$p$-ADIC DIFFERENTIAL EQUATIONS

$\quad p$-ADIC REPRESENTATIONS

AND

$\quad p$-ADIC DIFFERENCE EQUATIONS

Padova,

15 March, 2007
Summary:

- Introduction
- $p$-adic Representations
- (Abelian) $p$-adic differential equations
- $p$-adic difference equations
- Applications to $p$-adic Zeta function and to $p$-adic $L$ functions
Introduzione

Absolute values over $\mathbb{Q}$:

\[
\begin{cases}
| \cdot |_\infty = \text{Archimedean absolute value}: | \frac{m}{n} |_\infty = \max(\frac{m}{n}, -\frac{m}{n}), \\
| \cdot |_p = \text{p-adic absolute value}
\end{cases}
\]

For all integer $k \in \mathbb{Z}$, $k = np^v$, $(n, p) = 1$ let

\[v_p(k) := v = N^o \text{ of times that } p \text{ divides } k.\]

Let $0 < \varepsilon_p < 1$ be an arbitrary real number, then set $|p|_p := \varepsilon_p$ and, for all $k \in \mathbb{Z}$ as above, set

\[|k|_p = |p|_p^v = \varepsilon_p^{v_p(k)} \leq 1.\]

and more generally

\[\left| \frac{m}{n} \right|_p := \varepsilon_p^{v_p(m)-v_p(n)}.\]
Introduction

$|\cdot|_\infty : \mathbb{Q} \sim \mathbb{R} = \begin{cases} \text{complete} \\ \text{connected} \\ \text{dim}_{\mathbb{R}} \mathbb{C} = 2 \end{cases}$

$|\cdot|_p : \mathbb{Q} \sim \mathbb{Q}_p = \begin{cases} \text{complete} \\ \text{NON connected} \\ \text{dim}_{\mathbb{Q}_p} \mathbb{C}_p = +\infty \end{cases}$

Disks/Balls:

$D^-(a,r) := \{ x \in \mathbb{Q}_p \mid |x-a|_p < r \}$

$D^+(a,r) := \{ x \in \mathbb{Q}_p \mid |x-a|_p \leq r \}$

Pathologies:

- If $|x-a|_p = r$, then $D^-(x,r) \subset D^+(a,r)$;
- We set $\mathbb{Z}_p := D^+(0,1)$. One has $\mathbb{Z} \subseteq \mathbb{Z}_p$, and is dense.
$p$-adic representations

Let $k$ be an algebraically closed field of characteristic $p$.

The object of study is

$$G := \text{Gal}(k((t))^{\text{sep}}/k((t))).$$

- We want to study $G$ by classifying its representations, that is the groups homomorphisms

$$\rho : G \longrightarrow \text{GL}_n(K).$$

where $K/\mathbb{Q}_p$ is a finite extension of fields.

- **Remarkable Fact:** Every finite quotient of $G$ is solvable, more precisely

$$1 \rightarrow \mathcal{P} \rightarrow G \rightarrow G/\mathcal{P} \rightarrow 1 \ , \ G/\mathcal{P} = \prod_{\ell=\text{prime}, \ell \neq p} \mathbb{Z}_\ell.$$

- $\mathcal{P}$ is a pro-$p$-group essentially unknown.
The exemple of rank one representations of rank one of $\mathcal{P}$

The theory of Artin-Schreier describes the caratters of $\mathcal{P}$:

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbf{W}_n(k((t))) \xrightarrow{\text{F-1}} \mathbf{W}_n(k((t))) \to \text{Hom}(\mathcal{P}, \mathbb{Z}/p^n\mathbb{Z}) \to 0$$

- $\mathbf{W}_n(k((t)))$ is the group scheme of “Witt Vectors”.
  Its elements are vectors $(f_0(t), \ldots, f_n(t))$, with $f_i(t) \in k((t))$.

