\rho)} = \sup_{\substack{|x-c|\leq\rho \\ x\in K^{\mathrm{alg}}}} |f(x)| \ . \tag{6.2.2.1}$$

hence, even if the point $t_{c,\rho}$ is not uniquely determined by the property (6.2.1.1), the norm $|\cdot|_{c,\rho}$ (i.e. the Berkovch point $|\cdot|_{c,\rho}$) do not depends on the choice of $t_{c,\rho}$.

Remark 6.2.3. One has

$$|t_{c,\rho}|_\Omega = \sup(|c|, \rho) \ . \tag{6.2.3.1}$$

6.2.1 Generic radius of convergence

Definition 6.2.4 (Generic radius of convergence)**.** Let $q \in \mathcal{Q}(A) \cap \mathrm{D}^-(1,1)$ (resp. $q \in \mathcal{Q}(A) \cap \mathrm{D}^-(1,1)$, $q \notin \boldsymbol{\mu}(\mathcal{Q})$). Let M be the $(\sigma_q, \delta_q)-$module (resp. σ_q-module) defined by the system 6.1.0.4 (resp. 6.1.0.3). Suppose that ρ and q, satisfy the condition $|q-1| \cdot |c| < \ \rho \ \leq \rho_{c,A}$. Assume that M has a Taylor solution at $t_{c,\rho}$, and let

$$Y_{t_{c,\rho}}(T) \in GL_n(\mathcal{A}_\Omega(t_{c,\rho}, R_{t_{c,\rho}})) \tag{6.2.4.1}$$

be a convergent solution at $t_{c,\rho}$. We will call $(c,\rho)-$generic radius of convergence of M the real number

$$Ray_c(\mathrm{M}, \rho) := \min\big(\ R_{t_{c,\rho}} \ , \ \rho_{c,A} \ \big) \ , \tag{6.2.4.2}$$

where $R_{t_{c,\rho}}$ is the radius of convergence of $Y_{t_{c,\rho}}(T)$.

Remark 6.2.5. The number $Ray_c(\mathrm{M}, \rho)$ is invariant under change of basis in M, while the number $R_{t_{c,\rho}}$ depends on the chosen basis. Observe that $Ray_c(\mathrm{M}, \rho)$ depends on the affinoïd A, but not on the particular choice of the generic point $t_{c,\rho}$ (cf. remark 6.2.2).

Remark 6.2.6. The previous definition of generic radius of convergence is given for all $q \in \mathcal{Q} \cap \mathrm{D}^-(1,1)$ in the case of $(\sigma_q, \delta_q)-$modules, and only for $q \notin \boldsymbol{\mu}(\mathcal{Q}) \cap \mathrm{D}^-(1,1)$ in the case of σ_q-modules. Indeed, if $q = \xi \in \boldsymbol{\mu}(\mathcal{Q})$, the problem is that the ring of constants is too big and the set of solutions at $t_{c,\rho}$ is too big as shown by the following example.

Example 6.2.7. Let ξ be a $p-$th root of unity, with $\xi \neq 1$, and let $q = \xi$. If $Y_{t_{c,\rho}}(T) \in GL_n(\mathcal{A}_\Omega(t_{c,\rho}, R))$ is a matrix solution of a $\sigma_\xi-$module, then the set of all solutions, in this disk, is given by

$$Y_{t_{c,\rho}} \cdot GL_n(\mathcal{A}_\Omega(t_{c,\rho}, R)^{\sigma_\xi}) \ .$$

It may exist a solution $\widetilde{Y}_{t_{c,\rho}}$ with a radius of convergence different than $Y_{t_{c,\rho}}(T)$. Hence the radius of convergence depends on the chosen solution matrix. For example the solutions of the unit object at $t \in \Omega$ are the functions $y \in \mathcal{A}_\Omega(t, R)$ such that $y(\xi T) = y(T)$. One finds the function $\exp(T^p - t^p) \in \mathcal{A}_\Omega(t, |p|^{\frac{1}{p-1}})$, and $\exp(\alpha(T^p - t^p)) \in \mathcal{A}_\Omega(t, |p|^{\frac{1}{p-1}}/|\alpha|)$, which have different radius of convergence.

Remark 6.2.8.

A.– Let $q \notin \boldsymbol{\mu}(\mathcal{Q})$ and M be a σ_q–module. Then $R_{t_{c,\rho}}$ is equal to

$$R_{t_{c,\rho}} := \liminf_n \left(\frac{\|H_{[n]}(t_{c,\rho})\|_\Omega}{|[n]^!_q|} \right)^{-1/n} \tag{6.2.8.1}$$

defined in 6.1.7.2.

B.– Let now $q \in \mathcal{Q}$ and M be a (σ_q, δ_q)–module. Then $R_{t_{c,\rho}}$ is the radius of convergence of the solution $Y_{t_{c,\rho}}(T)$ of the differential equation 6.1.0.4 at the generic point :

$$R_{t_{c,\rho}} := \liminf_n \left(\frac{\|G_{[n]}(t_{c,\rho})\|_\Omega}{|n!|} \right)^{-1/n} , \tag{6.2.8.2}$$

where $G_{[1]}(T) := G(1, T)/T$, and $G_{[n+1]} := \frac{d}{dT}(G_{[n]}) + G_{[n]}G_{[1]}$. If $q \notin \boldsymbol{\mu}(\mathcal{Q})$ these two definitions coincide since, by assumption, $Y_{t_{c,\rho}}$ is simultaneously solution of both equations of 6.1.0.4.

Lemma 6.2.9 (Transfer principle)**.** *In the notation of definition 6.2.4 we have*

$$Ray_c(\mathrm{M}, \rho) = \min(\ \rho_{c,A}\ ,\ \inf_{\substack{|x-c|\leq\rho \\ x\in K^{\mathrm{alg}}}} Ray_x(Y_x(T))\ )\ , \tag{6.2.9.1}$$

where :

- *If* M *is a σ_q–module, with $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then $Y_x(T) := Y_{A(q,T)}(T, x)$ is the Taylor solution 6.1.3.1, of* M *at x ;*
- *If* M *is a (σ_q, δ_q)–module, and $q \in \mathcal{Q}$ is arbitrary, then $Y_x(T) := Y_G(T, x)$ is the usual Taylor solution at x of the differential equation 6.1.13.4 associated to* M *(which, by assumption, is equal to $Y_{A(q,T)}(T, x)$, if $q \notin \boldsymbol{\mu}(\mathcal{Q})$).*

Proof : The lemma follows from 6.2.2.1, and the definitions 6.1.3.1, and 6.1.13.4. □

Corollary 6.2.10. *We preserve the notations of 6.2.4. The module* M *has a Taylor solution at $t_{c,\rho}$, if and only if it has a Taylor solution at every $x \in K^{\mathrm{alg}}$, satisfying $|x - c| \leq \rho$.*

Proof : It is an immediate consequence of the transfer principle. □

6.2.2 Solvability

In this section, as above, $q \in \mathcal{Q} \cap \mathrm{D}^-(1,1)$ is arbitrary if M is a (σ_q, δ_q)−module, while, if M is a σ_q−module, then $q \in \mathcal{Q} \cap \mathrm{D}^-(1,1)$, and moreover $q \notin \boldsymbol{\mu}(\mathcal{Q})$.

Remark 6.2.11. Let $\mathrm{B} := \mathcal{A}_K(I)$, with $I :=]r_1, r_2[$. Let M be a σ_q−module (resp. a (σ_q, δ_q)−module) on $\mathcal{A}_K(I)$. Suppose that $c \in K$, $|c| = \rho \in I$. Fix an affinoïd $A \subseteq \mathcal{C}(I)$ containing c and containing the disk $\mathrm{D}^-(c, \rho)$. Then

$$\rho_{c,A} = \rho = |c| \ , \tag{6.2.11.1}$$

moreover the norm

$$|.|_{c,\rho} : \mathcal{A}_K(I) \longrightarrow \mathbb{R}_{\geq} \tag{6.2.11.2}$$

does not depend on the chosen c nor on A, since $t_{c,\rho} = t_{0,\rho}$. Hence the radius $Ray_c(\mathrm{M}, \rho)$ does not depend on c and on A. For this reason we will set

$$t_\rho \ := \ t_{c,\rho} \ , \tag{6.2.11.3}$$

$$Ray(\mathrm{M}, \rho) \ := \ Ray_c(\mathrm{M}, \rho) \ . \tag{6.2.11.4}$$

Definition 6.2.12 (solvability at ρ)**.** Let M be a (σ_q, δ_q)−module on $\mathcal{A}_K(I)$. We will say that M is solvable at $\rho \in I$ if M has a Taylor solution at t_ρ, and if

$$Ray(\mathrm{M}, \rho) = \rho \ . \tag{6.2.12.1}$$

The full subcategory of $\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))$ (resp. $(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))$) whose objects are solvable will be denoted by

$$\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{sol}(\rho)} \ , \quad q \notin \boldsymbol{\mu}(\mathcal{Q}) \tag{6.2.12.2}$$

$$(\text{resp.} \quad (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{sol}(\rho)} \qquad q \in \mathcal{Q} \cap \mathrm{D}^-(1,1) \quad) \tag{6.2.12.3}$$

Remark 6.2.13. Let M be a σ_q−module (resp. a (σ_q, δ_q)−module) over $\mathcal{R}_K$. By definition M comes, by scalar extension, from a module $\mathrm{M}_{\varepsilon_1}$ defined on an annulus $\mathcal{C}(]1-\varepsilon_1, 1[)$. If $\varepsilon_2 > 0$ and $\mathrm{M}_{\varepsilon_2}$ is another module on the annulus $\mathcal{C}(]1-\varepsilon_2, 1[)$ satisfying $\mathrm{M}_{\varepsilon_2} \otimes_{\mathcal{A}_K(]1-\varepsilon_2,1[)} \mathcal{R}_K \xrightarrow{\sim} \mathrm{M}$, then there exists a $\varepsilon_3 \leq \min(\varepsilon_1, \varepsilon_2)$ such that

$$\mathrm{M}_{\varepsilon_1} \otimes \mathcal{A}_K(]1-\varepsilon_3, 1[) \xrightarrow{\ \sim\ } \mathrm{M}_{\varepsilon_2} \otimes \mathcal{A}_K(]1-\varepsilon_3, 1[) \ . \tag{6.2.13.1}$$

Hence the limit $\lim_{\rho \to 1} Ray(\mathrm{M}_\varepsilon, \rho)$ is independent from the chosen module M_ε.

Definition 6.2.14. Let $\mathrm{B} = \mathcal{R}_K$ or $\mathrm{B} = \mathcal{H}_K^\dagger$. For all $|q-1| < r \leq 1$, we define the categories

$$\sigma_q - \mathrm{Mod}(\mathrm{B})^{[r]} \ , \quad q \notin \boldsymbol{\mu}(\mathcal{Q}) \tag{6.2.14.1}$$

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B})^{[r]} \ , \quad q \in \mathcal{Q} \cap \mathrm{D}^-(1,1) \tag{6.2.14.2}$$

as the full sub categories respectively of

$$\sigma_q - \mathrm{Mod}(\mathrm{B})\ , \quad q \notin \boldsymbol{\mu}(\mathcal{Q}) \tag{6.2.14.3}$$
$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B})\ , \quad q \in \mathcal{Q} \cap \mathrm{D}^-(1,1), \tag{6.2.14.4}$$

whose objects verify

$$\begin{aligned} Ray(\mathrm{M}, t_1) &\geq r\ , \quad \text{if M is a module over } \mathcal{H}_K^\dagger & (6.2.14.5)\\ \lim_{\rho \to 1^-} Ray(\mathrm{M}, t_\rho) &\geq r\ , \quad \text{if M is a module over } \mathcal{R}_K\ . & (6.2.14.6) \end{aligned}$$

Objects in

$$\sigma_q - \mathrm{Mod}(\mathrm{B})^{[1]}\ , \quad q \notin \boldsymbol{\mu}(\mathcal{Q}) \tag{6.2.14.7}$$
$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B})^{[1]}\ , \quad q \in \mathcal{Q} \cap \mathrm{D}^-(1,1) \tag{6.2.14.8}$$

will be called *solvable*.

6.2.3 Generic radius for discrete or analytic objects

Definition 6.2.15. For all subset $S \subseteq \mathrm{D}^-(1,1)$, for all $0 < \tau < 1$, we set

$$S_\tau := S \cap \mathrm{D}^-(1, \tau)\ . \tag{6.2.15.1}$$

Definition 6.2.16. Let again $\mathrm{B} = \mathcal{R}_K$ or $\mathrm{B} = \mathcal{H}_K^\dagger$, and for all $\varepsilon > 0$ let

$$I_\varepsilon := \begin{cases}]\,1-\varepsilon\ ,\ 1\,[\ , & \text{if} \quad \mathrm{B} = \mathcal{R}_K \\ \\ ]\,1-\varepsilon\ ,\ 1+\varepsilon\,[\ , & \text{if} \quad \mathrm{B} = \mathcal{H}_K^\dagger \end{cases} \quad . \tag{6.2.16.1}$$

Let $0 < r \leq 1$ and let $S \subseteq \mathrm{D}^-(1, r)$. We denote by

$$\sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]}\ , \quad \text{with } S^\circ \neq \emptyset \tag{6.2.16.2}$$
$$(\text{resp.} \quad (\sigma, \delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]}\ , \quad) \tag{6.2.16.3}$$

the full subcategories respectively of

$$\sigma - \mathrm{Mod}(\mathrm{B})_S\ , \quad \text{with } S^\circ \neq \emptyset \tag{6.2.16.4}$$
$$(\text{resp.} \quad (\sigma, \delta) - \mathrm{Mod}(\mathrm{B})_S\ , \quad) \tag{6.2.16.5}$$

whose objects M have the following properties :

1. For all τ such that $0 < \tau < r$, there exists $\varepsilon_\tau > 0$ such that the restriction
$$\mathrm{Res}_{S_\tau}^S(\mathrm{M}) \tag{6.2.16.6}$$
comes, by scalar extension, from an object
$$\mathrm{M}_{\varepsilon_\tau} \in \ \sigma - \mathrm{Mod}(\mathcal{A}_K(I_{\varepsilon_\tau}))_{S_\tau}^{\mathrm{disc}} \tag{6.2.16.7}$$

such that, for all $\rho \in I_{\varepsilon_\tau}$, and for all $q, q' \in S_\tau$ one has (cf. 6.1.3.1)

$$Y_{A(q,T)}(T, t_\rho) = Y_{A(q',T)}(T, t_\rho) \ . \tag{6.2.16.8}$$

(resp. for all $\rho \in I_{\varepsilon_\tau}$, and all $q \in S_\tau$, one has (cf. 6.1.12.1)

$$Y_{G(1,T)}(T, t_\rho) = Y_{A(q,T)}(T, t_\rho) \ . \tag{6.2.16.9}$$

2. The restriction of M to a $q \in S$ (and hence all $q \in S$) belongs to

$$\sigma_q - \mathrm{Mod}(\mathrm{B})^{[r]} \ .$$

Chapitre 7

Main theorem

7.1 Taylor admissible modules

In the sequel A will be a *bounded* affinoïd, hence

$$\mathcal{Q}(\mathcal{H}_K(A)) \subseteq \{q \in K \mid |q| = 1\} . \tag{7.1.0.1}$$

Definition 7.1.1 (Taylor admissible σ_q-modules)**.** Let $A \subseteq \mathbb{P}^1_K$ be a bounded affinoïd. Let $q \in (\mathcal{Q} - \boldsymbol{\mu}(\mathcal{Q})) \cap \mathrm{D}^-(1,1)$, and let

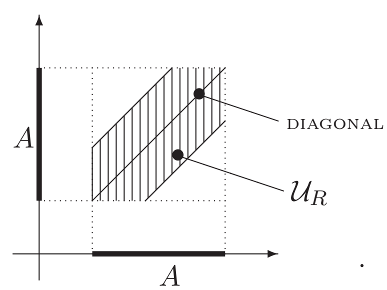
$$\sigma_q - A(q,T) \ , \quad A(q,T) \in GL_n(\mathcal{H}_K(A)) \tag{7.1.1.1}$$

be a $q-$difference equation defining a σ_q-module M. We will say that M is *Taylor admissible on the affinoïd* A if, in this (and hence all) basis of M, the matrix $Y_{A(q,T)}(x,y)$ (defined at 6.1.3.1) converges, and defines an analytic function in a neighborhood $\mathcal{U}_R$ of the diagonal Δ :

$$\Delta \subset \mathcal{U}_R \subseteq A \times A \ , \tag{7.1.1.2}$$

where $\mathcal{U}_R$ is a domain of the form

$$\mathcal{U}_R := \{(x,y) \mid |x-y| < R \} \ , \quad R > 0. \tag{7.1.1.3}$$



The full subcategory of $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(A))$ whose objects are Taylor admissible will be denoted by

$$\sigma_q - \mathrm{Mod}(\mathcal{H}_K(A))^{\mathrm{adm}} \ . \tag{7.1.1.4}$$

Definition 7.1.2 (Taylor admissible $(\sigma_q,\delta_q)-$modules)**.** Let $q \in \mathcal{Q}$ and let

$$\begin{cases} \sigma_q - A(q,T) \quad , \quad A(q,T) \in GL_n(\mathcal{H}_K(A)) \\ \\ \delta_q - G(q,T) \quad , \quad G(q,T) \in M_n(\mathcal{H}_K(A)) \end{cases} \tag{7.1.2.1}$$

be a system defining a $(\sigma_q,\delta_q)-$module M on A. We will say that M is *Taylor admissible on* A if, in this (and hence all) basis of M, the matrix $Y_G(x,y)$ defined at 6.1.12.1 is also solution of the equation $\sigma_q - A(q,T)$. The full subcategory of $(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}_K(A))$, whose objects are Taylor admissible, will be denoted by

$$(\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(A))^{\mathrm{adm}} \ . \tag{7.1.2.2}$$

Remark 7.1.3. 1.– Observe that, by assumption, the matrix $Y_{G(1,T)}(x,y)$ coincides with $Y_{A(q,T)}(x,y)$.

2.– Let $A = \mathbb{P}^1 - \bigcup_i \mathrm{D}^-(c_i,r_i)$. Let $r_A := \min_i(r_i)$. Then, by the Mittag-Leffler decomposition one has $\|f'\|_A \leq r_A^{-1}\|f\|_A$. Hence one obtain the estimation $\|G_{[n]}(1,T)\|_A \leq \max(r_A^{-1},\|G_{[1]}\|_A)^n$, where $G_{[n]}$ is the matrix of $(d/dT)^n$. This implies that the series $Y_G(x,y)$ (solution of $\delta_q - G(q,T)$) converges *automatically* in the neighborhood $\mathcal{U}_R$, where

$$R = \frac{|p|^{\frac{1}{p-1}}}{\max(r_A^{-1},\|G_{[1]}\|_A)} \ . \tag{7.1.3.1}$$

Definition 7.1.4 (Taylor admissible discrete modules on S)**.** Let $S \subset \mathcal{Q}$ be a subset (resp. S satisfy $S^\circ \neq \emptyset$). Let M be a discrete $(\sigma,\delta)-$module (resp. $\sigma-$module) represented by the family of equations

$$\begin{cases} \sigma_q - A(q,T) \quad , \quad A(q,T) \in GL_n(\mathcal{H}_K(A)) \ , \ \forall\, q \in S \ , \\ \\ \delta_1 - G(1,T) \quad , \quad G(1,T) \in M_n(\mathcal{H}_K(A)) \end{cases} \tag{7.1.4.1}$$

(resp.

$$\{\sigma_q - A(q,T)\}_{q\in S} \ , \ \ A(q,T) \in GL_n(\mathcal{H}_K(A)) \ , \ \forall\, q \in S \quad) \ . \tag{7.1.4.2}$$

We will say that $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ is *Taylor admissible on* A if, in this (and hence all) basis of M, the matrix $Y_{G(1,T)}(x,y)$ defined at 6.1.12.1 is simultaneously solution of every equation of 7.1.4.1.

(resp. $(\mathrm{M},\sigma^{\mathrm{M}})$ is *Taylor admissible on* A if, in this (and hence all) basis of M, the matrix $Y_{A(q,T)}(T,y)$, with $q \in S^\circ$, defined at 6.1.3.1, is convergent in a neighborhood of the diagonal of the type $\mathcal{U}_R$, $R > 0$, and is simultaneously solution of every equation of 7.1.4.2.).

The full subcategory of $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(A))_S^{\mathrm{disc}}$ (resp. $\sigma-\mathrm{Mod}(\mathcal{H}_K(A))_S^{\mathrm{disc}}$), whose objects are Taylor admissible, will be denoted by

$$(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(A))_S^{\mathrm{adm}} \tag{7.1.4.3}$$

$$(\text{resp.} \qquad \sigma - \mathrm{Mod}(\mathcal{H}_K(A))_S^{\mathrm{adm}} \quad) \tag{7.1.4.4}$$

7.2 Main theorem

Theorem 7.2.1 (Main Theorem first form)**.** *Let $A \subseteq \mathbb{P}^1_K$ be a bounded* affinoïd. *Then, if $q \in \mathcal{Q} \cap \mathrm{D}^-(1,1)$, and $q \notin \boldsymbol{\mu}(\mathcal{Q})$, the restriction functor (cf. def. 7.1.4)*

$$\bigcup_U \sigma - \mathrm{Mod}(\mathcal{H}_K(A))_U^{\mathrm{adm}} \xrightarrow{\bigcup_U \mathrm{Res}_q^U} \sigma_q - \mathrm{Mod}(\mathcal{H}_K(A))^{\mathrm{adm}} \tag{7.2.1.1}$$

is an equivalence, where U runs in the set of all open neighborhood of q. On the other hand, for all $q \in \mathcal{Q} \cap \mathrm{D}^-(1,1)$, the restriction functor

$$\bigcup_U (\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(A))_U^{\mathrm{adm}} \xrightarrow{\bigcup_U \mathrm{Res}_q^U} (\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(A))^{\mathrm{adm}} \tag{7.2.1.2}$$

is an equivalence, where U runs in the set of all open neighborhood of q.

