Infinitesimal deformation of p-adic differential equations on Berkovich curves

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Abstract

We show that if a differential equation $\mathscr{F}$ over a quasi-smooth Berkovich curve X has a certain compatibility condition with respect to an automorphism σ of X, then $\mathscr{F}$ acquires a semi-linear action of σ (i.e. lifting that on X). The compatibility condition forces the automorphism σ to be close to the identity of X, so the above construction applies to a certain class of automorphisms called *infinitesimal*. This generalizes [ADV04] and [Pul08]. We also obtain an application to Morita's p-adic Gamma function, and to related values of p-adic L-functions.

Introduction

Differential equations in the ultrametric world have been actively investigated since the proof of the rationality of the Zeta function by Dwork [Dwo60]. Most of the foundational ideas (as the over-convergence, the idea of Frobenius structure, the radius of convergence) come from the deep pioneering works of B.Dwork and P.Robba. Various languages have been used in the past half century to deal with the underling topological spaces, and with the several aspects of the original ideas of Dwork and Robba. We have (among others) the rigid geometry of J.Tate, the rigid cohomology of P. Berthelot, and the language of Meredith.

More recently the point of view of Berkovich geometry appeared as a powerful tool to describe some new features related to differential equations as shown in the works of F. Baldassarri [Bal10], K.Kedlaya [Ked13], and the author [Pul15], [PP15a], [PP15b], [PP13a], [PP13b].

This paper fits in this last sequence of works, and it is a generalization of [ADV04], [DV04], [Pul08] where one deals with q-difference equations. We generalize these last papers from two points of view:

i) We consider arbitrary quasi-smooth K-analytic Berkovich curves;

ii) We work with a larger class of automorphisms, called S-infinitesimal.

Let K be a complete valued ultrametric field, and let X be a quasi-smooth[1] K-analytic Berkovich curve over K. Such a curve always admits a so called weak triangulation S, which is a certain locally finite set in X such that $X - S$ is a disjoint union of virtual open annuli or discs. This is a nice way to cut the curve into local pieces. Each connected component of $X - S$ which is isomorphic to an annulus has a skeleton isomorphic to an open interval. The union of all those intervals in X, together with the points of S, form a locally finite graph Γ_S called the S-skeleton of X. The complement of Γ_S in X is a disjoint union of virtual open discs.

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[1]The terminology quasi-Smooth is that of [Duc], this corresponds to rig-smooth curves in the rigid analytic setting.

An automorphism $\sigma : X \to X$ of X is called *S-infinitesimal* if, after all base changes to a complete valued field extension L/K, it induces an automorphism of each such disc. We attach to σ a function $\mathcal{R}_S(-,\sigma) : X \to \mathbb{R}_{\geqslant 0}$ that measures how much σ is close to the identity. More precisely, if Ω/K is a large field extension such that $x \in X$ is Ω-rational, and if t is an Ω-rational point of X_Ω over x, then $\mathcal{R}_S(x,\sigma)$ controls (aftr a convenient normalization) the radius of the smallest closed disc centered at t which is stable under σ_Ω, and which does not intersect S.

A first result is the following:

Theorem 1 (Thm. 3.3.1)**.** *The function $x \mapsto \mathcal{R}_S(x,\sigma)$ is continuous. Moreover there exists a locally finite graph $\Gamma_S(\sigma) \subseteq X$, containing Γ_S, such that $\mathcal{R}_S(-,\sigma)$ is locally constant outside $\Gamma_S(\sigma)$. If $\sigma \neq \mathrm{Id}_X$, the end points of $\Gamma_S(\sigma)$ that do not belong to Γ_S are the rigid points of X that are fixed by σ.*

The statement and its proof are essentially derived from those of [Pul15], [PP15a], and [PP15b] where we have proved that the radius of convergence function $\mathcal{R}_{S,1}(-,\mathscr{F}) : X \to \mathbb{R}_{>0}$ of a differential equation $\mathscr{F}$ over X satisfies similar finiteness properties. We recall that $\mathcal{R}_{S,1}(x,\mathscr{F})$ controls (after normalization) the radius of the largest open disc centered at t where $\mathscr{F}$ is trivial.

Now, a differential equation $\mathscr{F}$ on X is called σ-compatible if for all $x \in X$ we have

$$\mathcal{R}_S(x,\sigma) \ < \ \mathcal{R}_{S,1}(x,\mathscr{F})\ . \tag{0.1}$$

This condition amounts to say that for all complete valued field extension L/K, σ stabilizes globally each virtual open disc $D \subset X_L$ on which $\mathscr{F}_L$ is trivial.