**Theorem 0.1 (Pulita)**  The group of characters $H^1 := \text{Hom}(\mathcal{P}, \mathbb{Q}/\mathbb{Z})$ is isomorphic to

$$H^1 \cong \bigoplus_{(n,p)=1} \left( \lim_{m \geq 0} (\mathbf{W}_m(k) \xrightarrow{\text{FV}} \mathbf{W}_{m+1}(k) \to \cdots) \right)$$

*The graduated of $H^1$ is isomorphic to

$$\text{Gr}_d(H^1) := \text{Fil}_d(H^1)/\text{Fil}_{d-1}(H^1) \cong k.$$*
From Representations to Equations

\[ \begin{array}{ccc}
| & \sim & |\\
\text{k}(\langle t \rangle)^{\text{sep}} & \overset{\sim}{\longrightarrow} & \tilde{R}_K \\
k(\langle t \rangle) & & R_K
\end{array} \]

- \( \mathcal{R}_K \) is the ring of (germs of) analytic functions on the wedge of the unit disk \( D^-(0,1) \), that is functions converging on an annulus \( \{1 - \varepsilon < |x|_p < 1 \} \) of \( \mathbb{C}_p \).
- We have a functor which is \textit{fully faithful}

\[
\text{Rep}^\text{fin}_K(G) \longrightarrow \left\{ \begin{array}{c}
\text{Diff.Eq.}/\mathcal{R}_K \\
\text{with a Frob. structure}
\end{array} \right\}
\]

\[
V \longrightarrow (V \otimes_K \tilde{R}_K)^G
\]

Obtained results:
- Computation of this functor in the Abelian case;
- Classification of all abelian Diff.Eq. over \( \mathcal{R}_K \);
- Criteria to say when a given differential Eq. comes From a repr.
$p$-adic Differential Equations

- The equations considered are *linear*, homogeneous, and in normal form: \( L := y^{(n)} + f_{n-1}y^{(n-1)} + \cdots + f_1 y' + f_0 y = 0. \)

- **Example**: The equation \( y' = y \) has solution \( \exp(T) := \sum \frac{T^n}{n!}. \)

- **Pathology**: The function \( \exp(T) \) does not converge everywhere, but converges only in a disk \( D^{-}(0, \omega_0) \), with \( \omega_0 < +\infty. \)

- **Invariant**: The *radius of convergence* of the Taylor solution in a point \( c \in K \) is an *invariant* of the equation.

- If the equation is defined on an annulus \( \{ r_1 < |T| < r_2 \} \), we can consider the function:

  \[
  r \mapsto R(L, r) := \min_{|c|_p = r} \left\{ \text{Radius of the Taylor solution of } L \text{ at } c \right\}
  \]

- The log-slopes of this function are *invariants* of the equation.
DWORK’S EXAMPLE

<table>
<thead>
<tr>
<th>Operator</th>
<th>Solution at $\infty$</th>
<th>Formal Irr.</th>
<th>$p$–adic Irr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1 := \partial_T + \pi_0 T^{-1}$</td>
<td>$\exp(\pi_0 T^{-1})$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$L_2 := \partial_T + p\pi_0 T^{-p}$</td>
<td>$\exp(\pi_0 T^{-p})$</td>
<td>$p$</td>
<td>1</td>
</tr>
<tr>
<td>$\partial_T := T \frac{d}{dT}$</td>
<td></td>
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</tbody>
</table>

- Since $\theta(T) := \exp(\pi_0 (T^{-p} - T^{-1})) = \frac{\exp(\pi_0 T^{-p})}{\exp(\pi_0 T^{-1})}$ is over-convergent, then these two operators are isomorphic (over $\mathcal{R}_K$).
Theorem 0.2 (Pulita) The rank one differential equations coming from a representation have a solution at $\infty$ of the type:

$$y = T^a \cdot \exp\left(\pi_m \phi_0(T) + \pi_{m-1} \frac{\phi_1(T)}{p} + \cdots + \pi_0 \frac{\phi_0(T)}{p^m}\right)$$

where $\phi_j(T) = f_0(T)^{p^j} + p \cdot f_1(T)^{p^{j-1}} + \cdots + p^j \cdot f_j(T)$, with $f_1, \ldots, f_j \in T^{-1}K[T^{-1}]$, and where $\{\pi_0, \ldots, \pi_m\}$ are $p^m$-torsion points of a Lubin-Tate group.