Proof : Let $(M, \sigma_q^{\mathrm{M}})$ be a σ_q-module. Fix a basis of M and a basis of discrete $q-$solutions $\alpha : \mathrm{M} \to \mathrm{C}$, $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$. We must show that there exist a disk $U := \mathrm{D}^-(q,\tau)$, $\tau > 0$, and structure on M of analytically constant $\sigma-$module over U, which extends to U the structure of Taylor admissible σ_q-module of M. This will result from theorem 7.2.2. This will show the essential surjectivity of the functor, while, if $q \notin \boldsymbol{\mu}(\mathcal{Q})$, the full faithfulness was proved at 5.1.5. $\square$

Theorem 7.2.2 (Main Theorem second form)**.** *Let A be a* bounded *affinoïd and let $q \in \mathcal{Q}(\mathcal{H}_K(A))$. Let*

$$Y(q\cdot T) = A_q(T)\cdot Y(T)\ , \quad A_q(T) \in GL_n(\mathcal{H}_K(A)) \tag{7.2.2.1}$$

be a Taylor admissible *$q-$difference equation (cf. 7.1.1) defining a σ_q-module* M. *Then there exist a $\tau > 0$, and a matrix $A(Q,T)$ uniquely determined by the following properties :*

1. *$A(Q,T)$ is analytic and invertible in the domain* $\mathrm{D}^-(q,\tau)\times A \subset \mathbb{A}^2_K$,
2. *$A(q,T) = A_q(T)$,*
3. *For all $q' \in \mathrm{D}^-_{K^{\mathrm{alg}}}(q,\tau)$, and all $c \in A(K^{\mathrm{alg}})$ the matrix $Y_A(x,y)$ verifies simultaneously*

$$Y_A(q'\cdot T, c) \ = \ A(q',T)\cdot Y_A(T,c) \ . \tag{7.2.2.2}$$

The matrix $A(Q,T)$ is independent from the chosen $c \in A(K^{\mathrm{alg}})$, and from the chosen solution $Y_A(x,y)$.

Proof : Let $c \in A(K)$. Since $Y_A(x,y)$ is invertible in its domain of convergence, then the matrix $A(Q,T)$ must be equal to

$$\begin{aligned} A(Q,T) &= Y_A(Q\cdot T, y)\cdot Y_A(T,y)^{-1}\ , && (7.2.2.3)\\ &= Y_A(Q\cdot T, y)\cdot Y_A(y,T)\ , && (7.2.2.4)\\ &= Y_A(Q\cdot T, T)\ , && (7.2.2.5)\end{aligned}$$

Hence $A(Q,T)$ converges at least in the domain of convergence of $Y_A(QT,T)$ and is invertible in that domain, since $Y_A(x,y)$ is invertible. We shall show that $A(Q,T)$ defines an analytic $\sigma-$module on an open sub-group of $\mathcal{Q}(\mathcal{H}_K(A))$ containing q. Since, by assumption, there exists $R > 0$ such that $Y_A(x,y)$ converges for $|x-y| < R$, $\forall\ x,y \in A$, then $Y_A(QT,T)$ converges for $|Q-1||T| < R$. Since A is a bounded affinoïd, then the matrix $A(Q,T)$ converges at least in the domain

$$\mathrm{D}^-(\ 1\ ,\ R/\max(A)\ )\ \times\ A\ , \tag{7.2.2.6}$$

where $\max(A) := \max_{x\in A(\Omega)}|x|$, and Ω is the field of 6.2.1. On the other hand, one has

$$A(Q,T) = A(Qq^{-1},qT)\cdot A(q,T)\ . \tag{7.2.2.7}$$

Now $A(q,T) = A_q(T)$ converges on the affinoïd A by assumption. On the other hand, $A(Qq^{-1},qT)$ converges in a domain of the type $\mathrm{D}^-(q,\tau)\times A$ if and only if $A(Qq^{-1},T)$ converges on the same domain, since σ_q^{-1} is an automorphism of $\mathcal{H}_K(A)$. We find, as above, that $A(Qq^{-1},T)$ converges in the domain $(Q,T) \in \mathrm{D}^-(q,R|q|/\max(A))\times A$. This shows that $A(Q,T)$ converges in $\mathrm{D}^-(q,R|q|/\max(A))\times A$ too.

The independence from c is evident (cf. 7.2.2.5), and since $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then all other solutions of M in this basis are of the form $Y_A(T,c)\cdot H$, with $H \in GL_n(K)$. Hence $A(Q,T)$ remains the same if $Y_A(T,c)$ is replaced by $Y_A(T,c)\cdot H$. $\square$

Corollary 7.2.3. *One has the following diagram*

$$\begin{tikzcd}
\bigcup_{q\in U}\sigma-\mathrm{Mod}(\mathcal{H}_K(A))_U^{\mathrm{an,adm}} \arrow[r, equal, "3.4.1.2"] \arrow[d, "\bigcup_{q\in U}\mathrm{Res}^U_{\{q\}}"'] \arrow[dr, phantom, "\odot"] & \bigcup_{q\in U}(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(A))_U^{\mathrm{an,adm}} \arrow[d, "\bigcup_{q\in U}\mathrm{Res}^U_{\{q\}}", "\wr"'] \\
\sigma_q-\mathrm{Mod}(\mathcal{H}_K(A))^{\mathrm{adm}} & (\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}_K(A))^{\mathrm{adm}} \arrow[l, "\text{Forget }\delta_q"]
\end{tikzcd} \tag{7.2.3.1}$$

in which :

1. *The right hand vertical functor $\bigcup_{q\in U}\mathrm{Res}^U_{\{q\}}$ is an equivalence for every value of $q \in \mathcal{Q}\cap\mathrm{D}^-(1,1)$;*
2. *If $q \in \mathcal{Q}\cap\mathrm{D}^-(1,1)$, but $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then the also the left hand functor $\bigcup_{q\in U}\mathrm{Res}^U_{\{q\}}$ is an equivalence, and hence every functor appearing in this diagram is an equivalence.*

Proof : $\square$

Remark 7.2.4. We can extend this result for all kind of ring of functions appearing in this paper. If $q \in \boldsymbol{\mu}(\mathcal{Q})$, then we will see that the entire category $\sigma_q-\mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$ formed by modules on $\mathcal{H}_K^\dagger$ having a Frobenius structure is trivial (i.e. every object in that category is direct sum of the trivial object, cf. 8.1.7). In that case the notion of σ_q-module is not interesting. For this reason it is convenable to use the notion of $(\sigma_q, \delta_q)-$module.

Remark 7.2.5. It is possible to generalize the main theorem to other kind of operators different from σ_q. In other words it is possible to "deform" the differential equation into an equation of the type $\sigma - A(T)$, where σ in an automorphism different from σ_q, but sufficiently close to the identity. In a work in progress we will study the action of a $p-$adic Lie group on differential equations.

7.3 Extension to the case $|q| = 1$, but $|q - 1| = 1$

In this section we shall show how to obtain the confluence in the case in which $q \in \mathcal{Q}$, with $|q| = 1$, but $|q - 1| = 1$.

For this we will assume that the reduction $\overline{q} \in k$ of q in the residue field k, has finite order n_0.

$$\overline{q}^{n_0} = \overline{1}\ , \qquad |q^{n_0} - 1| < 1\ . \tag{7.3.0.1}$$

The formal Taylor solution $Y(x, y)$ of a given $q-$difference equation

$$\sigma_q(Y) = A(q, T)Y \tag{7.3.0.2}$$

is also a solution of the iterate equation

$$\sigma_{q^{n_0}}(Y) = A(q^{n_0}, T)Y \tag{7.3.0.3}$$

under certain natural conditions on the radius of convergence of $Y(x, y)$, the Taylor $q-$series $Y(x, y)$ is the Taylor $q-$development in $K[[x - y]]_q$ of an analytic function over a poly-disk (cf. [DV04, Appendix A])

$$\mathrm{D} := \bigcup_{i \geq 0} q^i \mathrm{D}^-(y, \rho)\ . \tag{7.3.0.4}$$

Let $\mathcal{Y}(x, y)$ be the analytic function on D whose Taylor $q-$development is equal to $Y(x, y)$. In particular the function $\mathcal{Y}(x, y)$ converges in a neighborhood of the diagonal. Since $q^{n_0} \in \mathrm{D}^-(1, 1)$, hence the main theorem applies to $\mathcal{Y}(x, y)$. In other words the restriction of $\mathcal{Y}(x, y)$ to a single disc $\mathrm{D}^-(q^i y, \rho)$ is solution of a differential equation.

All details are well exposed in [DV04, Appendix A].

Remark 7.3.1. The precedent easy consideration shows how to obtain a differential equation from the $q-$difference equation in the case in which $|q| = 1$, but $|q - 1| = 1$. This gets the confluence.

The converse of this fact (i.e. the deformation of a differential equation into a $q-$difference equation with $|q| = 1$ and $|q - 1| = 1$) is actually difficult to prove, and is perhaps false.

In a work in progress with L. Di Vizio, we will study this and other related problems.

7.4 Main theorem for other rings

Remark 7.4.1. We observe that if I is open, and if M is a $(\sigma_q, \delta_q)-$module over $\mathrm{A}_K(I)$, then the required condition $Y_G(x,y)$ converges in a neighborhood of the diagonal, is not automatically verified since the norm $\sup_{\rho\in I}|G|_\rho$ can be equal to the infinity, and the relation 7.1.3.1 is no longer true.

We give in detail the extension of this theorem to the cases $\mathrm{B} = \mathcal{R}_K$ and $\mathrm{B} = \mathcal{H}_K^\dagger$. The definition of *admissible* modules needs an adaptation. We preserve notations of section 6.2.3. We recall that for all $\tau > 0$ one set

$$S_\tau := S \cap \mathrm{D}^-(1,\tau)\ . \tag{7.4.1.1}$$

Moreover, for all $\varepsilon > 0$, one set

$$I_\varepsilon := \begin{cases} \quad]1-\varepsilon, 1[& \text{if} \quad \mathrm{B} = \mathcal{R}_K \\ \\ ]1-\varepsilon, 1+\varepsilon[& \text{if} \quad \mathrm{B} = \mathcal{H}_K^\dagger \end{cases}\ . \tag{7.4.1.2}$$

Definition 7.4.2 (Admissibility over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$)**.** Let $\mathrm{B} := \mathcal{R}_K$ or $\mathrm{B} := \mathcal{H}_K^\dagger$. Let $S \subseteq \mathcal{Q} \cap \mathrm{D}^-(1,1)$, (resp. $S^\circ \neq \emptyset$). Let $\mathrm{M} \in (\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ (resp. $\mathrm{M} \in \sigma - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$). Then M is admissible if for all $\tau < 1$, there exists a $\varepsilon_\tau > 0$, such that the restriction

$$\mathrm{Res}^S_{S_\tau}(\mathrm{M}_{\varepsilon_\tau}) \tag{7.4.2.1}$$

is Taylor admissible over $\mathcal{C}(I_{\varepsilon_\tau})$.

Remark 7.4.3. In other words, for all $\tau < 1$, there exists $\varepsilon > 0$ such that $Y(x,y)$ is a simultaneous solution of every operator σ_q^{M} and of δ_1^{M} (resp. every operator σ_q^{M}) for all $q \in \mathrm{S}_\tau$ (resp. for all $q \in \mathrm{S}_\tau^\circ$).

Remark 7.4.4. Recall that the notion of *generic radius* has no meaning for σ_q-modules when q is a root of unity (cf. ex. 6.2.7).

Proposition 7.4.5. *Let $r > 0$, and let $S \subseteq \mathrm{D}^-(1,r)$ be a subset. Let $\mathrm{B} = \mathcal{R}_K$ or $\mathrm{B} = \mathcal{H}_K^\dagger$. Suppose that M belongs to one of the categories*

$$\sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]}\ , \quad \textit{with } S^\circ \neq \emptyset \tag{7.4.5.1}$$

$$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]}\ , \tag{7.4.5.2}$$

then M is the restriction to S of an analytically constant *module over all the disk $\mathrm{D}^-(1,r)$. Moreover, the restriction functors*

$$\sigma - \mathrm{Mod}(\mathrm{B})^{[r]}_{\mathrm{D}^-(1,r)} \xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}} \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]}\ , \quad (S^\circ \neq \emptyset) \tag{7.4.5.3}$$

$$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})^{[r]}_{\mathrm{D}^-(1,r)} \xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}} (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]}\ , \tag{7.4.5.4}$$

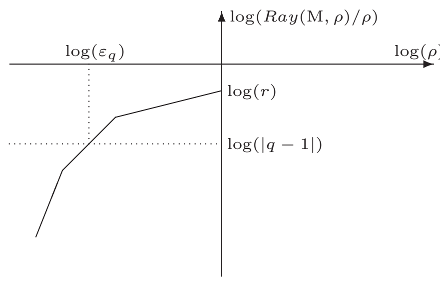
are equivalences. In particular solvable modules extend to all the disk $\mathrm{D}^-(1,1)$.

Proof : We prove only the case $\mathrm{B} = \mathcal{R}_K$, since the case $\mathrm{B} = \mathcal{H}_K^\dagger$ is completely analogous. By 5.1.5, it is enough to prove the essential surjectivity of $\mathrm{Res}_S^{\mathrm{D}^-(1,r)}$. Let M be a solvable object over $\mathcal{R}_K$. Since, by assumption, M is *constant* on S, one can assume $S = \{q\}$. Let $Y(x,y) := Y_A(x,y)$ (resp. $Y(x,y) := Y_G(x,y)$) be the matrix solution of $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ (resp. $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}})$) in a fixed basis. By the main theorem 7.2.1, there exists an open subset $U \subseteq \mathrm{D}^-(1,1)$, containing q, such that M is analytic over U. We will show that U contains $\mathrm{D}^-(1,r)$. Indeed, for all $q \in \mathrm{D}^-(1,r)$ there exists an interval $I_{\varepsilon_q} =]1-\varepsilon_q, 1[$, $\varepsilon_q > 0$, such that the matrix $A(Q,T) := Y(QT,T)$ converges in the domain

$$\mathrm{D}^+(1, |q-1|) \times \mathcal{C}_K(I_{\varepsilon_q}) \ . \tag{7.4.5.5}$$

This results from the fact that $\lim_{\rho\to 1^-} Ray(\mathrm{M},\rho) = r$. Indeed, since $|q-1| < r$, there exists $\varepsilon_q > 0$ such that $Ray(\mathrm{M},\rho) > |q-1|\cdot\rho$, for all $\rho \in I_{\varepsilon_q}$, as showed in the following picture, in which one finds the graphic of the function

$$\log(\rho) \longmapsto \log(Ray(\mathrm{M},\rho)/\rho) \tag{7.4.5.6}$$



This means that if $|y| \in I_{\varepsilon_q}$, then $Y(x,y)$ converges for all (x,y) satisfying $|x-y| < Ray(\mathrm{M},\rho)$. Since $|qx - x| < Ray(\mathrm{M},\rho)$, for all $(q,x) \in \mathrm{D}^+(1,|q-1|) \times \mathcal{C}_K(I_{\varepsilon_q})$, hence the proposition is proved. $\square$

Corollary 7.4.6. *For all $q, q' \in \mathrm{D}^-(1,1)$ (resp. $q, q' \notin \boldsymbol{\mu}(\mathrm{D}^-(1,1))$), and all r satisfying*

$$\max(|q-1|, |q'-1|) < r \leq 1 \ , \tag{7.4.6.1}$$

one has a well defined functor of deformation

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\mathrm{Def}_{q,q'}} (\sigma_{q'}, \delta_{q'}) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \tag{7.4.6.2}$$

$$(\textit{resp.} \quad \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\mathrm{Def}_{q,q'}} \sigma_{q'} - \mathrm{Mod}(\mathcal{R}_K)^{[r]}) \tag{7.4.6.3}$$

which is an equivalence. Moreover, if $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then the functor

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\text{“Forget } \delta_q\text{”}} \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \tag{7.4.6.4}$$

is an equivalence.

7.5 Examples

Example 7.5.1. 1.– We consider the rank one solvable differential equation

$$\delta_1 - \pi_0 T^{-1}, \tag{7.5.1.1}$$

where $\pi_0 = \xi_0 - 1 \in K$, with $\xi_0^p = 1$, and $\xi_0 \neq 1$. Then the solution of that equation is given by

$$y = \exp(\pi_0 T^{-1}) \ . \tag{7.5.1.2}$$

One finds that $A(q,T) = \exp(\pi_0(q^{-1}-1)T^{-1})$, which converges actually in the interval $]\varepsilon, 1[$, with $\varepsilon = |q^{-1}-1| < 1$.

2.– Let U_m be the differential module defined by the differential equation

$$\delta_1 - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \ . \tag{7.5.1.3}$$

Let $\ell_n := \log(T)^n/n!$, for all $n \geq 0$. Then the solution of U_m is given by

$$Y_{U_m} = \begin{pmatrix} 1 & \ell_1 & \cdots & \ell_{m-3} & \ell_{m-2} & \ell_{m-1} \\ 0 & 1 & \ell_1 & \cdots & \ell_{m-3} & \ell_{m-2} \\ 0 & 0 & 1 & \ell_1 & \cdots & \ell_{m-3} \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \ell_1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix} \ . \tag{7.5.1.4}$$

One has $\sigma_q(\ell_n) = \sigma_q(\log(T))^n/n! = (\log(q)+\log(T))^n/n! = \sum_{i=0}^n \frac{\log(q)^{n-k}}{(n-k)!} \cdot \ell_k$. This shows that the matrix of $\sigma_q^{U_m}$ is

$$\sigma_q(Y_{U_m}) = \begin{pmatrix} 1 & \log(q) & \frac{\log(q)^2}{2} & \frac{\log(q)^3}{3!} & \cdots & \frac{\log(q)^{m-1}}{(m-1)!} \\ 0 & 1 & \log(q) & \frac{\log(q)^2}{2} & \cdots & \frac{\log(q)^{m-2}}{(m-2)!} \\ 0 & 0 & 1 & \log(q) & \cdots & \frac{\log(q)^{m-3}}{(m-3)!} \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \log(q) \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \cdot Y_{U_m} \ . \tag{7.5.1.5}$$

7.6 Application to the classification of solvable Rank one $q-$difference equations

In this section we apply to rank one $q-$difference equations over $\mathcal{R}_K$, the results on the classification of rank one differential equations over $\mathcal{R}_K$, obtained in the first part of this thesis.

We fix a Lubin-Tate group $\mathfrak{G}_P$ isomorphic to $\mathbb{G}_m$ over $\mathbb{Q}_p$. We recall that $\mathfrak{G}_P$ is defined by an uniformizer w of $\mathbb{Z}_p$, and by a series $P(X) \in X\mathbb{Z}_p[[X]]$

satisfying $P(X) \equiv \mathrm{w}\cdot X \pmod{X^2\mathbb{Z}_p[[X]]}$ and $P(X) \equiv X^p \pmod{p\mathbb{Z}_p[[X]]}$. Such a formal series is called a *Lubin-Tate series*.

We fix now a sequence $\boldsymbol{\pi} := (\pi_m)_{m\geq 0}$, $\pi_m \in \mathbb{Q}_p^{\mathrm{alg}}$, such that $P(\pi_0) = 0$, $\pi_0 \neq 0$ and $P(\pi_{m+1}) = \pi_m$, for all $m \geq 0$. The element $(\pi_m)_{m\geq 0}$ is a generator of the Tate module of $\mathfrak{G}_P$ which is a free rank one $\mathbb{Z}_p-$module. For example one can chose $\mathfrak{G}_P = \mathbb{G}_m$, hence $P(X) = (X+1)^p - 1$, and $\pi_m = \xi_m - 1$, where ξ_m is a compatible sequence of $p^{m+1}-$th root of 1, i.e. $\xi_0^p = 1$ and $\xi_m^p = \xi_{m-1}$.

<u>One has the following facts :</u>

1. Every rank one *solvable* differential module over $\mathcal{R}_K$ has a basis in which the associated operator is
$$\mathrm{L}(a_0, \boldsymbol{f}^-(T)) := \delta_1 - \Big(a_0 - \sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T))\Big) \;, \tag{7.6.0.6}$$
where, $a_0 \in \mathbb{Z}_p$, and $\boldsymbol{f}^-(T) := (f_0^-(T), \ldots, f_s^-(T))$ is a Witt vector in $\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$, with $K_s := K(\pi_s)$.
2. In that basis in the matrix of the derivation lies in $\mathcal{O}_K[T^{-1}]$.
3. Note that, even though π_j does not belong to K, the Witt vector $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$ is such that the resulting polynomial $\sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T))$ appearing in 7.6.0.6 has coefficients in K.
4. The Taylor solution at ∞ of the differential module in this basis is given by the so called $\boldsymbol{\pi}$-exponential attached to $\boldsymbol{f}^-(T)$:
$$T^{a_0} \cdot \mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1) := T^{a_0} \cdot \exp(\sum_{j=0}^{s} \pi_{s-j} \frac{\phi_j^-(T)}{p^j}) \;, \tag{7.6.0.7}$$
where $\langle \phi_0^-(T), \ldots, \phi_s^-(T)\rangle \in (T^{-1}\mathcal{O}_K[T^{-1}])^s$ is the phantom vector of $\boldsymbol{f}^-(T)$, namely one has $\phi_j^-(T) = \sum_{i=0}^{j} p^i f_i^-(T)^{p^{j-i}}$.
5. The correspondence $\boldsymbol{f}^-(T) \mapsto \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ is a group morphism
$$\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}]) \xrightarrow{\mathrm{e}_{p^s}(-,1)} 1 + \pi_s T^{-1}\mathcal{O}_{K_s}[[T^{-1}]] \;. \tag{7.6.0.8}$$
6. Reciprocally, $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$ is solvable for all such $\boldsymbol{f}^-(T)$. Indeed using the Artin-Hasse exponential, one shows that, for all Witt vector $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$, the exponential $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ is a bounded power series in $1 + \pi_s T^{-1}\mathcal{O}_{K_s}[[T^{-1}]]$ and is then convergent in the disk $\{|T| > 1\}$. This implies, by transfer, that $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$ is solvable.
7. The operator $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$ has a Frobenius structure if and only if $a_0 \in \mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$.