If Σ is a family of automorphisms of X, a Σ-module over X is a pair $(\mathscr{F},\Sigma)$ where $\mathscr{F}$ is a locally free $\mathcal{O}_X$-module of finite type, and Σ is a family of $\mathcal{O}_X$-linear isomorphisms $\Sigma := \{\sigma : \mathscr{F} \xrightarrow{\sim} \sigma^*(\mathscr{F})\ ,\ \sigma \in \Sigma\}$. We say that a differential equation $(\mathscr{F},\nabla)$ is Σ-compatible if it is σ-compatible for all $\sigma \in \Sigma$. A second result of this paper is the following:

Theorem 2 (Thm. 4.3.1)**.** *Let Σ be a family of S-infinitesimal automorphisms of X, and let $\mathscr{F}$ be a Σ-compatible differential equation. Then there exists on $\mathscr{F}$ a canonical structure of Σ-module.*

If $\alpha : \mathscr{F} \to \mathscr{F}'$ is an $\mathcal{O}_X$-linear map between Σ-compatible differential equations commuting with the connections, then α also commutes with the corresponding action of Σ on $\mathscr{F}$ and $\mathscr{F}'$.

The theorem provides the existence of a faithful functor

$$\mathrm{Def}_{S,\Sigma}\ :\ \{\Sigma\text{-compatible differential equations}/X\}\ \longrightarrow\ \{\Sigma-\mathrm{Modules}/X\}\ . \tag{0.2}$$

In section 4.4 we provide a condition on the action of Σ on X that guarantee the fully faithfulness.

The existence of the Σ-module structure on $\mathscr{F}$ is obtained as a certain pull-back of the stratification associated to it (cf. Section 2.2). In down to earth terms if $D \subseteq X$ is a disc stable by Σ on which the differential equation $(\mathscr{F},\nabla)$ admits a complete basis of Taylor solutions $Y \in GL_r(\mathcal{O}(D))$, the Σ-module associated to ∇ also admits the same basis of Taylor solutions on D.

Example 3. *Let X be the open unit disc in $\mathbb{A}^{1,\mathrm{an}}_K$, with $S = \emptyset$. Let $\mathscr{F}$ be the differential equation $y' = y$ on X. For all complete valued field extension Ω/K, and all rational point t of X_Ω the solution of $\mathscr{F}$ around t is $\exp(T-t)$. Its radius function $\mathcal{R}_S(-,\mathscr{F}) : X \to \mathbb{R}_{>0}$ is constant, and its value ω is either equal to $|p|^{\frac{1}{p-1}}$ if K is a p-adic field, or 1 if the absolute value of K is trivial on $\mathbb{Z}$. Now, if $\sigma = \sigma_q : X \to X$ is the multiplication by $q \in K$, then $\mathcal{R}_S(-,\sigma_q) : X \to \mathbb{R}_{\geqslant 0}$ coincides with the function $x \mapsto |(q-1)T|(x)$, i.e. the function associating to $x \in X$ the absolute value of the function $(q-1)T : X \to \mathbb{A}^{1,\mathrm{an}}_K$ at x. Condition (0.1) is then equivalent to $|q-1| \leqslant \omega$.*

The σ_q-module corresponding to $y' = y$ is $y(qT) = A(q,T)y(T)$, where $A(q,T) = \exp((q-1)T)$.

Indeed $y = \exp(T)$ *is a solution, and* $A(q,T) = \exp((q-1)T) \in \mathcal{O}(X)$ *as soon as* $|q-1| \leqslant \omega$.

We also obtain an *analytic* version of the above result. If we have a K-analytic group G acting on X by S-infinitesimal automorphisms (here the general formalism is that of [Mum08, Section 12]), then the deformation functor furnishes an *analytic semi-linear* G*-module* structure of $\mathscr{F}$ (i.e. lifting that of G on $\mathcal{O}_X$) which is compatible with the connection (cf. Section 4.5.1 for more details). This structure is commonly known as G*-equivariant* D*-module on* X (cf. [MFK94], [Kas89]).

If the quotient $[X/G]$ of X by G exists, Theorem 2 amounts to say that a G-compatible differential equation on X gives a fiber bundle on $[X/G]$. In our context the quotient does not necessarily exist in the category of K-analytic spaces, and it has to be replaced by the usual simplicial object attached to the action of G on X (cf. (4.21)) i.e. by a Stack (as in [Tho87]).

From this point of view the result is similar to that of J.Sauloy [Sau09] where in the complex theory of q-difference equations he shows that a certain class of q-difference equations is equivalent to a category of fibered boundles over the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}} = [X/G]$.

In the case of automorphisms of the form $f(T) \mapsto f(qT+h)$ of the affine line (this includes as special cases q-difference, and difference equations) we are able to construct an explicit description of the image of the functor $\mathrm{Def}_{S,\Sigma}$, and a quasi-inverse functor called *Confluence* (cf. Section 6.3).

As an application, in section 7 we apply the previous theory to a particular difference equation satisfied by Γ_p. We prove the following results:

Theorem 4 (cf. Thm. 7.2.1)**.** *The Morita's p-adic Gamma function* Γ_p *is a solution of a first order differential equation over the open unit disc.*

Moreover we relate the radius of convergence of that differential equation to the p-adic valuation of certain values at *positive integers* of the L-functions appearing in the Taylor expansion of the function $\log(\Gamma_p)$. We then prove a family of congruences between these values (cf. Corollary 7.4.1).