- This exponential converges for $|T| > 1$.
- The isomorphism class of the equation is in bijection with the pair $(\bar{a}, \rho)$ where $\bar{a} \in \mathbb{Z}_p/\mathbb{Z}$ is the residue, and $\rho \in H^1$ is the character defined by the reduction of $(f_0, \ldots, f_m)$ in $W_m(k((t)))$:

$$(f_0, \ldots, f_m) \in W_m(T^{-1}K[T^{-1}]) \quad \rightarrow \quad (\bar{f}_0, \ldots, \bar{f}_m) \in W_m(k((t))) \quad \rightarrow \quad H^1 := \text{Hom}(G, \mathbb{Z}/p^m\mathbb{Z})$$
\textit{p-adic difference Equations}

\begin{itemize}
  \item Let \( q, h \in \mathbb{Q}_p \), be such that \(|q - 1| < 1\) and \(|h| < 1\). Let

  \[
  \sigma_{q,h}(f(T)) := f(qT + h), \quad \Delta_{q,h}(f) := \frac{f(qT + h) - f(T)}{(q - 1)T + h}.
  \]

  \[
  \left\{ q \rightarrow 1 \quad h \rightarrow 0 \right\} \quad \Rightarrow \quad \Delta_{q,h} \rightarrow d/dT.
  \]

  \item Difference equations (matrix form):

  \[
  \sigma_{q,h}(Y) = A(T) \cdot Y, \quad \iff \quad \Delta_{q,h}(Y) = G(T) \cdot Y(T),
  \]

  where \( G(T) = \frac{A(T) - 1}{(q - 1)T + h} \).

\end{itemize}

\textbf{Theorem 0.3 (Pulita)} A function \( Y(T) \) is solution of a differential equation if and only if it is solution of a difference equation.

In particular, for all differential equation, it exists an unique difference equation having the same solutions, and reciprocally.
Applications to $p$-adic Zeta and $L$ functions

Complex Zeta function: Values of the Zeta function $\zeta: (\mathbb{C} − 1) → \mathbb{C}$ at negative integers are known:

$$\zeta(1−n) = -\frac{B_n}{n}, \quad n ≥ 1.$$  

where $\{B_n\}_{n≥1}$ are the Bernulli numbers.

- Values of $\zeta$ at positive integers are unknown.

$p$-adic Zeta function: We know that $-\mathbb{N} \subseteq \mathbb{Z}_p$ is dense.

Theorem 0.4 (Kubota-Leopoldt, 1964) It exists a unique continuous function $\zeta_p: (\mathbb{Z}_p − \{1\}) → \mathbb{Q}_p$ such that:

$$\zeta_p(1−n) = -(1 − p^{n−1}) \cdot \frac{B_n}{n}, \quad n ≥ 1.$$  

- We define analogously $L(s, \chi)$ (complex) and $L_p(s, \chi)$ ($p$-adic) associate to a Dirichlet character $\chi$. 
Theorem 0.5 (Morita 1975) It exists a unique continuous function $\Gamma_p : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$, verifying the functional equation

$$\Gamma_p(x + 1) = \begin{cases} -x\Gamma_p(x) & \text{if } |x|_p = 1 \\ -\Gamma_p(x) & \text{if } |x|_p < 1 \end{cases}.$$ 

- $\Gamma_p(x + p) = A(x) \cdot \Gamma_p(x)$, with $A(x) = -(x + 1)(x + 2) \cdots (x + p - 1)$ is a difference equation, then:

Theorem 0.6 (Pulita) The function $\Gamma_p$ is solution of a $p$-adic differential equation with coefficients converging on $D^-(0, 1)$:

$$\Gamma_p(T)' = g(T) \cdot \Gamma_p(T) \quad \text{,} \quad g(T) \text{ converges on } D^-(0, 1).$$

- Interest of this: Let $\omega : \mathbb{Z} \rightarrow \mathbb{Z}_p$ be the Teichmüller char. Then

(Diamond 1979): $\quad g(T) = \lambda_0 + \sum_{m \geq 1} L_p(1 + 2m, \omega^{2m}) \cdot T^{1+2m}$

Corollary 0.1 (Pulita) We obtain congruences involving values $L_p(1 + 2m, \omega^{2m})$, in positive integers.