8. Let $K_\infty := K(\{\pi_m\}_{m\geq 0})$ and let k_∞ be its residue field. The operators $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$ and $\mathrm{L}(b_0, \boldsymbol{f}_2^-(T))$ define isomorphic differential modules if and only if the following two conditions are verified
   - $a_0 - b_0 \in \mathbb{Z}$,
   - The Witt vector $\boldsymbol{f}_1^-(T) - \boldsymbol{f}_2^-(T)$ is such that the Artin-Schreier equation
$$\bar{\mathrm{F}}(\overline{\boldsymbol{g}^-(T)}) - \overline{\boldsymbol{g}^-(T)} = \overline{\boldsymbol{f}_1^-(T) - \boldsymbol{f}_2^-(T)} \tag{7.6.0.9}$$
     has a solution $\overline{\boldsymbol{g}^-(T)}$ in $\mathbf{W}_s(k_\infty((t)))$, where t is the reduction of T, and $\bar{\mathrm{F}}$ is the Frobenius of $\mathbf{W}_s(k_\infty((t)))$ (sending $(\bar{g}_0, \ldots, \bar{g}_s)$ into $(\bar{g}_0^p, \ldots, \bar{g}_s^p)$).
9. In particular the most important fact concerning $\boldsymbol{\pi}-$exponentials is that the series $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ is over-convergent (i.e. belongs to $\mathcal{R}_K$) if and only if the Artin-Schreier equation
$$\bar{\mathrm{F}}(\overline{\boldsymbol{g}^-(T)}) - \overline{\boldsymbol{g}^-(T)} = \overline{\boldsymbol{f}^-(T)} \tag{7.6.0.10}$$
   has a solution $\overline{\boldsymbol{g}^-(T)}$ in $\mathbf{W}_s(k_\infty((t)))$.

Remark 7.6.1. All these facts are true also for $q-$difference equations, for all $q \in \mathrm{D}^-(1,1)$, since, by the main theorem 7.2.1 "solutions of $q-$difference equations" coincide with "solutions of differential equations". The fact that solutions of differential equations are solution also of $q-$difference equations can be seen directly in the rank one case. Indeed, by the point 9, for all $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$, the exponential
$$A(q,T) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(qT) - \boldsymbol{f}^-(T), 1) \tag{7.6.1.1}$$
is over-convergent and belongs to $\mathcal{R}_K$, since $\boldsymbol{f}^-(qT)$ and $\boldsymbol{f}^-(T)$ have the same reduction in $\mathbf{W}_s(k_\infty((t)))$.

The example 7.5.1 (point 1) is the particular case in which $\boldsymbol{f}^-(T)$ is the Witt vector of length 0 equal to
$$\boldsymbol{f}^-(T) = T^{-1} \quad \in \quad T^{-1}\mathcal{O}[T^{-1}] \quad = \quad \mathbf{W}_0(T^{-1}\mathcal{O}_K[T^{-1}]) \ .$$

Chapitre 8

Quasi unipotence and $p-$adic local monodromy theorem

In this chapter, we will suppose always that $q \in \mathrm{D}^-(1,1)$. We suppose moreover that K is discretely valued and that its residue field k is perfect, in order to have the $p-$adic local monodromy theorem (cf. [And02], [Ked04], [Meb02]).

We will deduce the *quasi-unipotence* of $q-$difference equations (and more generally of analytically constant $\sigma-$modules and $(\sigma,\delta)-$modules) over the Robba ring, from the *confluence* and from the *quasi-unipotence of differential equations*.

Definition 8.0.2. We set

$$\mathcal{E}_K := \{\sum_{i\in\mathbb{Z}} a_i T^i \mid a_i \in K, \lim_{i\to-\infty} |a_i| = 0, \sup_i |a_i| < +\infty\} \ . \tag{8.0.2.1}$$

The field $\mathcal{E}_K$ is complete with respect to the topology given by the Gauss norm $|\sum a_i T^i|_1 = \sup |a_i|$. Moreover, it is a field, since K has discrete valuation. We denote the so called *bounded Robba ring* by

$$\mathcal{E}_K^\dagger := \mathcal{E}_K \cap \mathcal{R}_K \ . \tag{8.0.2.2}$$

The field $\mathcal{E}_K^\dagger$ has two topologies arising from $\mathcal{R}_K$ and $\mathcal{E}_K$ respectively.

While $\mathcal{R}_K$ and $\mathcal{E}_K$ are complete rings, the field $\mathcal{E}_K^\dagger$ is dense in both $\mathcal{R}_K$ and $\mathcal{E}_K$, with respect to their respective topologies. One has

$$\mathcal{H}_K^\dagger \ \subset \ \mathcal{E}_K^\dagger \ \subset \ \mathcal{R}_K \ . \tag{8.0.2.3}$$

8.1 Frobenius Structure

Let $\varphi : K \to K$ be an absolute Frobenius (i.e. a lifting of the $p-$th power map of k). Since $\mathcal{R}_K$ is not a local ring, and does not have a residual ring, we need a particular definition :

Definition 8.1.1. An absolute *Frobenius* on $\mathcal{R}_K$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$) is a continuous ring morphism, again denoted by $\varphi : \mathcal{R}_K \to \mathcal{R}_K$, extending φ on K and such that

$$\varphi(\sum a_i T^i) = \sum \varphi(a_i)\varphi(T)^i \ , \tag{8.1.1.1}$$

where $\varphi(T) = \sum_{i\in\mathbb{Z}} b_i T^i \in \mathcal{R}_K$ (resp. $\varphi(T) \in \mathcal{H}_K^\dagger$, $\varphi(T) \in \mathcal{E}_K^\dagger$) verifies $|b_i| < 1$, for all $i \neq p$, and $|b_p - 1| < 1$.

Definition 8.1.2. We denote by ϕ the particular absolute Frobenius on $\mathcal{R}_K$ given by the choice

$$\begin{aligned} \phi(T) &:= T^p \ , && (8.1.2.1)\\ \phi(f(T)) &:= f^{\phi}(T^p) \ . && (8.1.2.2) \end{aligned}$$

where $f^{\varphi}(T)$ is the series obtained from $f(T)$ by applying $\varphi : K \to K$ on the coefficients.

Let B be one of the rings $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$, or $\mathcal{R}_K$. For all $q \in \mathrm{D}^-(1,1)$, one has

$$\begin{tikzcd} \mathrm{B} \arrow[r, "\phi"] \arrow[d, "\sigma_{q^p}"'] \arrow[dr, phantom, "\odot"] & \mathrm{B} \arrow[d, "\sigma_q"] \\ \mathrm{B} \arrow[r, "\phi"'] & \mathrm{B} \end{tikzcd} \quad ; \quad \begin{tikzcd} \mathrm{B} \arrow[r, "\phi"] \arrow[d, "p\cdot\delta_1"'] \arrow[dr, phantom, "\odot"] & \mathrm{B} \arrow[d, "\delta_1"] \\ \mathrm{B} \arrow[r, "\phi"'] & \mathrm{B} \end{tikzcd} \quad . \tag{8.1.2.3}$$

Definition 8.1.3 (Frobenius functor)**.** Let $S \subseteq \mathrm{D}^-(1,r)$, $r > 0$. Let

$$r' := \min(\ r^{1/p}\ ,\ r \cdot |p|^{-1}\ ) \ . \tag{8.1.3.1}$$

The Frobenius functor (cf. def. 6.2.16)

$$\begin{aligned} \phi^* : \quad (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]} &\longrightarrow (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r']} \ , && (8.1.3.2)\\ (\text{resp. } \phi^* : \quad \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]} &\longrightarrow \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r']}\ ) && (8.1.3.3) \end{aligned}$$

is defined as follows :

$$\phi^*(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}}) = (\ \phi^*(\mathrm{M})\ ,\ \sigma^{\phi^*(\mathrm{M})}\ ,\ \delta_1^{\phi^*(\mathrm{M})}\ ) \ , \tag{8.1.3.4}$$

where

1. $\phi^*(\mathrm{M}) := \mathrm{M} \otimes_{\mathrm{B},\phi} \mathrm{B}$ is the scalar extension of M via ϕ,
2. the morphism $\sigma^{\phi^*(\mathrm{M})}$ is given by :

$$q \longmapsto \sigma_{q^p}^{\mathrm{M}} \otimes \sigma_q \quad : \quad S \xrightarrow{\sigma^{\phi^*(\mathrm{M})}} \mathrm{Aut}_K^{\mathrm{cont}}(\phi^*(\mathrm{M})) \ , \tag{8.1.3.5}$$

3. the derivation is given by

$$\delta_1^{\phi^*(\mathrm{M})} = (p \cdot \delta_1^{\mathrm{M}}) \otimes \mathrm{Id}_{\mathrm{B}} + \mathrm{Id}_{\mathrm{M}} \otimes \delta_1^{\mathrm{B}} \ , \tag{8.1.3.6}$$

4. a morphism $\alpha : \mathrm{M} \to \mathrm{N}$ is sent into $\alpha \otimes 1 : \phi^*(M) \to \phi^*(\mathrm{N})$.

Remark 8.1.4. 1.– Let $\mathrm{M} \in (\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[r]}$. Let $Y_1(T) = \sum_{i\geq 0} Y_i(T-1)^i$, $Y_i \in M_n(K)$, be its Taylor solution at 1. Then the Taylor solution of $\phi^*(\mathrm{M})$ is

$$Y_1^\phi(T^p) \ := \ \sum_{i\geq 0} \varphi(Y_i)(T^p-1)^i \ . \tag{8.1.4.1}$$

2.– Fix a basis $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ of M. Let $\sigma_q - A(q,T)$ (resp. $\delta_1 - G(1,T)$) be the operator associated to σ_q^{M} (resp. δ_1^{M}) in this basis. Then the operators associated to $\phi^*(\mathrm{M})$ in the basis $\mathbf{e}\otimes 1$ are

$$\sigma_q - A^\phi(q^p, T^p) \ , \qquad \delta_1 - p\cdot G^\phi(1,T^p) \ , \tag{8.1.4.2}$$

where, according with 3.1.3.2, one has $A(q^p,T) = A(q,q^{p-1}T)\cdots A(q,qT)A(q,T)$.

3.– If M has a Frobenius structure, then $r = r'$ and hence M is solvable.

Definition 8.1.5 (Frobenius structure)**.** Let B be one of the rings $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$, or $\mathcal{R}_K$. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset. Let M be a discrete $\sigma-$module (resp. $(\sigma,\delta)-$module) over S. We will say that M has a Frobenius structure of order $h \geq 1$, if there exists an isomorphism

$$(\phi^*)^{(h)}(\mathrm{M}) \xrightarrow{\sim} \mathrm{M} \ , \tag{8.1.5.1}$$

where $(\phi^*)^{(h)} := \phi^* \circ \cdots \circ \phi^*$, $h-$times. We denote by

$$\sigma - \mathrm{Mod}(\mathrm{B})_S^{(\phi)} \tag{8.1.5.2}$$

$$(\text{resp.} \quad (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{(\phi)} \quad) \tag{8.1.5.3}$$

the full subcategory of $\sigma-\mathrm{Mod}(\mathrm{B})_S^{[1]}$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{[1]}$) whose objects have a Frobenius structure of some order.

Remark 8.1.6. Suppose that $\mathrm{M} \in (\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$ has a Frobenius structure of order $h \geq 1$. In terms of solution matrix, this means that there exists a matrix $H(T) \in GL_n(\mathcal{H}_K^\dagger)$ such that

$$Y^{\phi^h}(T^{p^h},1) = H(T)\cdot Y(T,1) \ , \tag{8.1.6.1}$$

where $Y_1(T)$ is the Taylor solution of M at 1.

Proposition 8.1.7. *Let $S = \{\xi\}$, with $\xi^{p^n} = 1$. Then every $\sigma_\xi-$module over $\mathcal{H}_K^\dagger$ with a Frobenius structure is trivial (i.e. isomorphic to a direct sum of the unit object).*

Proof : Let $\mathrm{M} \in \sigma_\xi - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$. Let $Y(T,1) \in GL_n(\mathcal{H}_K^\dagger)$ be its Taylor solution at 1, in the basis $\mathbf{e}$ of M. Then, by remark 8.1.6, there exists $H(T)$ such that $Y^{\phi^h}(T^{p^h},1) = H(T)\cdot Y(T,1)$. Hence, one has also $Y^{\phi^{nh}}(T^{p^{nh}},1) = H_n(T)\cdot Y(T,1)$, for some $H_n(T) \in GL_n(\mathcal{H}_K^\dagger)$. Since

$$\sigma_\xi(Y^{\phi^{nh}}(T^{p^{nh}})) = Y^{\phi^{nh}}(T^{p^{nh}}) \ , \tag{8.1.7.1}$$

then in the basis $\mathcal{H}_n(T)\cdot\mathbf{e}$ the matrix of σ_ξ is trivial : $A(\xi,T) = \mathrm{Id}$. □

8.2 Special coverings of $\mathcal{H}_K^\dagger$, canonical extension.

Since K has a discrete valuation, $\mathcal{E}_K^\dagger$ is an henselian local field (not complete) with residue field equal to $k(\!(t)\!)$ (with respect to the norm $|.|_1$). On the other hand the residue ring of $\mathcal{H}_K^\dagger$ (with respect to the Gauss norm $|.|_1$) is $k[t,t^{-1}]$. One has

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{H}_K^\dagger} & \subseteq & \mathcal{O}_{\mathcal{E}_K^\dagger} \\ \downarrow & \odot & \downarrow \\ k[t,t^{-1}] & \subseteq & k(\!(t)\!). \end{array} \tag{8.2.0.2}$$

Definition 8.2.1. We denote by $\mathcal{O}_K[T,T^{-1}]^\dagger$ the weak completion of $\mathcal{O}_K[T,T^{-1}]$, in the sense of Monsky and Washnitzer (cf. [MW68]).

One has

$$\mathcal{H}_K^\dagger = \mathcal{O}_K[T,T^{-1}]^\dagger \otimes_{\mathcal{O}_K} K\ . \tag{8.2.1.1}$$

8.2.1 Special coverings of $\mathbb{G}_{m,k}$.

Let us look at the residual situation. The morphism

$$\widehat{\eta} := \mathrm{Spec}(k(\!(t)\!)) \quad \hookrightarrow \quad \mathbb{G}_{m,k} = \mathrm{Spec}(k[t,t^{-1}]) \tag{8.2.1.2}$$

gives rise, by pull-back, to a map

$$\left\{ \begin{array}{l} \text{Finite Étale} \\ \text{coverings of } \widehat{\eta} \end{array} \right\} \xleftarrow{\text{Pull-back}} \left\{ \begin{array}{l} \text{Finite Étales} \\ \text{coverings of } \mathbb{G}_{m,k} \end{array} \right\} . \tag{8.2.1.3}$$

It is known (cf. [Kat86, 2.4.9]) that this map is surjective, and moreover that there exists a full sub-category of the right hand category, called *special coverings of* $\mathbb{G}_{m,k}$, which is equivalent, via pull-back, to the category on the left hand side. Special covers are defined by the property that they are tamely ramified at ∞, and that their geometric Galois group have a unique $p-$Sylow subgroup (cf. [Kat86, 1.3.1]).

On the other hand, if $\pi \in \mathcal{O}_K$ is an uniformizer element, then both $(\mathcal{O}_{\mathcal{E}_K^\dagger},(\pi))$ and $(\mathcal{O}_K[T,T^{-1}]^\dagger,(\pi))$ are Henselian couples in the sense of [Ray70, Ch.II] (cf. [Mat02, 5.1]). Hence the precedent situation lifts in characteristic

0 :

$$\begin{tikzcd}
\left\{\begin{array}{l}\text{Special}\\ \text{extensions of } \mathcal{H}_K^\dagger\end{array}\right\} \arrow[r, "-\otimes\mathcal{E}_K^\dagger", "\sim"'] & \left\{\begin{array}{l}\text{Finite unramified}\\ \text{extensions of } \mathcal{E}_K^\dagger\end{array}\right\} \arrow[r, "-\otimes\mathcal{R}_K", "\sim"'] & \left\{\begin{array}{l}\text{Special}\\ \text{ext. of } \mathcal{R}_K\end{array}\right\} \\
\left\{\begin{array}{l}\text{Special extensions}\\ \text{of } \mathcal{O}_K[T,T^{-1}]^\dagger\end{array}\right\} \arrow[u, "-\otimes K", "\wr"'] \arrow[r, "-\otimes\mathcal{O}_{\mathcal{E}_K^\dagger}", "\sim"'] \arrow[d, "-\otimes k"', "\wr"] \arrow[ur, phantom, "\odot"] & \left\{\begin{array}{l}\text{Finite onramified}\\ \text{extensions of } \mathcal{O}_{\mathcal{E}_K^\dagger}\end{array}\right\} \arrow[u, "\wr", "-\otimes K"'] \arrow[d, "\wr"', "-\otimes k"] \\
\left\{\begin{array}{l}\text{Special}\\ \text{coverings of } \mathbb{G}_{m,k}\end{array}\right\} \arrow[r, "\text{Pull-back}", "\sim"'] \arrow[ur, phantom, "\odot"] & \left\{\begin{array}{l}\text{Finite étale}\\ \text{coverings of } \hat{\eta}\end{array}\right\}
\end{tikzcd} \tag{8.2.1.4}$$

where, by special extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$) we means a finite étale Galois extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$) coming, by henselianity, from a special cover of $\mathbb{G}_{m,k}$.

Remark 8.2.2. 1.– One can show that every unramified extension $(\mathcal{E}_K^\dagger)'$ of $\mathcal{E}_K^\dagger$ (resp. special extension $\mathcal{R}_K'$ of $\mathcal{R}_K$) is *non canonically* isomorphic to $\mathcal{E}_{K'}^\dagger$ (resp. $\mathcal{R}_{K'}$), for some finite Galois unramified extension K'/K. This situation is analogue to the classical one, in which every extension of $\mathbb{C}(\!(T)\!)$ is of the form $\mathbb{C}(\!(T^{m/n})\!)$, for some integers $m, n \geq 0$, and hence it is isomorphic to $\mathbb{C}(\!(Z)\!)$, with $T = f(Z)$, $f(Z) := Z^{n/m}$.

In the case of special extensions of $\mathcal{E}_K^\dagger$ or $\mathcal{R}_K$, the relation between the new variable and the old one is highly non trivial, and essentially unknown. If k'/k is the residue field of K', and if $t = \overline{f}(z) \in k'(\!(z)\!)$ is the relation between t and z in characteristic p, then the relation between T and Z in characteristic 0 is given by $T = f(Z) \in \mathcal{O}_{\mathcal{E}_K^\dagger}$, where $f(Z)$ is an arbitrary Laurent series, obtained from $\overline{f}(z)$ by lifting coefficient by coefficient.

2.– Let us write $\mathcal{E}_{K,T}^\dagger := \mathcal{E}_K^\dagger$, and $\mathcal{E}_{K',Z}^\dagger := (\mathcal{E}_K^\dagger)'$. Let $q \in \mathrm{D}^-(1,1)$, the automorphism σ_q of $\mathcal{H}_{K,T}^\dagger$ extends uniquely to a continuous $K'-$linear automorphism of rings $\sigma_q' : \mathcal{E}_{K',Z}^\dagger \to \mathcal{E}_{K',Z}^\dagger$. Indeed let $P(Z) = 0$, $P(X) \in \mathcal{O}_{\mathcal{E}_{K,T}^\dagger}[X]$, be the minimal polynomial of Z. Let $P^{\sigma_q}(X)$ be the polynomial obtained from P by applying σ_q to the coefficients. Then, by henselianity, there is bijection between roots of $P(X)$ and of $P^{\sigma_q}(X)$. Hence σ_q' is the unique continuous $K'-$linear automorphism given by $\sigma_q'(Z) := Z'$, where Z' is the unique root of P^{σ_q} whose reduction in $k(\!(z)\!)$ is equal to z.

The real problem of the theory is that the extended automorphism does not send Z into qZ.

3.– The same considerations shows that σ extends uniquely to an automorphism of every special extension of $\mathcal{H}_K^\dagger$ and $\mathcal{R}_K$, and commutes with the action of $\mathrm{Gal}(k(\!(t)\!)^{sep}/k(\!(t)\!))$.

8.3 Quasi unipotence of $\sigma-$modules and $(\sigma,\delta)-$modules with Frobenius structure

Definition 8.3.1. We denote by $\widetilde{\mathcal{H}_K^\dagger}$ (resp. $\widetilde{\mathcal{E}_K^\dagger}$, $\widetilde{\mathcal{R}_K}$) the union of all special extensions of $\mathcal{H}_K^\dagger$ (resp. unramified extension of $\mathcal{E}_K^\dagger$; special extensions of $\mathcal{R}_K$)

Definition 8.3.2. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset (resp. $S \subseteq \mathrm{D}^-(1,1)$, with $S^\circ \neq \emptyset$). A discrete $(\sigma,\delta)-$module on S (resp. discrete $\sigma-$module on S) is called *quasi-unipotent* if it is trivialized by the discrete $(\sigma,\delta)-$algebra

$$\widetilde{\mathcal{H}_K^\dagger}[\log(T)] \qquad (\text{resp. } \widetilde{\mathcal{E}_K^\dagger}[\log(T)]\ ,\ \widetilde{\mathcal{R}_K}[\log(T)]\ )\ . \tag{8.3.2.1}$$

8.3.1 $p-$adic local monodromy and some corollaries

Theorem 8.3.3 ($p-$adic local monodromy theorem)**.** *If* $\mathrm{M} \in \delta_1-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$, *then there exists a finite extension* K'/K *such that* $\mathrm{M}\otimes_K K'$ *is quasi unipotent (i.e. trivialized by* $\widetilde{\mathcal{H}_{K'}^\dagger}[\log(T)]$*). In other words, objects in* $\delta_1-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ *become quasi-unipotent after an eventual extension of the field of constants.*

Proof : See [And02],[Ked04],[Meb02]. □

Theorem 8.3.4. *If a differential equation* $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)$ *is quasi-unipotent, then it has a Frobenius structure. Moreover, the scalar extension functor*

$$\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \xrightarrow{-\otimes\mathcal{R}_K} \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \tag{8.3.4.1}$$

is essentially surjective.

Proof : [Mat02, 7.10], [Mat02, 7.15]. □

Theorem 8.3.5. *There exists a full sub-category of* $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$ *which is equivalent to* $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ *via the scalar extension functor 8.3.4.1. Moreover, objects in that category are trivialized by* $\widetilde{\mathcal{H}_K^\dagger}[\log(T)]$.

Proof : [Mat02, 7.15]. □

Definition 8.3.6 (Canonical extension)**.** Objects in the category above will be called *special objects*. We will denote by

$$\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \xrightarrow{\mathrm{Can}} \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \tag{8.3.6.1}$$

the section of the functor 8.3.4.1, whose image is the category of special objects (cf. 8.3.5). We will call it the *canonical extension functor*.

Corollary 8.3.7. *Let* $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$*, then there exists a finite extension* K'/K *such that* $\mathrm{M} \otimes_K K'$ *decomposes in a direct sum of submodules of the form*

$$\mathrm{N} \otimes U_m \;, \tag{8.3.7.1}$$

where N *is a module trivialized by a special extension* $\mathcal{R}_K$*, and* U_m *is the* $m-$*dimensional object defined by the operator (cf. ex. 7.5.1)*

$$\delta_1 - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} . \tag{8.3.7.2}$$

Proof : [And02, 7.1.6]□

Remark 8.3.8. The $\log(T)$ appearing in 8.3.2.1, is present uniquely to trivialize the module U_m, $m \geq 2$ (cf. ex. 7.5.1).

Lemma 8.3.9. *Let* $\mathrm{N} \in \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$ *be a (special) object trivialized by* $\widetilde{\mathcal{H}_K^\dagger}$*. Let* $\widetilde{Y} = (\widetilde{y}_{i,j}) \in GL_n(\widetilde{\mathcal{H}^\dagger})$ *be a fundamental matrix of solution of* N*. Let* $(\mathcal{E}^\dagger)'$ *(resp.* $\mathcal{R}'$*) be the smallest special extension of* $\mathcal{E}_K^\dagger$ *(resp.* $\mathcal{R}_K$*), such that* $\mathrm{N} \otimes \mathcal{E}_K^\dagger$ *is trivialized by* $(\mathcal{E}^\dagger)'$ *(*$\mathrm{N} \otimes \mathcal{R}_K$ *is trivialized by* $\mathcal{R}'$*). Then one has*

$$(\mathcal{E}^\dagger)' = \mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \;, \qquad \mathcal{R}' = \mathcal{R}_K[\{\widetilde{y}_{i,j}\}_{i,j}] \;. \tag{8.3.9.1}$$

In other words, the smallest special extension of $\mathcal{E}_K^\dagger$ *(resp.* $\mathcal{R}_K$*) trivializing* M *is generated by the solutions of* M*.*

Proof : Since M is trivialized by $\widetilde{\mathcal{E}^\dagger}$, then $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \subseteq (\mathcal{E}^\dagger)'$. Hence $\mathcal{E}_{K'}^\dagger[\widetilde{y}_{i,j}]$ is an unramified extension, and is then a special extension. This shows that $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] = (\mathcal{E}^\dagger)'$. The case over $\mathcal{R}_K$ follows from the case over $\mathcal{E}_K^\dagger$. □

Corollary 8.3.10. *Let us preserve the hypothesis of lemma 8.3.9. There exists a unique* $K-$*linear ring automorphism* σ_q *of* $\mathcal{E}_K^\dagger[\widetilde{y}_{i,j}]$*, which induces the identity on the residual field.*

Proof : By 8.3.9, $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ is a special extension (i.e. Henselian). Hence, by 8.2.2 point 2., then the extension of σ_q to $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ extension is unique.□

8.3.2 Quasi unipotence of $\sigma-$modules and $(\sigma, \delta)-$modules

In this section we shall prove the quasi-unipotence of every object in $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$.

Corollary 8.3.11. *Let $S \subseteq \mathrm{D}^-(1,1)$ (resp. $S^\circ \neq \emptyset$). The scalar extension functor*

$$(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{\mathrm{disc},(\phi)} \xrightarrow{-\otimes\mathcal{R}_K} (\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{disc},(\phi)} \qquad (8.3.11.1)$$

$$(\textit{resp.} \qquad \sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{\mathrm{disc},(\phi)} \xrightarrow{-\otimes\mathcal{R}_K} \qquad \sigma-\mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{disc},(\phi)} \quad)(8.3.11.2)$$

is essentially surjective.

By 7.4.5, it is sufficient to prove that the functor

$$(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \xrightarrow{-\otimes\mathcal{R}_K} (\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \qquad (8.3.11.3)$$

$$(\text{resp.} \qquad \sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \xrightarrow{-\otimes\mathcal{R}_K} \qquad \sigma-\mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \quad)(8.3.11.4)$$

is essentially surjective. By 8.3.4 there exists a basis of M in which the matrix $G(1,T)$ of δ_1^{M} lies in $M_n(\mathcal{H}_K^\dagger)$. Moreover, $\mathrm{Can}(\mathrm{M},\delta_1^{\mathrm{M}})$ is Taylor admissible, since all solvable differential equations are Taylor admissible. By 7.4.5, for all $q \in \mathrm{D}^-(1,1)$, the matrix $A(Q,T) := Y_G(QT,T)$ defines an analytic function in a domain of the form

$$\mathrm{D}^+(1,|q-1|) \times \mathcal{C}_K(]1-\varepsilon_q, 1+\varepsilon_q[) \;, \qquad (8.3.11.5)$$

with $\varepsilon_q > 0$. Hence, in this basis, the matrix $A(q,T)$ belongs to $GL_n(\mathcal{H}_K^\dagger)$, for all $q \in \mathrm{D}^-(1,1)$. $\Box$

Lemma 8.3.12. *Let $\mathrm{M} \in \delta_1-\mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$. Let K'/K be a finite extension such that $\mathrm{M}\otimes_K K'$ is quasi unipotent. Let $(\mathcal{E}^\dagger)'$ be the smallest special extension of $\mathcal{E}_{K'}^\dagger$ such that $\mathrm{M}\otimes_K K'$ is trivialized by $(\mathcal{E}^\dagger)'[\log(T)]$. Let $\widetilde{Y} \in GL_n(\widetilde{\mathcal{E}_{K'}^\dagger}[\log(T)])$ be a fundamental matrix solution of* M*. Then there exists a K''/K such that the matrix*

$$\widetilde{A}(q,T) := \sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1} \qquad (8.3.12.1)$$

belongs to $\mathcal{E}_{K''}^\dagger$, for all $q \in \mathrm{D}_{K''}^-(1,1)$.

Proof : We can suppose that $K = K'$. By 8.3.7, we can suppose that M is equal to N, or that M is equal to U_m, where N and U_m are defined by the theorem 8.3.7. Indeed, if the lemma is verified by every such N and every such U_m, then the lemma holds for M which is sum of tensor products of such modules. The case "$\mathrm{M} = U_m$" is trivial, since both the matrices of $\delta_1^{U_m}$ and of $\sigma_q^{U_m}$ are constant (cf. 7.5.1). Let now M = N. In this case

$$\widetilde{Y} \in GL_n((\mathcal{E}^\dagger)') \;. \qquad (8.3.12.2)$$

Then, for all $\gamma \in \mathrm{Gal}((\mathcal{E}^\dagger)'/\mathcal{E}_K^\dagger)$, one has

$$\gamma(\widetilde{Y}) = \widetilde{Y} \cdot H_\gamma \;, \quad H_\gamma \in GL_n(K) \;. \qquad (8.3.12.3)$$

Since σ_q commutes with every such $\gamma \in \mathrm{Gal}((\mathcal{E}^\dagger)'/\mathcal{E}_K^\dagger)$ (cf. 8.2.2), then one find

$$\begin{aligned}\gamma(\widetilde{A}(q,T)) &= \gamma(\sigma_q(\widetilde{Y})\cdot\widetilde{Y}^{-1}) &(8.3.12.4)\\ &= \sigma_q(\widetilde{Y})\cdot H_\gamma\cdot(\widetilde{Y}\cdot H_\gamma)^{-1} = \widetilde{A}(q,T)\ . &(8.3.12.5)\end{aligned}$$

Hence $\widetilde{A}(q,T)$ belongs to $\mathcal{E}_K^\dagger$, for all $|q-1|<1$. $\square$

Corollary 8.3.13 ($p-$adic local monodromy theorem (generalized form))**.** *Let $S \subset \mathrm{D}^-(1,1)$, be a subset (resp. $S^\circ \neq \emptyset$). Then every object in $(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)}$ (resp. $\sigma-\mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)}$) becomes quasi unipotent after an eventual finite extension K' of K.*

Proof : By 7.4.5, one has $(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} = (\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)}$, (resp. $\sigma-\mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} = \sigma-\mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)}$). On the other hand, $(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} = \sigma-\mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)}$ (cf. 3.4.1.2). Hence, we can suppose that M is an analytic $(\sigma,\delta)-$module on the disk $\mathrm{D}^-(1,1)$, with a Frobenius structure. We can suppose that $K'=K$, i.e. M is quasi unipotent and is trivialized by the extension $\mathcal{R}'[\log(T)]$, where $\mathcal{R}'/\mathcal{R}_K$ comes from a totally ramified extension of $k(\!(t)\!)$.

Let us consider a basis of M in which the matrices of δ_1^{M} and σ_q^{M} have coefficients in $\mathcal{H}_K^\dagger$ (cf. 8.3.11).

Now we start from the analytic $(\sigma,\delta)-$module $(\mathrm{M},\delta_1^{\mathrm{M}},\sigma^{\mathrm{M}})$ on $\mathrm{D}^-(1,1)$. We forget σ^{M}, and consider only the differential equation $(\mathrm{M},\delta_1^{\mathrm{M}})$. Then, by 8.3.12, there exists a second structure of discrete $\sigma-$module on M on $\mathrm{D}^-(1,1)$, arising from the fact that M is trivialized by $\widetilde{\mathcal{H}_K^\dagger}[\log(T)]$. Let us call $(\mathrm{M},\delta_1^{\mathrm{M}},\widetilde{\sigma}^{\mathrm{M}})$ this second structure. We shall show that

$$\sigma^{\mathrm{M}} = \widetilde{\sigma}^{\mathrm{M}}\ . \tag{8.3.13.1}$$

We consider the Taylor solution of M at the point 1 :

$$Y(T,1) = (y_{i,j})_{i,j} \in GL_n(\mathcal{A}_K(1,1))\ . \tag{8.3.13.2}$$

On the other hand, in the same basis, we consider the solution $\widetilde{Y}$ of M in $\widetilde{\mathcal{H}_K^\dagger}$:

$$\widetilde{Y} = (\widetilde{y}_{i,j})_{i,j} \in GL_n(\widetilde{\mathcal{H}_K^\dagger}[\log(T)])\ . \tag{8.3.13.3}$$

We shall verify that, for all $q \in \mathrm{D}^-(1,1)$, the matrix $A(q,T) \in GL_n(\mathcal{H}_K^\dagger)$ defined by the main theorem 7.2.2

$$A(q,T) := Y(qT,1)\cdot Y(T,1)^{-1}\ , \tag{8.3.13.4}$$

is actually equal to the matrix $\widetilde{A}(q,T) \in GL_n(\mathcal{E}_K^\dagger)$

$$\widetilde{A}(q,T) := \sigma_q(\widetilde{Y})\cdot\widetilde{Y}^{-1}\ , \tag{8.3.13.5}$$

defined by the lemma 8.3.12. To do this we need the following lemma :

Lemma 8.3.14. *Let* $(\mathrm{M},\delta_1^{\mathrm{M}}) \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$*. Then, after an eventual finite extension* K'/K*, the decomposition given at 8.3.7 extends to a decomposition of the analytic* $(\sigma,\delta)-$*module* $(\mathrm{M},\delta_1^{\mathrm{M}},\sigma^{\mathrm{M}})$ *on* $\mathrm{D}^-(1,1)$*, attached to* M *via the proposition 7.4.5, and also to a decomposition of the discrete* $(\sigma,\delta)-$*module* $(\mathrm{M},\delta_1^{\mathrm{M}},\widetilde{\sigma}^{\mathrm{M}})$ *on* $\mathrm{D}^-(1,1)$*, attached to* M *via the proposition 8.3.12.*

Proof : By 7.4.5, 8.3.12 and 5.1.5, the forget functors

$$(\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{\mathrm{an,const},(\phi)}_{\mathrm{D}^-(1,1)} \xrightarrow[\sim]{\mathrm{Res}_1^{\mathrm{D}^-(1,1)}} \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$$

$$\text{and } (\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}},\mathrm{C})^{\mathrm{disc,const},(\phi)}_{\mathrm{D}^-(1,1)} \xrightarrow[\sim]{\mathrm{Res}_1^{\mathrm{D}^-(1,1)}} \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$$

are equivalences, where $\mathcal{R}_{K^{\mathrm{alg}}} := \mathcal{R}_K \otimes_K K^{\mathrm{alg}}$ and $\mathrm{C} := \widetilde{\mathcal{R}_{K^{\mathrm{alg}}}}[\log(T)]$. Indeed every object on these categories comes by scalar extension from a module over $\mathcal{R}_{K'}$, for some finite extension K'/K. $\square$

Continuation of proof of 8.3.13 : By 8.3.14, it is sufficient to discuss the case $\mathrm{M} = U_m$ and $\mathrm{M} = \mathrm{N}$ (cf. 8.3.7). The case $\mathrm{M} = U_m$ is trivial (cf. 7.5.1) : one has that $A(q,T) = \widetilde{A}(q,T)$ has constant coefficients.

Let us suppose that $\mathrm{M} = \mathrm{N}$ is trivialized by $\widetilde{\mathcal{R}_K}$. We recall that N is defined over $\mathcal{H}_K^\dagger$, and hence $\widetilde{Y} \in GL_n(\widetilde{\mathcal{H}_K^\dagger}) \subseteq GL_n(\widetilde{\mathcal{E}_K^\dagger})$, and $Y(T,1) \in GL_n(\mathcal{A}_K(1,1))$. In other words we are working with Can(N) (cf. 8.3.6).

We consider then the discrete $(\sigma,\delta)-$algebras $\mathcal{E}_K^\dagger[\{y_{i,j}\}_{i,j}]$ and $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$. By assumption, one has an isomorphism commuting with δ_1^{M}

$$\mathcal{E}_K^\dagger[\{y_{i,j}\}_{i,j}] \xrightarrow[y_{i,j}\longmapsto\widetilde{y}_{i,j}]{\sim} \mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]\ . \tag{8.3.14.1}$$

We must show that this morphism commutes also with σ_q. This follows immediately from the fact that there is a unique $K-$linear automorphism σ_q on $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$, inducing the identity on $k(\!(t)\!)$ (cf. 8.3.10). $\square$

8.4 The confluence of André-Di Vizio

In [ADV04] authors consider σ_q-modules over $\mathcal{R}_K$ with a Frobenius structure (i.e. $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$). Moreover they restrict themselves to a fixed $q \in \mathrm{D}^-(1,1)$ such that $|q-1| < |p|^{\frac{1}{p-1}}$, hence have no trouble with roots of unity. Their aim is to find a "theory of slopes" (*filtration de type Hasse-Arf*) for this kind of $q-$difference equations in order to apply the main theorem of [And02] and deduce an equivalence :[1]

$$\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \xrightarrow[\sim]{} \left\{ \begin{array}{l} \text{Repr. of the inertia} \\ \text{of } \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t))) \\ \text{with finite image} \end{array} \right\} . \qquad (8.4.0.2)$$

The proof of this fact needs a remarkable effort and is the central result of the paper. On the other hand by the $p-$adic local monodromy theorem for differential equations, one has another equivalence

$$\left\{ \begin{array}{l} \text{Repr. of the inertia} \\ \text{of } \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t))) \\ \text{with finite image} \end{array} \right\} \xrightarrow[\sim]{\text{Local monodromy}} \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} . \qquad (8.4.0.3)$$

They call “confluence” the composite functor

$$\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \xrightarrow[\sim]{\text{“confluence”}} \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} . \qquad (8.4.0.4)$$

In a second time they try to describe explicitly this equivalence. For this they introduce the notion of “*confluent weak Frobenius structure*” using the fact that the sequence q^{p^n} goes to 1 ([ADV04, 12.3]) :

Remark 8.4.1. We recall that an *antecedent* of a σ_q-module M over $\mathcal{R}_K$ is a σ_q-module M_{-s} together with an isomorphism

$$\Phi : (\phi^*)^s \mathrm{M}_{-s} \xrightarrow{\sim} \mathrm{M} . \qquad (8.4.1.1)$$

for some $s \geq 1$.

Definition 8.4.2 ([ADV04, 12.11])**.** Let $|q-1| < |p|^{\frac{1}{p-1}}$ and let $s \geq 1$ be a natural number. A *confluent weak Frobenius structure* on a σ_q-module $\mathrm{M} \in \sigma_q - \mathrm{Mod}(\mathcal{R}_K)$ is a sequence of “antecedents” $\{\mathrm{M}_{-sm} = (\mathrm{M}, \sigma_{q^{p^{sm}}}^{\mathrm{M}_{-sm}})\}_{m\geq 0}$, such that $\mathrm{M}_0 = (\mathrm{M}, \sigma_q^{\mathrm{M}})$, and satisfying

1. The operators $\Delta_{q^{p^{sm}}}^{\mathrm{M}_{-sm}} := \frac{\sigma_{q^{p^{sm}}}^{\mathrm{M}_{-sm}} - 1}{q^{p^{sm}} - 1}$ converge to a derivation $\Delta^{\mathrm{M}_\infty}$ on M.
2. If $\mathrm{M}_\infty := (\mathrm{M}, \Delta^{\mathrm{M}_\infty})$ is that differential module, then the sequence of isomorphisms

   $$\Phi_{-sm} : (\phi^*)^s(\mathrm{M}_{-sm}) \xrightarrow{\sim} \mathrm{M}_{-s(m-1)} , \qquad (8.4.2.1)$$

   converge to a Frobenius isomorphism

   $$\Phi_\infty : \phi^*(\mathrm{M}_\infty) \xrightarrow{\sim} \mathrm{M}_\infty . \qquad (8.4.2.2)$$

[1] Recall that $\mathcal{R}_{K^{\mathrm{alg}}} := \mathcal{R}_K \otimes K^{\mathrm{alg}}$ (cf. proof of 8.3.14).

Actually, if M has a Frobenius structure, then the notion of "Confluent weak Frobenius structure" is nothing else that the strong confluence :

Lemma 8.4.3. *Let* $|q-1| < |p|^{\frac{1}{p-1}}$. *If* $\mathrm{M} \in \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$, *then* M *has a "Confluent weak Frobenius structure" and moreover*

$$\mathrm{M}_{-sm} = \mathrm{Def}_{q,q^{p^s}}(\mathrm{M}) \ , \quad \mathrm{M}_\infty = \mathrm{Def}_{q,1}(\mathrm{M}) \ . \qquad (8.4.3.1)$$

Proof : Suppose that M has a Frobenius structure of order $s \geq 1$. Hence for all $s \geq 1$ there exists an isomorphisms $\psi : \mathrm{M} \xrightarrow{\sim} (\phi^*)^s(\mathrm{M})$. Let $\widetilde{\mathrm{M}}$ be the analytic $\sigma-$module defined by M (cf. 7.4.5). We identify $\widetilde{\mathrm{M}}$ with $\widetilde{\mathrm{M}}_{-sm}$, for all $m \geq 0$, in order that

1. One has $\Delta^{\widetilde{\mathrm{M}}_{-sm}}_{q^{p^{sm}}} = \Delta^{\widetilde{\mathrm{M}}}_{q^{p^{sm}}}$,
2. One has $\Phi_{-sm} = \Phi_{-1}$.

By this identification it is evident that M_{-sm} is the restriction to

$$(\mathrm{M}_{-sm}, (\sigma_q^{\mathrm{M}_{-sm}})^{p^s}) \xrightarrow{\sim} \mathrm{Def}_{q,q^{p^s}}(\mathrm{M}, \sigma_q^{\mathrm{M}}) \ . \qquad (8.4.3.2)$$

Then evidently one has $\lim_{m\to\infty} \frac{\Delta^{\mathrm{M}}_{q^{p^{sm}}}-1}{q^{p^{sm}}-1} = \delta_1^{\mathrm{M}}$ (cf. 6.1.10). □

Remark 8.4.4. If one start from a σ_q-module with Frobenius structure, and if the matrix of $\sigma_q^{\mathrm{M}_{-sm}}$ is known, then, by the relation 8.4.3.2, one can *compute* the differential equation $\mathrm{Conf}_q(\mathrm{M}) \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$.

Remark 8.4.5. A.— The lemma 8.4.3 gives a straightforward and explicit construction of the functor $D^{\mathrm{conf}(\phi)} \circ V^{(\phi)}_{\sigma_q}$ of [ADV04].
B.— By our result it is evident that the "slopes" of a σ_q-module are the "slopes" of the attached differential module, by strong confluence.
C.— The quasi-unipotence of $q-$difference equations is a straightforward consequence of the "strong confluence" and of the $p-$adic local monodromy theorem for differential equations.

Annexe A

Rank one differential equations over the Amice ring

A.1 Criterion of solvability for differential equations over $\mathcal{E}_K$

In this section we obtain a criterion of solvability for differential equations over $\mathcal{E}_K$. After a technical part (cf. A.1.3), this is actually an immediate consequence of the Lemma A.1.8.

As a corollary we obtain that every differential equation over $\mathcal{E}_K$ has a basis in which the associated operator has coefficients in $\mathcal{O}_K[[T^{-1}]]$. Hence it will converge in an affinoïd and we can apply the theory of confluence developed in the previous sections.

Definition A.1.1. We set

$$\omega := |p|^{\frac{1}{p-1}} . \tag{A.1.1.1}$$

Lemma A.1.2 (Small radius)**.** *Let $\partial_T - g(T)$, $g(T) \in \mathcal{E}_K$. Then $Ray(\partial_T - g(T), 1) < \omega$ if and only if $|g(T)|_1 > 1$. In this case we have*

$$Ray(\partial_T - g(T), 1) = \omega \cdot |g(T)|_1^{-1} .$$

A.1.1 Technical lemma

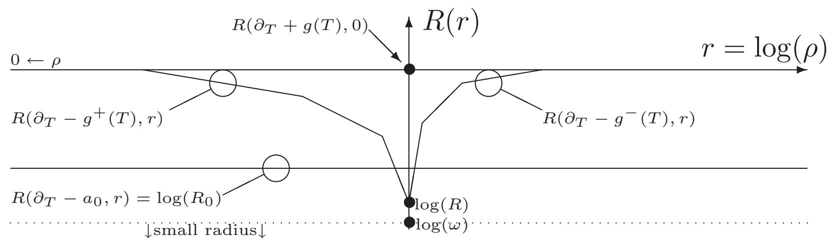
For all functions $g(T) = \sum_{i \in \mathbb{Z}} a_i T^i$ we set as usual $g^-(T) := \sum_{i \leq -1} a_i T^i$, $g^+(T) := \sum_{i \geq 1} a_i T^i$. In this subsection we shall prove the proposition A.1.3 below. The "*Step* 4" of this proposition has been proved by G.Christol.

Proposition A.1.3. *Let $\partial_T - g(T)$, $g(T) \in \mathcal{E}_K$, be a solvable equation. Then $\partial_T - g^-(T)$, $\partial_T - a_0$, and $\partial_T - g^+(T)$ are all solvable.*

Proof : —*Step 1 :* The equation $\partial_T - g^-(T)$ (resp. $\partial_T - g^+(T)$) has a convergent solution at ∞ (resp. at 0), hence $Ray(\partial_T - g^-(T), \rho) = \rho$, for large values of ρ (resp. $Ray(\partial_T - g^-(T), \rho) = \rho$, for ρ close to 0). On the other hand, there is a $R^0 > 0$ such that $Ray(\partial_T - a_0, \rho) = R^0 \cdot \rho$, for all ρ. Let

$$R^- := Ray(\partial_T - g^-(T), 1) \;, \tag{A.1.3.1}$$
$$R^+ := Ray(\partial_T - g^+(T), 1) \;, \tag{A.1.3.2}$$
$$R^0 := Ray(\partial_T - a_0, 1) \;. \tag{A.1.3.3}$$

—*Step 2 :* We shall show that $R^+ = R^-$ and $R^0 > R^- = R^+$, as in the picture in which $R := R^- = R^+$. For all operators L, we let $r := \log(\rho)$ and $R(L, r) := \log(Ray(L, \rho)/\rho)$.



Recall that $Ray(M \otimes N, \rho) \geq \min(Ray(M, \rho), Ray(N, \rho))$, and equality holds if $Ray(M, \rho) \neq Ray(N, \rho)$. For $\rho > 1$ close to 1, the slope of $Ray(\partial_T - g^-(T), \rho)$ is strictly positive. Hence $Ray(\partial_T - g^-(T), \rho) \neq Ray(\partial_T - a_0, \rho) = R^0$ with the exception of a isolated ρ, then $Ray(\partial_T - (a_0 + g^-(T)), \rho) = \min(Ray(\partial_T - g^-(T), \rho), R^0)$, for all $\rho > 1$ close to 1. By continuity, this equality holds for $\rho = 1$, that is $Ray(\partial_T - (a_0 + g^-(T)), 1) = \min(R^-, R^0)$. Since $\partial_T - g(T)$ is the tensor product of $\partial_T - g^+(T)$ and $\partial_T - (a_0 + g^-(T))$, and since $Ray(\partial_T - g(T), 1) = 1$, hence

$$R^+ := Ray(\partial_T - g^+(T), 1) = Ray(\partial_T - (a_0 + g^-(T)), 1) = \min(R^-, R^0) \;.$$

On the other hand, one has $R^0 \geq R^-$, and the previous equality implies $R^+ = R^-$. Indeed if $R^- > R^0$, then $R^+ = R^0$. Hence, for all $\rho < 1$, one has $Ray(\partial_T - g^+(T), \rho) \neq Ray(\partial_T - a_0, \rho)$, and then $Ray(\partial_T - (a_0 + g^+(T)), \rho) = Ray(\partial_T - a_0, \rho) = R^0$, for all $\rho < 1$. By continuity $Ray(\partial_T - (a_0 + g^+(T)), 1) = R^0 < R^-$. Hence $Ray(\partial_T - g(T), 1) = \min(R^0, R^-) = R^0 < 1$, which is in contradiction with the solvability.

—*Step 3 :* We have $R \geq \omega$. Indeed if $R^- < \omega$ or $R^+ < \omega$, then, by A.1.2, $|g^-(T)|_1 > 1$ or $|g^+(T)|_1 > 1$, hence $|g(T)|_1 > 1$ which is in contradiction with the small radius lemma, since the equation $\partial_T - g(T)$ is solvable (cf. A.1.2).

—*Step 4 :* We shall show that $R > \omega$. We need two lemmas :

Lemma A.1.4 ([Chr83, 4.8.5]). *Let $\partial_T - g(T)$, $g(T) \in \mathcal{E}_K$, $|g(T)|_1 \leq 1$ be some equations. Then $Ray(\partial_T - g(T), 1) > \omega$ if and only if $|g_{[s]}(T)| < 1$, for some $s \geq 1$.*

Proof : [Chr83, 4.8.5]. See B.3.9 for the q-analogue of this lemma. □

Lemma A.1.5. *If $Ray(\partial_T - g(T), 1) > \omega$, where $g(T) = \sum a_i T^i$, then $|a_i| < 1$, for all $i \leq -1$.*

Proof : The matrix of d/dT is $g_{[1]} := g(T)/T$. By definition one has

$$\begin{aligned} Ray(\partial_T - g(T), 1) = Ray(d/dT - g_{[1]}(T), 1) &= \min\big(1, \liminf_s (|g_{[s]}(T)|_1/|s!|)^{-\frac{1}{s}}\big) \\ &= \min\big(1, \omega \cdot \liminf_s (|g_{[s]}(T)|_1)^{-\frac{1}{s}}\big) , \end{aligned}$$

where $g_{[s]}(T)$ is associated to the derivation $(\frac{d}{dT})^s$. Since $Ray(\partial_T - g(T), 1) > \omega$, hence $\lim_{s\to\infty} |g_{[s]}(T)|_1 = 0$. In particular $|g_{[s]}(T)|_1 < 1$, for some $s \geq 1$. Moreover, by the small radius lemma, we have $|g(T)|_1 \leq 1$ (cf. A.1.2). Let $-d$ be the smallest index such that $|a_{-d}| = 1$. The reduction of $g_{[1]}(T) = g(T)/T$ in $k(\!(t)\!)$ is of the form $\overline{g_{[1]}(T)} = \overline{a}_{-d} t^{-d-1} + \cdots$. If $-d \leq -1$, then an induction on the equation $g_{[s+1]} = \frac{d}{dx}(g_{[s]}) + g_{[s]} g_{[1]}$ shows that $\overline{g_{[s]}(T)} = \overline{a}^s_{-d} t^{(-d-1)s} + \cdots \neq 0$. This is in contradiction with the fact that $|g_{[s]}(T)|_1 < 1$, for some $s \geq 1$.□

Let us show now that $R > \omega$. Since $R^+ = R^- = R$, it is sufficient to show that $R^- > \omega$. By *A.1.5*, we have $|a_i| < 1$, for all $i \leq -1$. Since $\lim_{i\to-\infty} |a_i| = 0$, hence we have $|g^-(T)|_1 < 1$. Then Lemma A.1.4 implies $R^- > \omega$.

—Step 5 : Since $R > \omega$, then we can take the antecedent by Frobenius of $\partial_T - g^-(T)$, $\partial_T - g^+(T)$, $\partial_T - a_0$. More precisely, there exists $f^+(T) = \sum_{i\geq 0} b_i^+ T^i \in \mathcal{A}([0,1[)^\times$, $f^-(T) = \sum_{i\leq 0} b_i^- T^i \in \mathcal{A}([1,\infty])^\times$, and there are functions $g^{(1),-}(T) = \sum_{i\leq 0} a_i^{(1),-} T^i$, $g^{(1),+}(T) = \sum_{i\geq 0} a_i^{(1),+} T^i$, $b_0 \in K$ such that

$$\begin{aligned} pb_0 &= a_0 + n \,, \quad \text{for some } n \in \mathbb{Z} \,; \\ pg^{(1),-}(T^p)^\sigma &= g^-(T) + \frac{\partial_T(f^-(T))}{f^-(T)} \,; \\ pg^{(1),+}(T^p)^\sigma &= g^+(T) + \frac{\partial_T(f^+(T))}{f^+(T)} \,. \end{aligned}$$

We see immediately that $b_0^+ \neq 0$ and $b_0^- \neq 0$, and that $v_T(\partial_T(f^+)/f^+) \geq 1$ and $v_{T^{-1}}(\partial_T(f^-)/f^-) \geq 1$, where v_T is the $T-$adic valuation, and $v_{T^{-1}}$ is the $T^{-1}-$adic valuation. Since $g^-(T)$ and $g^+(T)$ have no constant term, this implies that $a_0^{(1),+} = 0$ and $a_0^{(1),-} = 0$. Observe now that both f^- and f^+ belong to $\mathcal{E}_K^\times$, hence $\partial_T - \big(g^{(1),-}(T) + b_0 + g^{(1),+}(T)\big)$ is an antecedent of Frobenius of $\partial_T - g(T)$, and is then solvable.

— *Step 6 :* Steps 1, 2, 3, 4 are still true for the antecedent. In particular, if we set

$$\begin{aligned} R^-(1) &:= Ray(\partial_T - g^{(1),-}(T), 1) \ , && \text{(A.1.5.1)} \\ R^+(1) &:= Ray(\partial_T - g^{(1),+}(T), 1) \ , && \text{(A.1.5.2)} \\ R^0(1) &:= Ray(\partial_T - b_0, 1) \ , && \text{(A.1.5.3)} \end{aligned}$$

then we must have $R^-(1) = R^+(1) > \omega$. Let $R(1) := R^-(1) = R^+(1)$, then $R(1) = R^{1/p}$ by the property of the antecedent. This implies $R > \omega^{1/p}$. Moreover, one has again $R(1) > \omega$, then we can again take the antecedent. This process can be iterated indefinitely. This shows that $R > \omega^{1/p^h}$ for all $h \geq 0$, that is $R = 1$. □

Corollary A.1.6. *Let $\partial_T - g(T)$, $g(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{E}_K$ be a solvable equation, then $a_0 \in \mathbb{Z}_p$ and $\partial_T - g^+(T)$ is trivial.* □

Proof : By the transfer theorem, the Taylor solution at 0 of $\partial_T - g^+(T)$ is convergent in the open unit disk. This solution is invertible with inverse the solution of the dual differential module, hence it is bounded and belongs to $\mathcal{E}_K$. □

A.1.2 Criterion of solvability

Definition A.1.7. We set $\mathrm{J}_p := \{n \in \mathbb{Z} \mid (n,p) = 1, n \geq 1\}$.

Lemma A.1.8. *The differential equation $\partial_T - g^+(T)$, $g^+(T) = \sum_{i\geq 1} a_i T^i \in \mathcal{A}([0,1[)$ is solvable if and only if there exists a family $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$, $\boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$, with phantom components $\phi_n = (\phi_{n,0}, \phi_{n,1}, \ldots)$ satisfying*

$$a_{np^m} = n\phi_{n,m} \ , \qquad \textit{for all } n \in \mathrm{J}_p,\ m \geq 0 \ . \qquad \text{(A.1.8.1)}$$

In other words, we have $\exp(\sum_{i\geq 1} a_i \frac{T^i}{i}) = E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1)$, *where*

$$E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1) := \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} \phi_{n,m} T^{np^m}/p^m) \ . \qquad \text{(A.1.8.2)}$$

Proof : The formal series $E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1) \in 1 + T\mathcal{O}_K[[T]]$ is solution of the equation $L := \partial_T - \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} n\phi_{n,m} T^{np^m}$. Since this exponential converges in the unit disk, then $Ray(L,\rho) = \rho$, for all $\rho < 1$, and L is solvable. Conversely, if $\partial_T - g^+(T)$ is solvable, then the Witt vectors $\boldsymbol{\lambda}_n =$

$(\lambda_{n,0}, \lambda_{n,1}, \ldots)$ are defined by the relation A.1.8.1. For example, for all $n \in \mathrm{J}_p$ we have

$$\lambda_{n,0} = \frac{a_n}{n} \quad , \qquad \lambda_{n,1} = \frac{1}{p}\Big(\frac{a_{np}}{n} - (\frac{a_n}{n})^p\Big) \ . \tag{A.1.8.3}$$

We must show that $|\lambda_{n,m}| \leq 1$, for all $n \in \mathrm{J}_p$, $m \geq 0$.

—*Step 1 :* By the small radius lemma A.1.2, we have $|a_i| \leq 1$, for all $i \geq 1$. Hence, by A.1.8.3, for all $n \in \mathrm{J}_p$, we have $|\lambda_{n,0}| \leq 1$. Then the exponential

$$E(\sum_{n\in\mathrm{J}_p}(\lambda_{n,0}, 0, 0, \ldots)T^n, 1) = \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}\frac{T^{np^m}}{p^m})$$

converges in the unit disk and is solution of the operator $Q^{(0)} := \partial_T - h^{(0)}(T)$, where $h^{(0)}(T) = \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}T^{np^m}$. By transfer, $Q^{(0)}$ is then solvable.

—*Step 2 :* The tensor product operator $\partial_T - (g^+(T) - h^{(0)}(T))$ is again solvable and satisfies $g^+(T) - h^{(0)}(T) = p \cdot g^{(1)}(T^p)$, for some $g^{(1)}(T) \in TK[[T]]$. In other words, the "antecedent by ramification" φ_p^* of the equation $\partial_T - (g^+(T) - h^{(0)}(T))$ is given by $\partial_T - g^{(1)}(T)$, which is then solvable.

—*Step 3 :* We observe that $g^{(1)}(T) = \frac{1}{p}\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}(a_{np^{m+1}} - n(\frac{a_n}{n})^{p^{m+1}})T^{np^m}$, and again by the small radius lemma, we have $|a_{np} - n(\frac{a_n}{n})^p| \leq 1$, which implies $|\lambda_{n,1}| \leq 1$. The process can be iterated indefinitely. □

Remark A.1.9. We shall now consider the general case of an equation $\partial_T - g(T)$, with $g(T) = \sum_{i\in\mathbb{Z}} a_iT^i \in \mathcal{E}_K$, and get a criterion of solvability. Suppose that $\partial_T - g(T)$ is solvable. We know that $\partial_T - g^-(T)$, $\partial_T - a_0$ and $\partial_T - g^+(T)$ are all solvable (cf. A.1.3). We can then consider $\partial_T - g^-(T)$ as an operator on $]1,\infty]$ (instead of $[1,\infty]$), and the precedent lemma A.1.8 give us the existence of a family of Witt vector $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p} \subset \mathbf{W}(\mathcal{O}_K)$, satisfying $a_{-np^m} = -n\phi_{-n,m}$, for all $n \in \mathrm{J}_p$, and all $m \geq 0$. Conversely, suppose given two families $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_{n}\}_{n\in\mathrm{J}_p}$, with $\boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$. Since the phantom components of $\boldsymbol{\lambda}_n$ are bounded by 1, then $|a_i|$ is bounded by 1, and then $g^+(T)$ belongs to $\mathcal{E}_K$.

But now we need conditions on the family $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ in order that the series $g^-(T) := \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} -n\phi_{-n,m}T^{-np^m}$ belongs to $\mathcal{E}_K$.

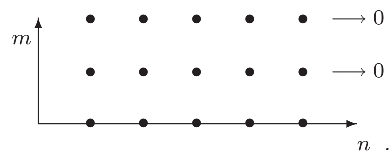
Proposition A.1.10. *Let $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$, $\boldsymbol{\lambda}_{-n} \in \mathbf{W}(\mathcal{O}_K)$, be a family of Witt vectors. Let $\langle\phi_{-n,0}, \phi_{-n,1}, \ldots\rangle$ be the phantom vector of $\boldsymbol{\lambda}_{-n} := (\lambda_{-n,0}, \lambda_{-n,1}, \ldots)$. The series*

$$g^-(T) := \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} -n\phi_{-n,m}T^{-np^m} \ ,$$

belongs to $\mathcal{E}_K$ if and only if

$$\begin{cases} |\lambda_{-n,m}| < 1 \quad , \quad \text{for all } n \in \mathrm{J}_p \ , \ \text{for all } m \geq 0 \ ; \\ \lim_{n\in\mathrm{J}_p, n\to\infty} \lambda_{-n,m} = 0 \quad , \quad \text{for all } m \geq 0 \ , \end{cases} \tag{A.1.10.1}$$

as in the picture



Lemma A.1.11. *Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathcal{O}_K)$ be a Witt vector, and let $\langle \phi_0, \phi_1, \ldots \rangle \in \mathcal{O}_K^{\mathbb{N}}$ be its phantom vector. Then $\phi_j \to 0$ in $\mathcal{O}_K$ if and only if $|\lambda_j| < 1$, for all $j \geq 0$.*

Proof : The set of Witt vectors whose phantom components go to 0 is clearly an ideal $I \subset \mathbf{W}(\mathcal{O}_K)$ containing $\mathbf{W}(\mathfrak{p}_K)$. Reciprocally, suppose $\phi_j \to 0$, since $\phi_j = \lambda_0^{p^j} + p\lambda_1^{p^{j-1}} + \cdots + p^j\lambda_j$, hence $|\lambda_0| < 1$. Then $\boldsymbol{\lambda}^{(1)} := (0, \lambda_1, \lambda_2, \ldots) = \boldsymbol{\lambda} - (\lambda_0, 0, \ldots)$ lies again in the ideal I, and hence $\phi_j(\boldsymbol{\lambda}^{(1)}) = p\lambda_1^{p^{j-1}} + p^2\lambda_2^{p^{j-2}} + \cdots + p^j\lambda_j \to 0$. This shows that $|\lambda_1| < 1$, and, inductively, that $|\lambda_j| < 1$, for all $j \geq 0$. $\square$

Proof of A.1.10 : Assume that $g^-(T) = \sum_{n \in \mathrm{J}_p} \sum_{m \geq 0} -n\phi_{-n,m} T^{-np^m}$ lies in $\mathcal{E}_K$. This happens if and only if $\lim_{np^m \to \infty} \phi_{-n,m} = 0$, and implies $\lim_{m \to \infty} \phi_{-n,m} = 0$ for all $n \in \mathrm{J}_p$. By A.1.11, we have $|\lambda_{-n,m}| < 1$, for all $n \in \mathrm{J}_p$ and all $m \geq 0$. An easy induction shows that $\lim_{n \in \mathrm{J}_p, n \to \infty} \lambda_{-n,m} = 0$, for all $m \geq 0$.

Reciprocally, assume that $\{\boldsymbol{\lambda}_{-n}\}_{n \in \mathrm{J}_p}$ satisfies the condition A.1.10.1. We must show that $\lim_{np^m \to \infty} \phi_{-n,m} = 0$. For all $\varepsilon > 0$, we choose $k \geq 0$ such that $|p^{k+1}| < \varepsilon$. By assumption, for all $0 \leq m \leq k$, there exists N_m such that $|\lambda_{-n,m}| < \varepsilon$, for all $n \geq N_m$. Let $N := \max(N_0, \ldots, N_k)$. Then

$$\phi_{-n,m} = \underbrace{\lambda_{-n,0}^{p^m} + \cdots + p^k \lambda_{-n,k}^{p^{m-k}}}_{<\varepsilon \text{ , if } n \geq N} + \underbrace{p^{k+1}\lambda_{-n,k+1}^{p^{m-k-1}} + \cdots + p^m \lambda_{-n,m}}_{<\varepsilon} \ .$$

Hence $|\phi_{-n,m}| < \varepsilon$, if $n \geq N$. On the other hand, by assumption, there is $\delta < 1$ such that $|\lambda_{-n,m}| \leq \delta < 1$, for all $m = 0, \ldots, k$, $n = 0, \ldots, N$. Hence there exists M such that $|\lambda_{-n,0}^{p^m}|, \ldots, |\lambda_{-n,k}^{p^{m-k}}| < \varepsilon$, for all $m \geq M$. Then $|\phi_{-n,m}| \leq \varepsilon$, for all $n \geq N$, $m \geq M$. Hence $\lim_{np^m \to \infty} \phi_{-n,m} = 0$.$\square$

Definition A.1.12. We denote by $\mathrm{Conv}(\mathcal{E})$ the set of families $\{\boldsymbol{\lambda}_{-n}\}_{n \in \mathrm{J}_p}$, with $\boldsymbol{\lambda}_{-n} = (\lambda_{-n,0}, \lambda_{-n,1}, \ldots) \in \mathbf{W}(\mathcal{O}_K)$, satisfying the condition A.1.10.1.

Corollary A.1.13 (Criterion of solvability)**.** *The equation $\partial_T - g(T)$, $g(T) = \sum_{i \in \mathbb{Z}} a_i T^i \in \mathcal{E}_K$, is solvable if and only if $a_0 \in \mathbb{Z}_p$, and there exist two families $\{\boldsymbol{\lambda}_{-n}\}_{n \in \mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_{-n}\}_{n \in \mathrm{J}_p}$ with $\boldsymbol{\lambda}_{-n}, \boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$, for all $n \in \mathrm{J}_p$, such that $\{\boldsymbol{\lambda}_{-n}\}_{n \in \mathrm{J}_p} \in \mathrm{Conv}(\mathcal{E}_K)$, and moreover*

$$a_{np^m} = n\phi_{n,m} \quad , \quad a_{-np^m} = -n\phi_{-n,m} \ . \tag{A.1.13.1}$$

In other words, its formal solution $T^{a_0}\exp(\sum_{i\leq -1} a_i\frac{T^i}{i})\exp(\sum_{i\geq 1} a_i\frac{T^i}{i})$ *can be represented by the symbol*

$$T^{a_0}\cdot\exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}\frac{T^{-np^m}}{p^m})\cdot\exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{n,m}\frac{T^{np^m}}{p^m})\ , \qquad (\mathrm{A}.1.13.2)$$

where $(\phi_{-n,0},\phi_{-n,1},\ldots)$ *(resp.* $(\phi_{n,0},\phi_{n,1},\ldots)$*) is the phantom vector of* $\boldsymbol{\lambda}_{-n}$ *(resp.* $\boldsymbol{\lambda}_n$*), and hence*

$$g(T)=\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}-n\phi_{-n,m}T^{-np^m}+a_0+\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}n\phi_{n,m}T^{np^m}\ . \qquad (\mathrm{A}.1.13.3)$$

Corollary A.1.14 (canonical extension)**.** *Let* $\partial_T\text{-Mod}(\mathcal{A}_K([1,\infty]))^{\mathrm{sol}}_{rk=1}$ *be the category of rank one differential modules over* $\mathcal{A}_K([1,\infty])$*, solvable at all* $\rho\geq 1$*, with a regular singularity at* ∞ *(i.e. the matrix of* ∂_T *converge at* ∞ *and hence belongs to* $\mathcal{A}_K([1,\infty])$*). The scalar extension functor*

$$\partial_T\text{-Mod}(\mathcal{A}_K([1,\infty]))^{\mathrm{sol}}_{rk=1}\longrightarrow\partial_T\text{-Mod}(\mathcal{E}_K)^{\mathrm{sol}}_{rk=1}$$

is an equivalence.

Proof : The corollary A.1.13 shows that there is bijection between objects. Indeed, all differential equations $\partial_T-g(T)$ over $g(T)=g^-(T)+a_0+g^+(T)\in\mathcal{E}_K$ is isomorphic over $\mathcal{E}_K$ to the equation $\partial_T-(g^-(T)+a_0)$. On the other hand, let $M,N\in\partial_T-Mod(\mathcal{A}_K([1,\infty]))^{\mathrm{sol}}_{rk=1}$, and let ∂_T-g_M, ∂_T-g_N be the operators corresponding to a chosen basis of M and N. An element of $\mathrm{Hom}(M,N)\xrightarrow{\sim}M^\vee\otimes N$ is then a solution of the operator $\partial_T-(g_N-g_M)$. This solution will be of the form $y(T)=T^{a_0}\exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}T^{-np^m}/p^m)$, for some ϕ. Since we are supposing that this solution belongs to $\mathcal{A}_K([1,\infty])$, then $a_0\in\mathbb{Z}$ and this exponential lies in $\mathcal{A}_K([1,\infty])$. Since the same argument works for $\mathrm{Hom}_{\partial_T}(M\otimes\mathcal{E}_K,N\otimes\mathcal{E}_K)$, and since $\mathcal{A}_K([1,\infty])\subset\mathcal{E}_K$, then the map $\mathrm{Hom}_{\partial_T}(M,N)\to\mathrm{Hom}_{\partial_T}(M\otimes\mathcal{E}_K,N\otimes\mathcal{E}_K)$ is bijective. $\Box$

Remark A.1.15. We are not able to obtain a complete description of the isomorphism class of a given differential equation over $\mathcal{E}_K$. Namely, over $\mathcal{R}_K$, we know that a solution of a differential equation lies in $\mathcal{R}_K$ if and only if the corresponding Witt vector satisfies the Artin-Schreier condition 7.6.0.10. But we do not have the analogous result over $\mathcal{E}_K$. In other words, we do not have a necessary and sufficient condition on the Witt vector $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ in order that $E(\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n},1)$ belongs to $\mathcal{E}_K$.

Annexe B

Rank one $q-$difference equations over the Amice ring

We shall establish the $q-$analogue of the results of section A.1. In order to do that, we will need some numerical lemmas (cf. section B.1) and a generalization of the result of E.Motzkin (cf. [Mot77], and section B.2 below).

We will prove that, if $q \in \mathrm{D}^-(1,|p|^{\frac{1}{p-1}})$, then solvable rank one $q-$difference equations are strongly confluent. Nevertheless, almost all results are true for $q \in \mathrm{D}^-(1,1)$. The principal obstructions are the existence of the "antecedent by Frobenius" (used in "Step 5" of B.3.6), which is proved in [DV04] only for $|q-1|<|p|^{\frac{1}{p-1}}$, and the "*Step* 0" of theorem B.3.13 (see B.3.14). The author hopes that these difficulties will be overcoming in future.

For these reasons the hypothesis $q \in \mathrm{D}^-(1,1)$ will be introduced systematically starting from B.3.10 on. Until B.3.10 we will suppose that $q \in \mathrm{D}^-(1,1)$.

B.1 Numerical Lemmas

Definition B.1.1. We set

$$\omega := |p|^{\frac{1}{p-1}} \ . \tag{B.1.1.1}$$

Lemma B.1.2. *Let us fix a $j \geq 0$. Let $\omega^{1/p^{j-1}} < \rho < \omega^{1/p^j}$, if $j \geq 1$, and let $\rho < \omega$, if $j=0$. Then $\rho^{p^j}/|p^j| > \sup(\rho^r/|r| \ : \ r \geq 1, r \neq p^j)$. Moreover, we have*

$$\rho < \frac{\rho^p}{|p|} < \cdots < \frac{\rho^{p^{j-1}}}{|p^{j-1}|} < \frac{\rho^{p^j}}{|p^j|} \quad ; \quad \frac{\rho^{p^j}}{|p^j|} > \frac{\rho^{p^{j+1}}}{|p^{j+1}|} > \frac{\rho^{p^{j+2}}}{|p^{j+2}|} > \cdots \ . \tag{B.1.2.1}$$

Proof : If $r \neq p^k$, for all $k \geq 0$, then $|r| = |p|^v$, with $v := v_p(r)$, hence $\rho^r/|r| < \rho^{p^v}/|p|^v$. The condition $\rho^{p^{k-1}}/|p^{k-1}| < \rho^{p^k}/|p^k|$ is equivalent to $\rho_1 < \frac{\rho_1^p}{p}$, with $\rho_1 := \rho^{p^{k-1}}$, and is verified if and only if $\rho_1 > |\pi_0|$, that is

$\rho > |\pi_{k-1}|$. On the other hand, the inequality $\rho^{p^{k-1}}/|p^{k-1}| > \rho^{p^k}/|p^k|$ is equivalent to $\rho < |\pi_k|$.□

Lemma B.1.3. *Let $n \geq 1$ be a natural number. Let $l := [\log_p(n)]$, where $[x]$ is the greatest integer smaller than or equal to the real number x. If $c \leq |p|^{l+1}$, then*

$$\frac{c^n}{|n!|} \geq \frac{c^k}{|k!|} \ , \qquad \text{for all } k \geq n \ . \tag{B.1.3.1}$$

Proof : If $k = n$, the relation is trivial; suppose $k > n$. The equation B.1.3.1 is equivalent to $c \leq |\frac{k!}{n!}|^{\frac{1}{k-n}}$. Since $|n!| = |\pi_0|^{n-S_n}$, where S_n is the sum of the digits of the base p expansion of n, then $|\frac{k!}{n!}|^{\frac{1}{k-n}} = |\pi_0|^{1+\frac{S_n-S_k}{k-n}}$. If $n = n_0+n_1p+n_2p^2+\cdots+n_lp^l$, with $0 \leq n_i \leq p-1$, then $S_n = n_0+n_1+\cdots+n_l$, hence $1 \leq S_n \leq (p-1)(l+1)$. This shows that

$$1+\frac{S_n - S_k}{k-n} \leq 1+\frac{(p-1)(l+1)-1}{k-n} \leq 1+(p-1)(l+1)-1 = (p-1)(l+1) \ . \tag{B.1.3.2}$$

Hence $|\frac{k!}{n!}|^{\frac{1}{k-n}} \geq |\pi_0|^{(p-1)(l+1)} = |p|^{l+1}$, for all $k > n$. □

Definition B.1.4. Let $q \in \mathrm{D}^-(1,1)$. For all α in any extension of K we define

$$q^\alpha = ((q-1)+1)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} (q-1)^k \ . \tag{B.1.4.1}$$

Clearly, if $|\alpha| > 1$, then $|\binom{\alpha}{k}| = \frac{|\alpha|^k}{|k!|}$, hence q^α converges exactly for $|q-1| < |\pi_0|/|\alpha|$. While if $\alpha \leq 1$, then q^α converges at least for $|q-1| < |\pi_0|$, in particular if $\alpha \in \mathbb{Z}_p$, then q^α converges at least for $|q-1| < 1$. For a detailed discussion on the radius of convergence of q^α see [DGS94, Ch.IV,Prop.7.3].

Lemma B.1.5. *For all α in some extension of K, we have*

$$\lim_{q\to 1} \frac{q^\alpha - 1}{q-1} = \alpha \ . \tag{B.1.5.1}$$

Proof : Write $q^\alpha - 1 = ((q-1)+1)^\alpha - 1 = \sum_{k\geq 1} \binom{\alpha}{k}(q-1)^k$. We see that $|\binom{\alpha}{k}|$ is bounded by $s^k/|k!|$, where $s := \sup(\alpha, 1)$. Hence $\frac{q^\alpha-1}{q-1} = \alpha + \sum_{k\geq 2}\binom{\alpha}{k}(q-1)^{k-1}$. We shall show that if $|q-1|$ is sufficiently small, then $|\binom{\alpha}{k}(q-1)^{k-1}| \leq |\binom{\alpha}{2}(q-1)|$. Indeed, if $n \geq 1$ and $k \geq n$, then $|\binom{\alpha}{k}(q-1)^{k-1}| \leq |\binom{\alpha}{n}(q-1)^{n-1}|$ if and only if

$$|q-1| \leq |\tbinom{\alpha}{n}/\tbinom{\alpha}{k}|^{\frac{1}{k-n}} = \left(\frac{|k!|}{|n!|}\frac{1}{|(\alpha-n)\cdots(\alpha-k+1)|}\right)^{\frac{1}{k-n}} \ . \tag{B.1.5.2}$$

By lemma B.1.3 we know that, if $l := [\log_p(n)]$, then $(\frac{k!}{n!})^{\frac{1}{k-n}} \geq |p|^{l+1}$, for all $k \geq n$. On the other hand, it is clear that $|(\alpha-n)\cdots(\alpha-k+1)| \leq$

s^{k-n}, hence the right hand side of B.1.5.2 is bigger than $|p|^{l+1}/s$. Now if $|q-1| < |p|^{l+1}/s$, then $|\binom{\alpha}{k}(q-1)^{k-1}| \leq |\binom{\alpha}{2}(q-1)|$, for all $k \geq 2$, and $\lim_{q\to 1}\sum_{k\geq 2}\binom{\alpha}{k}(q-1)^k = 0$.$\square$

Lemma B.1.6. *For all $j \geq 1$ let $\omega^{1/p^{j-1}} < |q-1| < \omega^{1/p^j}$, or, if $j=0$, let $|q-1| < \omega$. Let $d := \alpha p^m \in \mathbb{Z}_p$, $(\alpha, p) = 1$, $\alpha \in \mathbb{Z}_p$. Let $i := \min(m, j)$. Then*

$$|q^d - 1| = |d| \cdot \frac{|q-1|^{p^i}}{|p|^i} = |p^{m-i}||q-1|^{p^i} \ . \tag{B.1.6.1}$$

Proof : Since $(\alpha, p) = 1$, hence $\left|\binom{\alpha}{1}\right| = 1$. Then

$$|q^\alpha - 1| = |((q-1)+1)^\alpha - 1| = |\sum_{k=1}^{\infty}\binom{\alpha}{k}(q-1)^k| = |q-1| \ , \tag{B.1.6.2}$$

because $|q-1| = |\binom{\alpha}{1}(q-1)| > |\binom{\alpha}{k}(q-1)^k|$, for all $k \geq 2$. Moreover, one has $|q^{\alpha p^m} - 1| = |((q^\alpha - 1)+1)^{p^m} - 1| = |\sum_{k=1}^{p^m}\binom{p^m}{k}(q^\alpha - 1)^k|$. Since $|\binom{p^m}{k}| = \frac{|p|^m}{|k|}$, for $k \leq p^m$, hence apply B.1.2 to $\rho = |q-1| = |q^\alpha - 1|$, then $|\binom{p^m}{k}(q^\alpha - 1)^k| = |p^m|\frac{\rho^k}{|k|}$.$\square$

Lemma B.1.7. *Let $q \in \mathrm{D}^-(1,1)$. Denote the $q-$factorial by*

$$[n]_q^! := \frac{\prod_{i=1}^n (q^i - 1)}{(q-1)^n}.$$

Then the sequence $|[n]_q^!|^{1/n}$ has a limit $\omega_q < 1$. Moreover, let κ be the smallest integer such that $|q^\kappa - 1| < \omega$, then

$$\omega_q = \begin{cases} \omega & \text{if} \quad \kappa = 1 \ , \\ (|\frac{q^\kappa - 1}{q-1}| \cdot \omega)^{\frac{1}{\kappa}} & \text{if} \quad \kappa \geq 2 \ . \end{cases}$$

Proof : [DV04, 3.5]. $\square$

Lemma B.1.8. *Let $q \in \mathrm{D}^-(1,1)$. For all $f(T) \in \mathcal{A}_K(I)$, all $\rho \in I$ and all $k \geq 1$, we have $|\frac{d_q^k}{[k]_q^!}(f)|_\rho \leq \rho^{-k}|f|_\rho$. The same result is true for $f \in \mathcal{E}_K$ and $\rho = 1$.*

Proof : [DV04, 2.1].$\square$

B.2 The Motzkin decomposition

In [Mot77] a decomposition theorem for analytic element over an affinoïd (i.e. a set of type $\mathbb{P}^1_K - \cup_{i=1,\ldots,n}\mathrm{D}^-_K(a_i, r_i)$) is proved. In [Chr81] G.Christol generalizes this decomposition for matrices with coefficients in analytic functions. We need to generalize this theorem for series in $\mathcal{E}_K$ (cf. B.2.3).

Lemma B.2.1. *Let* $a^-(T) = 1 + \alpha_{-1}T^{-1} + \alpha_{-2}T^{-2} + \cdots$ *be an invertible function in* $1 + T^{-1}\mathcal{A}_K([r,\infty])$*. Then* $|\alpha_i| < r^{-i}$*, for all* $i \leq -1$.

Proof : By replacing T with $\gamma_r T$, with $|\gamma_r| = r$, we can suppose $r = 1$. Let $X = T^{-1}$, and let $P(X) := a^-(X^{-1}) = 1 + \alpha_{-1}X + \alpha_{-2}X^2 + \cdots$. Since $P(X)$ has no roots in the unit disks and since $P(0) = 1$, then $|P(X)|_\rho = 1$, for all $\rho \leq 1$. If $|\alpha_{-i}| = 1$ for some $i \geq 1$, then the reduced series $\overline{P(X)}$ is a polynomial in $k[X]$. Then there exists a root $\bar{\alpha}$ of $\overline{P(X)}$ in k^{alg}. This shows that there is ζ in K^{alg}, with $|\zeta| = 1$ and such that $|P(\zeta)| < 1$. If ζ is a zero of $P(X)$, then $P(X)$ is not invertible, this is in contradiction with the hypothesis. Hence ζ is not a zero of $P(X)$. Let us write $P(X) = \sum_{i\geq 0} b_i(X-\zeta)^i$. Since $|P(\zeta)| < 1$, then $|b_0| < 1$. Since $|P(X)|_1 = 1$, then $\lim_{\rho\to 1}\sup_{i\geq 0}|b_i|\rho^i = 1$. This implies that the polygon of valuation of $|P(X)|$ has a change of slope. Hence $P(X)$ has a zero in the closed unit ball $|T| \leq 1$. This is the desired contradiction.□

Lemma B.2.2. *Let* $r_1 \leq r_2$ *and* $\rho \in [r_1, r_2]$*. Let* $a^-(T) = 1 + \alpha_{-1}T^{-1} + \alpha_{-2}T^{-2} + \cdots \in \mathcal{A}_K([r_1,\infty])^\times$*, and* $a^+(T) = 1 + \alpha_1 T + \alpha_2 T^2 + \cdots \in \mathcal{A}_K([0,r_2[)^\times$*. Then*

$$|a^-(T)\cdot a^+(T) - 1|_\rho < 1 \ . \tag{B.2.2.1}$$

Proof : Write $a^-(T)a^+(T) = \sum_{n\in\mathbb{Z}} c_n T^n$. If we let $\alpha_0 = 1$, then, for all $n \geq 0$, $c_n = \sum_{k=0}^\infty \alpha_{n+k}\alpha_{-k}$, and $c_{-n} = \sum_{k=0}^\infty \alpha_{-n-k}\alpha_k$. By B.2.1, one has $|\alpha_{-k}|\rho^{-k} < 1$, for all $k \geq 1$, and $\lim_{k\to\infty}|\alpha_{-k}|\rho^{-k} = 0$. On the other hand, $|\alpha_k|\rho^k \leq 1$ for all $k \geq 0$, since $a^+(T)$ is invertible. Hence, if $n \geq 1$, then $|c_n|\rho^n < 1$, and $|c_{-n}|\rho^{-n} < 1$, for all $\rho \in]r_1, r_2[$. Moreover, $|c_0 - 1| \leq 1$, since $c_0 = 1 + \alpha_1\alpha_{-1} + \alpha_2\alpha_{-2} + \cdots$. □

Theorem B.2.3. *Let* K *be discrete valuated. Let* $a(T) \in \mathcal{E}_K$*. Then there exist* $\lambda \in K$*,* $N \in \mathbb{Z}$*,* $a^-(T) = 1 + h^-(T)$ *invertible in* $1 + T^{-1}\mathcal{A}_K([1,\infty])$*, with* $h^-(T) = \sum_{i\leq -1}\alpha_i T^i$*, and* $a^+(T) = 1 + h^+(T)$ *invertible in* $1 + T\mathcal{A}_K([0,1[)$*, with* $h^+(T) = \sum_{i\geq 1}\alpha_i T^i$*, such that*

$$a(T) = \lambda \cdot T^N \cdot a^-(T)\cdot a^+(T).$$

Moreover, such a decomposition is unique.

Proof : The theorem is evident for rational fractions. Notice that the theorem can not be deduced immediately "by density" because rational fractions are not dense in $\mathcal{E}_K$ with respect to the Gauss norm $|.|_1$. Actually, rational fractions are dense on $\mathcal{E}_K^\dagger$ with respect to the limit Frechet topology induced by the Robba ring $\mathcal{R}_K$, and subsequently $\mathcal{E}_K^\dagger$ is dense in $\mathcal{E}_K$ with respect to the Gauss norm.

Lemma B.2.4. *Suppose that* K *has a discrete valuation. Let* $I = [r_1, r_2]$ *be a closed interval. If* $a(T) \in \mathcal{A}_K(I)$ *has no poles nor zeros on* I*, then* *there exist a* $\lambda \in K$*, a* $N \in \mathbb{Z}$*, a function* $a^-(T) = 1 + h^-(T)$ *invertible in* $1 + T^{-1}\mathcal{A}_K([r_1, \infty])$*, with* $h^-(T) = \sum_{i\leq -1} \alpha_i T^i$*, and a function* $a^+(T) = 1+h^+(T)$ *invertible in* $1+T\mathcal{A}_K([0, r_2])$*, with* $h^+(T) = \sum_{i\geq 1} \alpha_i T^i$*, such that*

$$a(T) = \lambda \cdot T^N \cdot a^-(T) \cdot a^+(T) \ . \tag{B.2.4.1}$$

Moreover, such a decomposition is unique.

Proof : The assertion is evident on rational fractions. By B.2.2, if $a(T) = \sum_{i\in\mathbb{Z}} b_i T^i$, then $N = \min(i \mid |b_i| = |a(T)|_1)$, and $\lambda = b_N$. Since K has a discrete valuation, then $N < \infty$. We can suppose $\lambda = 1$ and $N = 0$. We shall now prove this theorem by “density”. If $a_n(T) \in \mathcal{A}_K(I)$ is a sequence of rational fractions convergent to $a(T)$, then for n sufficiently large $a_n(T)$ has no poles nor zeros on I, hence $a_n(T)$ admits such a decomposition : $a_n(T) = \lambda_n T^{N_n} a_n^-(T) a_n^+(T)$. Moreover, $N_n = 0$ and $\lim_n \lambda_n = 1$. Since $1 + T^{-1}\mathcal{A}_K([r_1, \infty])$ and $1 + T\mathcal{A}_K([0, r_2])$ are closed sets in $\mathcal{A}_K(I)$, it is sufficient to show that the maps $a \mapsto h^-$ and $a \mapsto h^+$ are both continuous on the set of rational fractions without zeros nor poles on I. Let $a(T) = \lambda_a T^{N_a} a^- a^+$, $b(T) = \lambda_b T^{N_b} b^- b^+$ be rational fractions without zeros nor poles on I. Suppose that $\|a - b\|_I < \varepsilon < \|a\|_I$. Again we can suppose $N_a = N_b = 0$, $\lambda_a = 1$, $|\lambda_b - 1| < \varepsilon$. Write then $a = a^- a^+$ and $b = \lambda_b b^- b^+$. Since a^+, a^-, b^+, b^- are all invertible, then $\|a^+\|_I = \|a^-\|_I = \|b^+\|_I = \|b^-\|_I = 1$. We have the equality

$$\|a - b\|_I = \|a^- a^+ - \lambda_b b^- b^+\|_I = \Big\|\frac{a^- a^+ - \lambda_b b^- b^+}{a^+ b^-}\Big\|_I = \Big\|\frac{a^-}{b^-} - \lambda_b \frac{b^+}{a^+}\Big\|_I \ .$$

Now write $a^- = 1 + h_a^-$, $a^+ = 1 + h_a^+$, $b^- = 1 + h_b^-$, $b^+ = 1 + h_b^+$. Let $1+h^- := a^-/b^- = (1+h_a^-)/(1+h_b^-)$ and $1+h^+ := b^+/a^+ = (1+h_b^+)/(1+h_a^+)$, then

$$\|a - b\|_I = \|1 - \lambda_b + h^- - \lambda_b h^+\|_I \ .$$

Since $h^- = \sum_{i\leq -1} \beta_i T^i$ and $h^+ = \sum_{i\geq 1} \beta_i T^i$, then for all $\rho \in I$ one has $|(1 - \lambda) + h^- - \lambda_b h^+|_\rho = \sup(|1 - \lambda_b|, |h^-|_\rho, |h^+|_\rho)$, then $\|(1 - \lambda_b) + h^- - \lambda_b h^+\|_I = \sup(|1 - \lambda_b|, \|h^-\|_I, \|h^+\|_I)$. In other words, we have

$$\|a-b\| = \sup(|1-\lambda_b|, \|h^-\|, \|h^+\|) \geq \|h^+\| = \Big\|\frac{1 + h_b^+}{1 + h_a^+} - 1\Big\| = \Big\|\frac{h_b^+ - h_a^+}{1 + h_a^+}\Big\| = \|h_b^+ - h_a^+\| \ .$$

In the same way we have $\|h_b^- - h_a^-\|_I \leq \|a - b\|_I$. This shows the continuity. □

Proof of B.2.3 : Since K is discrete valued, then $\mathcal{E}_K^\dagger$ is a field, that is all $f(T) \in \mathcal{E}_K$ have no zeros nor poles in a small open interval $J :=]1 - \varepsilon, 1[$. We express the open interval J as union of increasing sequence of closed sub-intervals $I_n \subseteq I_{n+1} \subset J$. For all n we have a decomposition $a(T) = \lambda_n T^{N_n} a_n^-(T) a_n^+(T)$. By unicity, for all n, we must have $\lambda_{n+1} = \lambda_n$,

$N_{n+1} = N_n$, $a^-_{n+1}(T) = a^-_n(T)$, $a^+_{n+1}(T) = a^+_n(T)$. This shows the theorem for $\mathcal{E}^\dagger_K$. With the same proof of B.2.4, the two maps $a(T) \mapsto h^-(T)$ and $a(T) \mapsto h^+(T)$ are continuous on $\mathcal{E}^\dagger_K$ with respect to the Gauss norm $|.|_1$. The theorem results then by density. $\square$

B.3 Criterion of solvability for q-difference equations over $\mathcal{E}_K$

Hypothesis B.3.1. From now on the valuation on K will be discrete valuation, in order to have theorem B.2.3.

We denote by

$$\sigma_q : f(T) \mapsto f(qT) \ , \qquad \partial_T := T\frac{d}{dT} \ , \tag{B.3.1.1}$$

$$d_q := \frac{\sigma_q - 1}{(q-1)T} \ , \qquad \Delta_q := \frac{\sigma_q - 1}{q-1} \ . \tag{B.3.1.2}$$

B.3.1 Preliminary lemmas

Lemma B.3.2 (q-Small Radius, $q-$analogue of A.1.2)**.** *Let $q \in \mathrm{D}^-(1,1)$ and let I be any interval. Let $\sigma_q - a(q,T)$, $a(q,T) \in \mathcal{A}_K(I)$ be some equation. Let $R_\rho := Ray(\sigma_q - a(q,T), \rho)$ be the radius of convergence of the equation at $\rho \in I$. Then*

$$R_\rho \geq \frac{\omega_q \cdot \rho}{\max(|g(q,T)|_\rho, 1)} = \frac{\omega_q \cdot \rho \cdot |q-1|}{\max(|a(q,T)-1|_\rho, |q-1|)} \tag{B.3.2.1}$$

Moreover $R_\rho < \omega_q \cdot \rho$ if and only if $|a(q,T)-1|_\rho > |q-1|$, and in this case

$$R_\rho = \frac{\omega_q \cdot \rho \cdot |q-1|}{|a(q,T)-1|_\rho} \ . \tag{B.3.2.2}$$

The same assertions hold for solvable q-difference equations over $\mathcal{E}_K$, with $\rho = 1$.

Proof : Let $g_{[s]}(q,T) \in \mathcal{A}_K(I)$ be the 1×1 matrix of $(d_q)^s$. By definition

$$\begin{aligned} Ray(d_q - g_{[1]}(q,T), 1) &= \min\big(\rho \ , \ \liminf_s (|g_{[s]}(q,T)|_1 / |[s]^!_q|)^{-\frac{1}{s}}\big) \\ &= \min\big(\rho \ , \ \omega_q \cdot \liminf_s (|g_{[s]}(q,T)|_1)^{-\frac{1}{s}}\big) \ . \end{aligned} \tag{B.3.2.3}$$

One has inductively $|g_{[s]}|_\rho \leq \max(|g_{[1]}|_\rho, \rho^{-1})^s$, this shows B.3.2.1. Moreover, if $|g_{[1]}|_\rho > \rho^{-1}$, then $|g_{[s]}|_\rho = \max(|g_{[1]}|_\rho, \rho^{-1})^s$ and B.3.2.2 holds. Reciprocally, if $R_\rho < \omega_q \cdot \rho$, then, by B.3.2.1, one has $|a(q,T)-1| > |q-1|$. $\square$

Lemma B.3.3. *Let $q \in \mathrm{D}^-(1,1)$. Let $\sigma_q - a(q,T)$ be a* solvable *equation such that $a(q,T) \in \mathcal{R}_K$ or $a(q,T) \in \mathcal{E}_K$. Let $a(q,T) = \lambda_q T^N a^-(q,T) a^+(q,T)$ be the Motzkin decomposition of $a(q,T)$ (cf. B.2.3), then $N = 0$ and $|\lambda_q - 1| < 1$.*

Proof : The solvability implies $|a(q,T) - 1|_1 \leq |q-1| < 1$ (cf. B.3.2), hence $|a(q,T)|_1 = 1$. More precisely, in the notations of B.2.2, one has $|\lambda_q \sum_{n\in\mathbb{Z}} c_n T^{n+N} - 1|_1 \leq |q-1| < 1$. We know that $\sup_{n\neq 0} |c_n| < 1$ and $|c_0 - 1| < 1$ (cf. B.2.2). If $N \neq 0$, then $|\lambda_q c_0 T^N|_1 < 1$ and $|\lambda_q c_{-N} - 1| < 1$. The first one implies $|\lambda_q| < 1$, which contrasts the second one. Hence $N = 0$. Then $|\lambda_q c_0 - 1| < 1$ implies $|\lambda_q - 1| < 1$.□

Lemma B.3.4. *Let $q \in \mathrm{D}^-(1,1)$. There exists $R_0 > 0$ such that $Ray(\sigma_q - \lambda_q, \rho) = R_0 \cdot \rho$, for all $\rho \in [0, \infty]$.*

Proof : By [DV04, 1.2.4], one has

$$\left|g_{[n]}(T)\right|_1^{\frac{1}{n}} = \frac{|\sum_{j=0}^{n}(-1)^j \binom{n}{j}_{q^{-1}} q^{\frac{-j(j-1)}{2}} \lambda_q^j|^{1/n}}{|q-1| \cdot \rho} \;.$$

Since the numerator does not depend on ρ, the lemma is proved. □

B.3.2 The settings

As for differential equations, we shall find a description of the formal solution of a given solvable $q-$difference equation

$$\sigma_q(y_q) = a(q,T) \cdot y_q \;, \qquad \text{(B.3.4.1)}$$

with $a(q,T) \in \mathcal{E}_K$. We will show that solutions of $q-$difference equations are actually solutions of differential equation (cf. A.1.13.2). By B.3.3, we know that

$$a(q,T) = \lambda_q \cdot a^-(q,T) a^+(q,T) \;, \qquad \text{(B.3.4.2)}$$

with $a^-(q,T) := 1 + \sum_{-\infty}^{-1} \alpha_i T^i$, and $a^+(q,T) := 1 + \sum_{i=1}^{+\infty} \alpha_i T^i$. Now write formally

$$a^-(q,T) := \exp(\sum_{i\leq -1} a_i T^i) \;, \qquad a^+(q,T) := \exp(\sum_{i\geq 1} a_i T^i) \;. \qquad \text{(B.3.4.3)}$$

Then the formal solution of B.3.4.1 is

$$y_q(T) := \exp\big(\sum_{i\leq -1} \frac{a_i}{q^i - 1} T^i\big) \cdot q^{a_0} \cdot \exp\big(\sum_{i\geq 1} \frac{a_i}{q^i - 1} T^i\big) \;. \qquad \text{(B.3.4.4)}$$

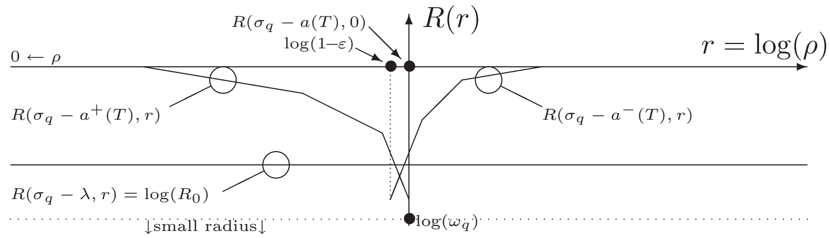
We are interested to study this exponential in the case in which the equation B.3.4.1 is solvable. The main result is the Criterion of solvability B.3.19.

B.3.3 Technical lemma

In this section $q \in \mathrm{D}^-(1,1)$ is fixed. We will omit the index q in the series.

Proposition B.3.5 (q-analogue of A.1.3)**.** *Let $q \in \mathrm{D}^-(1,1)$. Let $\sigma_q - a(T)$, $a(T) = \lambda a^-(T)a^+(T) \in \mathcal{R}_K$, be a solvable equation. Then $\sigma_q - a^-(T)$, $\sigma_q - \lambda$, $\sigma_q - a^+(T)$ are all solvable.*

Proof : With analogous notations of A.1.3, we find the following picture :



Since there exists a common interval $I :=]1-\varepsilon, 1[$ in which all operators exist, and since the slope of $Ray(\sigma_q - a^-, \rho)$ (resp. $Ray(\sigma_q - a^+, \rho)$) is strictly positive (resp. negative) in I, hence there are at most 3 points in which these graphics cross. Hence

$$Ray(\sigma_q - a, \rho) = \min(\ Ray(\sigma_q - a^-, \rho)\ ,\ Ray(\sigma_q - a^+, \rho)\ ,\ Ray(\sigma_q - \lambda, \rho)\ )\ , \tag{B.3.5.1}$$

for all $\rho \in I$. Since, by assumption, $\lim_{\rho \to 1^-} Ray(\sigma_q - a, \rho) = 1$, then equation B.3.5.1 implies the proposition. $\Box$

Proposition B.3.6 (q-analogue of A.1.3)**.** *Let $|q-1| < \omega$. Let $\sigma_q - a(T)$, $a(T) = \lambda a^-(T)a^+(T) \in \mathcal{E}_K$, be a solvable equation. Then $\sigma_q - a^-(T)$, $\sigma_q - \lambda$, $\sigma_q - a^+(T)$ are all solvable.*

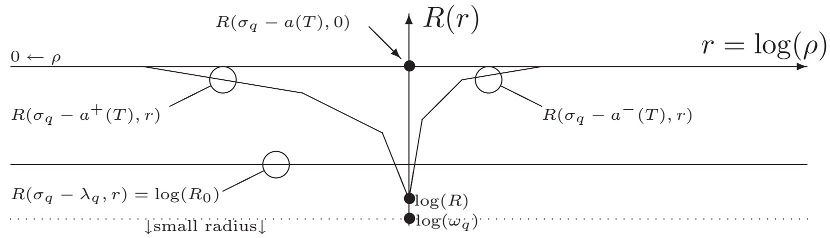
Proof : Steps 1 and 2 of this proof coincide with the same steps of the proof of A.1.3. However, we will expose it without proofs for fixing notation. The first part of this proposition does not use the hypothesis $|q-1| < \omega$, so we will assume this hypothesis starting from B.3.10.

— *Step 1 :* By [DV04, 3.6], the equation $\sigma_q - a^-(T)$ (resp. $\sigma_q - a^+(T)$) has a convergent solution at ∞ (resp. at 0), hence $Ray(\sigma_q - a^-(T), \rho) = \rho$, for large values of ρ (resp. $Ray(\sigma_q - a^-(T), \rho) = \rho$ for ρ close to 0). Let R^0 be as in B.3.4,

$$R^- := Ray(\sigma_q - a^-(T), 1)\ , \tag{B.3.6.1}$$
$$R^+ := Ray(\sigma_q - a^+(T), 1)\ . \tag{B.3.6.2}$$

— *Step 2 : We have $R^+ = R^-$ and $R^0 \geq R^- = R^+$ (as in the following picture in which $R := R^- = R^+$).*

We set $r := \log(\rho)$, and $R(r) := \log(Ray(\sigma_q - a(T), \rho)/\rho)$.



— *Step 3 : We have $R \geq \omega_q$.*

Indeed, if $R^- = R^+ < \omega_q$, then, by B.3.2, $|a^-(T) - 1|_1 > |q - 1|$ and $|a^+(T) - 1|_1 > |q - 1|$. We shall now show that this implies that $|a(T) - 1|_1 > |q - 1|$, which is in contradiction with the small radius lemma, since the equation $\sigma_q - a(T)$ is solvable (cf. B.3.2). This will results from the following lemma :

Lemma B.3.7. *Let $(R, |.|)$ be an ultrametric valued ring. Let $h^-, h^+ \in U$ be two elements satisfying $|h^-| < 1$ and $|h^- + h^+| = \sup(|h^-|, |h^+|)$. Then*

$$|h^- + h^+ + h^- h^+| = \sup(|h^-|, |h^+|).$$

Proof : If $|h^+| > |h^-|$, then $|h^- + h^+ + h^- h^+| = |h^+|$. If $|h^+| \leq |h^-| < 1$, then $|h^- h^+| < |h^-| = \max(|h^-|, |h^+|) = |h^- + h^+|$. $\square$

Proof of Step 3 *:* Write $a^-(T) = 1 + h_q^-(T)$ and $\lambda_q \cdot a^+(T) = 1 + (\lambda_q - 1) + \lambda_q \cdot h_q^+(T)$. Namely, in the notations of B.2.3, we have $h_q^-(T) = \sum_{i \leq -1} \alpha_i T^i$ and $h_q^+(T) = \sum_{i \geq 1} \alpha_i T^i$. We apply Lemma B.3.7 to the field $U := \mathcal{E}_K$, $h^- := h_q^-(T)$ and $h^+ := (\lambda_q - 1) + \lambda_q h_q^+(T)$. Indeed, $|h^- + h^+|_1 = \sup(|h^-|_1, |h^+|_1)$ and moreover, by B.3.8 below, we have $|h^-|_1 < 1$. Lemma B.3.7 then implies

$$|a(T) - 1|_1 = |(1 + h^-)(1 + h^+) - 1|_1 = |h^- + h^+ + h^- h^+|_1 \overset{B.3.7}{=} \sup(|h^-|_1, |h^+|_1). \tag{B.3.7.1}$$

Now, if $R^- < \omega_q$, then $|a^-(T) - 1| > |q - 1|$, that is $|h^-(T)| > |q - 1|$. Hence $|a(T) - 1|_1 > |q - 1|$, which implies that the radius of $\sigma_q - a(T)$ is small (cf. Lemma B.3.2). Since, by assumption, $Ray(\sigma_q - a(T), 1) = 1$, this is absurd and then $R \geq \omega_q$.

— *Step 4 : We shall show that $R > \omega_q$.*

Lemma B.3.8 (q-analogue of A.1.5)**.** *Let $\sigma_q - a(T)$, $a(T) \in \mathcal{E}_K$. Let $a(T) := \lambda_q a^-(T) a^+(T)$ be the Motzkin decomposition of $a(T)$. If $Ray(\sigma_q - a(T), 1) > \omega_q$, then $a^-(T) = 1 + h_q^-(T)$, where $h_q^-(T) = \sum_{i \leq -1} \alpha_i T^i$ satisfies $|h_q^-|_1 < |q - 1|$.*

Proof : Consider the operator $d_q - g_{[1]}(T)$, with $g_{[1]}(T) := \frac{a(T)-1}{(q-1)T}$, and write

$$g_{[1]}(T) = \frac{a^-(T)-1}{(q-1)T} + a^-(T)\frac{\lambda_q a^+(T)-1}{(q-1)T} = g^-_{[1]}(T) + a^-(T)\frac{\lambda_q a^+(T)-1}{(q-1)T}\ , \tag{B.3.8.1}$$

with $g^-_{[1]}(T) := \frac{a^-(T)-1}{(q-1)T}$. Since $Ray(d_q - g_{[1]}(T),1) > \omega_q$, hence, by B.3.2.3 and B.1.7, one has $\lim_{s\to\infty}|g_{[s]}(T)|_1 = 0$. In particular $|g_{[s]}(T)|_1 < 1$, for some $s \geq 1$. Moreover, by the Small Radius Lemma B.3.2, we have $|g_{[1]}(T)|_1 \leq 1$. These facts implies the lemma in the following way. Suppose that $|a^-(T)-1|_1 \geq |q-1|$. By the same process of step 3, we see that $Ray(\sigma_q - a^-(T),1) \geq \omega_q$, then, by B.3.2, $|g^-_{[1]}(T)|_1 \leq 1$, and hence $|a^-(T)-1|_1 = |q-1|$. Let $-d$ be the smallest index such that $|\alpha_{-d}| = |q-1|$. The reduction of $g_{[1]}(T)$ in $k(\!(t)\!)$ is of the form $\overline{g_{[1]}(T)} = \overline{\alpha}t^{-d-1} + \cdots$, where $\overline{\alpha}$ is the reduction of $\alpha_{-d}/(q-1)$. This results form the equation B.3.8.1. Indeed, by B.2.1, the reduction $a^-(T)$ is 1 and the reduction of $\frac{\lambda_q a^+(T)-1}{(q-1)T}$ lies in $t^{-1}k[[t]]$. If $-d \leq -1$, then a simple induction on the equation $g_{[s+1]} = d_q(g_{[s]}) + \sigma_q(g_{[s]})g_{[1]}$ shows that $\overline{g_{[s]}(T)} = \overline{\alpha}^s t^{(-d-1)s} + \cdots$, this is in contradiction with the fact that $|g_{[s]}(T)|_1 < 1$, for some $s \geq 1$.□

Lemma B.3.9 (q-analogue of A.1.4)**.** *Let $q \in \mathrm{D}^-(1,1)$. Let $\Delta_q - g(T)$, $g(T) \in \mathcal{E}_K$, be some equation. Suppose that $|g(T)|_1 \leq 1$. Then $Ray(\Delta_q - g(T),1) > \omega_q$ if and only if $|g_{[s]}(T)| < 1$, for some $s \geq 1$.*

Proof : The condition $|g(T)|_1 \leq 1$ implies that the sequence $|g_{[n]}|_1$ is decreasing. Indeed, $|g_{[1]}|_1 = |T^{-1}g(T)|_1 \leq 1$ and inductively $|g_{[n+1]}|_1 = |d_q(g_{[n]})+\sigma_q(g_{[n]})g_{[1]}|_1 \leq \sup(|g_{[n]}|_1, |g_{[n]}g_{[1]}|_1) = |g_{[n]}|_1 \sup(1,|g_{[1]}|_1) = |g_{[n]}|_1$.

Let $Ray(d_q - g(T),1) > \omega_q$. Since $\lim_{n\to\infty}|[s]^!_q|^{1/s} = \omega_q$, hence, by B.3.2.3, $|g_{[s]}(T)|_1$ goes to 0. Assume now that $|g_{[n]}|_1 < 1$, for some $n \geq 1$. Since the sequence is decreasing, then $|g_{[p^h]}|_1 < 1$, for some $h \geq 1$. By [DV04, 1.2.2], one has

$$d_q^{(m+1)p^h}(y) = d_q^{p^h}(d_q^{mp^h}(y)) = \sum_{r=0}^{p^h}\binom{p^h}{r}_q d_q^r(g_{[mp^h]})\cdot\sigma_q^r(g_{[p^h-r]})\sigma_q^r(y)\ .$$

Then $g_{[(m+1)p^h]} = \sum_{r=0}^{p^h}\binom{p^h}{r}_q d_q^r(g_{[mp^h]})\cdot\sigma_q^r(g_{[p^h-r]})a(T)a(qT)\cdots a(q^{r-1}T)$. One has $|a(q^jT)|_1 = |a(T)|_1 = 1$, for all $j \geq 0$, and $|d_q^k(f)|_1 \leq |[k]^!_q||f|_1$ (cf. B.1.8). Moreover, $|\binom{p^h}{r}_q| = |[p^h]_q[p^h-1]_q\cdots[p^h-r+1]_q|/|[r]^!_q|$, where $[n]_q := \frac{q^n-1}{q-1}$. Since $|[p^h]_q| < |[p]_q|$, hence

$$|g_{[(m+1)p^h]}|_1 \leq \sup(|[p]_q|,|g_{[p^h]}|_1)\cdot|g_{[mp^h]}|_1\ . \tag{B.3.9.1}$$

Let $s := \sup(|[p]_q|,|g_{[p^h]}|_1) < 1$. Then, by B.3.9.1, one has $|g_{[mp^h]}|_1 \leq s^m$, for all $m \geq 1$. For all integer $n \geq p^h$ we let $m(n) := [n/p^h] \geq 1$, where $[a]$

is the greatest integer smaller than or equal to a. Then $m(n)p^h \leq n$ and $|g_{[n]}|_1 \leq |g_{[m(n)p^h]}|_1 \leq s^{m(n)}$. So

$$\left|\frac{g_{[n]}}{[n]_q^!}\right|_1^{\frac{1}{n}} \leq \frac{s^{m(n)/n}}{|[n]_q^!|^{1/n}} \leq \frac{s^{1/p^h}}{s^{1/n}\cdot|[n]_q^!|^{1/n}} \xrightarrow{n\mapsto\infty} \frac{s^{1/p^h}}{\omega_q} . \tag{B.3.9.2}$$

Indeed $\frac{1}{p^h}-\frac{1}{n} \leq \frac{[n/p^h]}{n} \leq \frac{1}{p^h}$. By B.3.2.3, one has $Ray(\Delta_q - g_{[1]},1) \geq \omega_q/s^{1/p^h} > \omega_q$. $\square$

Proof of Step 4 *:* Let us show now that $(R^+ = R^- =)R > \omega_q$. It is sufficient to show that $R^- > \omega_q$. Recall that R^- is the radius of the operator $d_q - g_{[1]}^-(T)$, $g_{[1]}^-(T) := \frac{a^-(T)-1}{T(q-1)}$. Since $Ray(\sigma_q - a(T),1) = 1 > \omega_q$, then, by B.3.8, one has $|a^-(T)-1|_1 < |q-1|$, that is $|g_{[1]}^-(T)|_1 < 1$. Hence, by B.3.9, we have $R^- > \omega_q$. $\square$

Hypothesis B.3.10. From now on we will suppose that $q \in \mathrm{D}^-(1,|p|^{\frac{1}{p-1}})$, hence $\omega_q = \omega$.

The reason of the precedent hypothesis is that it is necessary to have the theorem [DV04] which state the existence of the antecedent of Frobenius only for $|q-1| < \omega$.

— *Step 5 : Since $|q-1|<1$, and since $R>\omega$, then, by [DV04], we can take the antecedent by Frobenius of $\sigma_q - a^-(T)$, $\sigma_q - a^+(T)$ and $\sigma_q - \lambda_q$.*

More precisely, there exist a finite extension $K^{(1)}/K$, an $f^+(T) = \sum_{i\geq 0} b_i^+ T^i \in \mathcal{A}_{K^{(1)}}([0,1[)^\times$, $f^-(T) = \sum_{i\leq 0} b_i^- T^i \in \mathcal{A}_{K^{(1)}}([1,\infty])^\times$, and there are functions $a^{(1),-}(T) = \sum_{i\leq 0}\alpha_i^{(1),-}T^i \in \mathcal{E}_{K^{(1)}}$, $a^{(1),+}(T) = \sum_{i\geq 0}\alpha_i^{(1),+}T^i \in \mathcal{E}_{K^{(1)}}$, and $\lambda_q^{(1)} \in K^{(1)}$ such that

$$\begin{aligned}
(\lambda_q^{(1)})^p &= \lambda_q ;\\
a^{(1),-}(T^p)^\sigma \cdot a^{(1),-}(q\cdot T^p)^\sigma \cdots a^{(1),-}(q^{p-1}T^p)^\sigma &= a^-(T)\cdot\frac{f^-(q\cdot T)}{f^-(T)} ;\\
a^{(1),+}(T^p)^\sigma \cdot a^{(1),+}(q\cdot T^p)^\sigma \cdots a^{(1),+}(q^{p-1}T^p)^\sigma &= a^+(T)\cdot\frac{f^+(q\cdot T)}{f^+(T)} ,
\end{aligned}$$

where, for all functions $a(T) := \sum \alpha_i T^i \in \mathcal{E}_K$, we let $a(T)^\sigma := \sum \sigma(\alpha_i)T^i$. We see immediately that $b_0^+ \neq 0$, $b_0^- \neq 0$, and that $f^+(qT)/f^+(T)) = 1 + u_1T + u_2T^2 + \cdots$, and $f^-(qT)/f^-(T) = 1 + u_{-1}T^{-1} + u_{-2}T^{-2} + \cdots$. Since $a^-(T) = 1 + \alpha_{-1}T^{-1} + \cdots$, and $a^+(T) = 1 + \alpha_1 T + \cdots$, this implies that $\alpha_0^{(1),+} = 1$ and $\alpha_0^{(1),-} = 1$. Hence

$$a^{(1)}(T) := \lambda_q^{(1)} \cdot a^{(1),-}(T)\cdot a^{(1),+}(T)$$

lies in $\mathcal{E}_K$; moreover this decomposition is the unique one of Theorem B.2.3. Observe now that both f^- and f^+ belong to $\mathcal{E}_K^\times$, hence $\sigma_q - a^{(1)}(T)$ is an antecedent of Frobenius of $\sigma_q - a(T)$ over $\mathcal{E}_K$, and it is then solvable.

— *Step 6 :* Steps 1, 2, 3, 4 are still true for the antecedent. In particular if

$$R^-(1) \;:=\; Ray(\partial_T - g^{(1),-}(T), 1) \;, \tag{B.3.10.1}$$
$$R^+(1) \;:=\; Ray(\partial_T - g^{(1),+}(T), 1) \;, \tag{B.3.10.2}$$
$$R^0(1) \;:=\; Ray(\partial_T - b_0, 1) \;. \tag{B.3.10.3}$$

we must have $R^-(1) = R^+(1) > \omega$. Let $R(1) := R^-(1) = R^+(1)$, then $R(1) = R^{1/p}$, by the property of the antecedent. This implies $R > \omega^{1/p}$. Since again $R(1) > \omega$, then we can take again the antecedent. This process can be iterated indefinitely. This shows that $R > \omega^{1/p^h}$, for all $h \geq 0$, that is $R = 1$. This proves B.3.6. □

Corollary B.3.11 (q-analogue of A.1.6)**.** *Let $q \in \mathrm{D}^-(1,1)$. Let $\sigma_q - a(T)$ be a solvable differential equation. Let $a(T) = \lambda \cdot a^-(T) \cdot a^+(T)$ be the Motzkin decomposition of $a(T)$. Then $\lambda = q^{a_0}$, for some $a_0 \in K$. Moreover, this operator is isomorphic to $\sigma_q - \lambda \cdot a^-(T)$.*

Proof : See the proof of A.1.6. □

Remark B.3.12. Observe that the unique obstruction to generalize B.3.6 and B.3.11 to all $q \in \mathrm{D}^-(1,1)$ is represented by the so called Weak Frobenius structure for $q-$difference modules over a disk with $q \in \mathrm{D}^-(1,1)$, which is still missing in the literature. We hope that in the future this fundamental result will be proved.

B.3.4 Criterion of Solvability

Lemma B.3.13 (q-analogue of A.1.8)**.** *Let $q \in \mathrm{D}^-(1,\omega)$. Suppose that $a(T) = a^+(T)$ is the Motzkin decomposition of $a(T)$. Write $a^+(T) = \exp(\sum_{i\geq 1} a_i T^i) \in \mathcal{A}([0,1[)^\times$ (cf. the settings of B.3.2). Then the $q-$difference equation $\sigma_q - a^+(T)$ is solvable if and only if there exists a family $\{\boldsymbol{\lambda}_n\}_{n\in \mathrm{J}_p}$, where $\boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$ has phantom components $\phi_n = (\phi_{n,0}, \phi_{n,1}, \ldots)$ satisfying*

$$a_{np^m} = \frac{(q^{np^m} - 1)}{p^m} \cdot \phi_{n,m} \;, \qquad \text{for all } n \in \mathrm{J}_p,\ m \geq 0 \;, \tag{B.3.13.1}$$

for all $n \in \mathrm{J}_p$, and all $m \geq 0$. In other words, the formal solution of this equation is $y(T) = E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1) := \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} \phi_{n,m}\frac{T^{np^m}}{p^m})$ (cf. B.3.4.4).

Proof : This proof is an adaptation of the proof of A.1.8 in the q-difference framework. The formal series $E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1)$ belongs to $1 + T\cdot\mathfrak{p}_K[[T]] \subset$

$\mathcal{E}_K$, and is solution of the equation $L := \sigma_q - \exp(\sum_{n\in \mathrm{J}_p}\sum_{m\geq 0}\phi_{n,m}(q^{np^m} - 1)T^{np^m})$. Since this exponential converges in the unit disk, then $Ray(L,\rho) = \rho$, for all $\rho < 1$, hence, by continuity of the radius, $Ray(L,1) = 1$ and L is solvable. Conversely, suppose that $\sigma_q - a^+(T)$ is solvable, then the Witt vectors $\boldsymbol{\lambda}_n = (\lambda_{n,0},\lambda_{n,1},\ldots)$ is defined by the relation B.3.13.1. For example, for all $n \in \mathrm{J}_p$ we have

$$\lambda_{n,0} = \frac{a_n}{(q^n-1)} \quad , \qquad \lambda_{n,1} = \frac{1}{p}\left(\frac{p\cdot a_{np}}{(q^{np}-1)} - \left(\frac{a_n}{(q^n-1)}\right)^p\right) \ . \qquad \text{(B.3.13.2)}$$

We must show that $|\lambda_{n,m}| \leq 1$, for all $n \in \mathrm{J}_p$, $m \geq 0$.

— *Step* 0 *: We have* $|\lambda_{n,0}| = |\phi_{n,0}| \leq 1$ *for all* $n \in \mathrm{J}_p$.

This results by the small radius lemma B.3.2 as follows : denote by $\phi_q^+(T) := \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{n,m}(q^{np^m}-1)T^{p^m}/p^m$ the argument of the exponential $a^+(T)$. By the small radius B.3.2, one has $|a^+(T)-1|_1 = |\exp(\phi_q^+)-1|_1 \leq |q-1|$. Since $|q-1| < \omega$, then $|\exp(\phi_q^+)-1|_1 < \omega$, hence $\phi_q^+ = \log(\exp(\phi_q^+))$ and $|\phi_q^+|_1 = |\exp(\phi_q^+)-1|_1 \leq |q-1|$. This implies $|\phi_{n,m}(q^{np^m}-1)/p^m| \leq |q-1|$, for all $n \in \mathrm{J}_p$ and all $m \geq 0$. In particular, for $m = 0$ we have $|\lambda_{n,0}| = |\phi_{n,0}| \leq 1$, for all $n \in \mathrm{J}_p$.

— *Step* 1 *:* This implies that the exponential

$$E(\sum_{n\in\mathrm{J}_p}(\lambda_{n,0},0,0,\ldots)T^n,1) = \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}\frac{T^{np^m}}{p^m})$$

converges in the unit disk and is solution of the operator $Q^{(0)} := \sigma_q - a^{(0)}(T)$, with $a^{(0)}(T) = \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}(q^{np^m}-1)\frac{T^{np^m}}{p^m})$. By transfer, $Q^{(0)}$ is then solvable.

— *Step 2 :* The tensor product operator $\sigma_q - a^+(T)/a^{(0)}(T)$ is again solvable. We have explicitly

$$\frac{a^+(T)}{a^{(0)}(T)} = \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}(\phi_{n,m} - \lambda_{n,0}^{p^m})(q^{np^m}-1)\frac{T^{np^m}}{p^m}) \ .$$

This operator corresponds to the family of Witt vectors $\{\boldsymbol{\lambda}_n - (\lambda_{n,0},0,0,\ldots) = (0,\lambda_{n,1},\lambda_{n,2},\ldots)\}_{n\in\mathrm{J}_p}$. Observe that the coefficient corresponding to $m = 0$ is equal to 0, for all $n \in \mathrm{J}_p$. This leads us to compute easily the "antecedent by ramification" of $\sigma_q - a^+(T)/a^{(0)}(T)$, namely this antecedent is given by $\sigma_q - a^{(1)}(T)$, with

$$a^{(1)}(T) := \exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}(\phi_{n,m} - \lambda_{n,0}^{p^m})(q^{np^m}-1)\frac{(q-1)}{(q^p-1)}\frac{T^{np^{m-1}}}{p^m}) \ .$$

In other words, we have

$$a^{(1)}(T^p)\cdot a^{(1)}(qT^p)\cdot a^{(1)}(q^2T^p)\cdots a^{(1)}(q^{p-1}T^p)=\frac{a^+(T)}{a^{(0)}(T)}\;.$$

— *Step 3 :* The antecedent is again solvable, hence, as in Step 0, we find $|\phi_{n,1}-\lambda_{n,0}^p|\leq|q^p-1|=|p|$, which implies $|\lambda_{n,1}|\leq 1$. The process can be iterated indefinitely. □

Remark B.3.14. It seems that for $|q-1|\geq\omega$ there exist solvable rank one $q-$difference equations that are not strongly confluent, i.e. such that their solution $y(T)$ is an exponential of the type $E(\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_nT^n,1)$, which lies in $\mathcal{O}_K[[T]]$ but such that $\boldsymbol{\lambda}_n\in\mathbf{W}(K)-\mathbf{W}(\mathcal{O}_K)$, for some $n\in\mathrm{J}_p$. The author does not know examples of such Witt vectors.

Remark B.3.15 (q-analogue of A.1.9)*.* We shall now consider the general case of an equation $\sigma_q-a(T)$, with $a(T)=\lambda\cdot a^-(T)a^+(T)\in\mathcal{E}_K$, and get a criteria of solvability. We proceed as in A.1.9. Suppose given two families $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$, with $\boldsymbol{\lambda}_n\in\mathbf{W}(\mathcal{O}_K)$. By B.3.17, $a^+(T)$ belongs always to $\mathcal{E}_K$. On the other hand, we will see (cf. B.3.18) that the series $a^-(T)$ belongs to $\mathcal{E}_K$ if and only if the family $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ belongs to $\mathrm{Conv}(\mathcal{E}_K)$.

Notation B.3.16. Let $\sigma_q-a(q,T)$, $a(q,T)\in\mathcal{E}_K$ be a solvable differential equation. Let $a(q,T):=q^{a_0}\cdot a^-(q,T)\cdot a^+(q,T)$, $a_0\in\mathbb{Z}_p$, be the Motzkin decomposition of $a(q,T)$. In the notations of Lemma B.3.13 we can write

$$a^-(q,T)=\exp(\phi_q^-(T))\quad,\quad a^+(q,T)=\exp(\phi_q^+(T))\;, \tag{B.3.16.1}$$

$$\phi_q^-(T)\;:=\;\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}(q^{-np^m}-1)\frac{T^{-np^m}}{p^m}\;, \tag{B.3.16.2}$$

$$\phi_q^+(T)\;:=\;\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{n,m}(q^{np^m}-1)\frac{T^{np^m}}{p^m}\;. \tag{B.3.16.3}$$

In other words, the solution of $\sigma_q-a(q,T)$ can be represented by the symbol

$$y(T):=T^{a_0}\cdot\exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}\frac{T^{-np^m}}{p^m})\cdot\exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{n,m}\frac{T^{np^m}}{p^m})\;, \tag{B.3.16.4}$$

as well as for differential equations.

Lemma B.3.17. *Let $|q-1|<\omega$. Let $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$ be a family of Witt vectors such that $\boldsymbol{\lambda}_n\in\mathbf{W}(\mathcal{O}_K)$. Then $a^+(T)$ belongs to $\mathcal{E}_K$.*

Proof : We use the notations of part one of this thesis. Let $P(X)=(X+1)^p-1$ be the Lubin-Tate series corresponding to the formal multiplicative group $\hat{\mathbb{G}}_m$. The phantom vector of $[q^n-1]_P\in\mathbf{W}(\mathcal{O}_K)$ is then $\langle q^n-1,q^{np}-$

$1, q^{np^2}-1, \cdots\rangle$, for all $n \in \mathbb{Z}$ (cf. part one of the thesis). Then, for all $n \in \mathrm{J}_p$, the phantom vector of $[q^n-1]_P \cdot \boldsymbol{\lambda}_n$ is

$$\langle (q^n-1)\phi_{n,0}\ ,\ (q^{np}-1)\phi_{n,1}\ ,\ (q^{np^2}-1)\phi_{n,2}\ ,\ \ldots\rangle\ .$$

Hence we can express $a^+(q,T)$ as a product of Artin-Hasse exponentials

$$a^+(q,T) = \prod_{n\in \mathrm{J}_p} E([q^n-1]_P \cdot \boldsymbol{\lambda}_n\ ,\ T)\ .$$

Since $[q^n-1]_P \cdot \boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$, hence, for all $n \in \mathrm{J}_p$, the Artin-Hasse exponential $E([q^n-1]_P \cdot \boldsymbol{\lambda}_n\ ,\ T)$ belongs to $1+T\mathcal{O}_K[[T]]$, which is contained in $\mathcal{E}_K$. $\square$

Lemma B.3.18 (q-analogue of A.1.10)**.** *Let* $|q-1|<\omega$. *Let* $\{\boldsymbol{\lambda}_{-n}\}_{n\in \mathrm{J}_p} \in \mathbf{W}(\mathcal{O}_K)$ *be a family of Witt vectors. Then the following assertions are equivalent :*
(1) *The series* $a^-(T) = \exp(\phi_q^-(T))$ *belongs to* $\mathcal{E}_K$ *;*
(2) $\phi_q^-(T)$ *belongs to* $\mathcal{E}_K$ *;*
(3) $\{\boldsymbol{\lambda}_{-n}\}_{n\in \mathrm{J}_p} \in \mathrm{Conv}(\mathcal{E})$.

Proof : The equivalence (2) $\Leftrightarrow$ (3) is known (cf. A.1.10). Since, by assumption, we have $\boldsymbol{\lambda}_{-n} \in \mathbf{W}(\mathcal{O}_K)$, then $|\phi_{-n,m}| \leq 1$, and hence

$$|(q^{-np^m}-1)\phi_{-n,m}p^{-m}| = |q-1|\cdot|\phi_{-n,m}| \leq |q-1| < \omega\ . \qquad \text{(B.3.18.1)}$$

Then $|\phi_q^-(T)|_1 \leq |q-1| < \omega$. Assume that $\phi_q^-(T) \in \mathcal{E}_K$. Since the exponential series converges in the disk $D^-_{\mathcal{E}_K}(0,\omega) := \{f \in \mathcal{E}_K \mid |f|_1 < \omega\}$, then $\exp(\phi_q^-(T)) \in \mathcal{E}_K$. Conversely, assume that $\exp(\phi_q^-(T)) \in \mathcal{E}_K$. Since, for all $\rho > 1$, $|\phi_q^-(T)|_\rho < |q-1|$, then $\phi_q^-(T) \in \mathrm{D}^-_{\mathcal{A}_K(]\rho,\infty])}(0,\omega) := \{f \in \mathcal{A}_K(]\rho,\infty]) \mid |f|_\rho < \omega\}$, and hence $\exp(\phi_q^-(T))$ converge in $\mathcal{A}_K([\rho,\infty])$, for all $\rho > 1$. Moreover, $|\exp(\phi_q^-(T))-1|_\rho = |\phi_q^-(T)|_\rho \leq |q-1| < \omega$, for all $\rho > 1$. By continuity, we have $|\exp(\phi_q^-(T))|_1 = |\phi_q^-(T)|_1 \leq |q-1| < \omega$. Now the logarithm converges in the disk $D_{\mathcal{E}_K}(1,1^-) := \{f \in \mathcal{E}_K \mid |f|_1 < 1\}$, hence $\phi_q^-(T) = \log\exp(\phi_q^-(T))$. Then $\phi_q^-(T)$ belongs to $\mathcal{E}_K$.$\square$

Corollary B.3.19 (Criterion of solvability for q-difference equations)**.** *The equation* $\sigma_q - a(q,T)$, *with* $a(q,T) = \lambda_q T^N a^-(q,T)a^+(q,T)$, *with*

$$a^-(T) := \exp(\sum_{i\leq -1} a_i T^i) \quad , \quad a^+(T) := \exp(\sum_{i\geq 1} a_i T^i)\ ,$$

is solvable if and only if the following conditions are verified

1. $\lambda = q^{a_0}$, *with* $a_0 \in \mathbb{Z}_p$ *;*
2. $N = 0$ *;*

3. *There exist two families* $\{\boldsymbol{\lambda}_{-n}\}_{n\in \mathrm{J}_p}$ *and* $\{\boldsymbol{\lambda}_n\}_{n\in \mathrm{J}_p}$, *with* $\boldsymbol{\lambda}_{-n}, \boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$, *for all* $n \in \mathrm{J}_p$, *such that*

$$a_{-np^m} = \frac{(q^{-np^m}-1)}{p^m}\cdot\phi_{-n,m} \quad , \quad a_{np^m} = \frac{(q^{np^m}-1)}{p^m}\cdot\phi_{n,m} \ , \tag{B.3.19.1}$$

for all $n \in \mathrm{J}_p$ *and all* $m \geq 0$ *;*

4. $\{\boldsymbol{\lambda}_{-n}\}_{n\in \mathrm{J}_p} \in \mathrm{Conv}(\mathcal{E})$ *(cf. A.1.12).*

In other words, the formal solution of this equation can be represented by the symbol B.3.16.4 in which the family $\{\boldsymbol{\lambda}_{-n}\}_{n\in \mathrm{J}_p}$ *belongs to* $\mathrm{Conv}(\mathcal{E}_K)$, *and* $a(T) = \exp(\phi_q^-(T))\cdot q^{a_0}\cdot\exp(\phi_q^+(T))$, *where* $\phi_q^-(T)$, $\phi_q^+(T)$ *are defined in B.3.16.2 and B.3.16.3.*

Corollary B.3.20 (canonical extension for $q-$difference)**.** *Let* $\sigma_q{-}Mod(\mathcal{A}_K([1,\infty]))^{\mathrm{sol}}_{rk=1}$ *be the category of rank one* σ_q-*modules over* $\mathcal{A}_K([1,\infty])$, *solvable at all* $\rho \geq 1$. *The scalar extension functor* $\sigma_q\text{-}\mathrm{Mod}(\mathcal{A}_K([1,\infty]))^{\mathrm{sol}}_{rk=1} \to \sigma_q\text{-}\mathrm{Mod}(\mathcal{E}_K)^{\mathrm{sol}}_{rk=1}$ *is an equivalence.*

Proof : The proof is formally equal to A.1.14. $\square$

B.4 Strong confluence

The computation we have obtained shows that the solutions of differential equations and of $q-$difference equation over $\mathcal{E}_K$ coincides.

Moreover by the canonical extension theorem for differential and $q-$difference equations one knows that, if $|q-1| < \omega$, then every rank one object comes by scalar extension from an object over the affinoid $A := \mathbb{P}^1 - \mathrm{D}^-(0,1) = \{|x| \geq 1\}$. In particular, for all $r > 1$, every object comes by scalar extension from an object over the *bounded* affinoid $\{|x| \in [1,r]\}$. Hence the main theorem 7.2.1 applies and we have the strong confluence over $\mathcal{E}_K$.

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