In section 5 we focus our attention to a local situation: that of differential equations over the Robba ring. In this situation an important classification result is the so called p-adic local monodromy theorem [And02], [Ked04], [Meb02], saying that each differential equation with an (unspecified) Frobenius structure is quasi-unipotent.

From Theorem 2 we deduce the following analogue of the p-adic local monodromy theorem:

Theorem 5 (Thm. 5.5.2)**.** *Each* Σ*-module over the Robba ring, that is the deformation of a differential equation with a Frobenius structure is quasi unipotent.*

In particular this allows us to obtain a characterization of the essential image of the deformation functor for equations over the Robba ring (cf. Cor. 5.5.5).

We mention that the proof of Theorem 5 requires a Σ-analogue of the Katz's existence of a canonical extension functor [Kat87], [Mat02] (cf. Section 5.4).

In the case of q-difference equations, Theorem 5 is the central result of [ADV04], which is the foundational paper of the theory. The difference with [ADV04] is that they deduce the existence of the deformation functor from the quasi-unipotence of q-difference equations by showing that the Tannakian groups of the two categories are isomorphic. Our approach goes in the opposite direction, we deduce the quasi unipotence from the deformation.

Finally we wish to quote two recently appeared papers of B.Le Stum and A.Quiros [LSQ15a, LSQ15b], where the authors obtain with different methods similar results, allowing for instance positive characteristic. The point of view of those papers is more related to that of Rigid cohomology of varieties of positive characteristic.

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1. Notations

All rings are commutative with unit element. $\mathbb{R}$ is the field of real numbers, and $\mathbb{R}_{\geqslant 0} := \{r \in \mathbb{R} \mid r \geqslant 0\}$. For all field L we denote its algebraic closure by L^{alg}, by $L[T]$ the ring of polynomial with coefficients in L, and by $L(T)$ the fraction field of $L[T]$. If L is valued, $\widehat{L}$ will be its completion.

In all the paper $(K, |.|)$ will be a complete field of characteristic 0 with respect to an ultrametric absolute value $|.| : K \to \mathbb{R}_{\geqslant 0}$ i.e. verifying $|1| = 1$, $|a \cdot b| = |a||b|$, $|a + b| \leqslant \max(|a|, |b|)$ for all $a, b \in K$, and $|a| = 0$ if and only if $a = 0$. We denote by $|K| := \{r \in \mathbb{R}_{\geqslant 0}$ such that $r = |t|,\ \exists t \in K\}$. The ring of integers of K will be $K^\circ := \{x \in K$, such that $|x| \leqslant 1\}$, its maximal ideal $K^{\circ\circ} := \{x \in K$, such that $|x| < 1\}$, and its residue field $\widetilde{K} := K^\circ/K^{\circ\circ}$. We refer to [Ber90] for the definition of Berkovich spaces. But we chose to use [Duc] as a uniform reference for Berkovich curves.

Let $I \subset \mathbb{R}_{\geqslant 0}$ be an interval, and let $t \in K$. The annulus (resp. disc) centered at t relative to I will be denoted by

$$C(t, I) := \{x \in \mathbb{A}^{1,\mathrm{an}}_K \mid |T - t|(x) \in I\} \qquad (\text{resp. } D^-(t,\rho) := C(t, [0, \rho[),\ D^+(t, \rho) := C(t, [0, \rho])) . \tag{1.1}$$

We will use the word *annulus* if and only if $0 < \inf I$ and $\sup I < +\infty$. Analogously a disc will always have a finite radius by definition. When $\inf I = 0$ and $0 \notin I$, we say that $C(t, I)$ is either a *punctured disc* if $\sup I < \infty$, and multiplicative group $\mathbb{G}^{\mathrm{an}}_{m,K}$ if $\sup I = +\infty$.

The ring of *analytic functions* $\mathcal{O}(C(t, I))$ over $C(t, I)$ is formed by power series

$$\mathcal{O}(C(t, I)) := \Big\{\sum_{i\in\mathbb{Z}} a_i(T - t)^i,\ a_i \in K,\ \text{such that } \forall \rho \in I \text{ one has } \lim_{i\to\pm\infty} |a_i|\rho^i = 0\Big\} . \tag{1.2}$$

It is understood that if $0 \in I$, then $a_i = 0$ for all $i \leqslant -1$. The ring $\mathcal{B}(C(t, I))$ of *Bounded analytic functions* on $C(t, I)$ is formed by analytic functions in $\mathcal{O}(C(t, I))$ satisfying $\sup_{i\in\mathbb{Z}} |a_i|\rho^i \leqslant c < +\infty$ for all $\rho \in I$, where c is a convenient constant depending on the power series.

A *virtual* open (resp. closed) disc (resp. annulus) is a connected subset of $\mathbb{A}^{1,\mathrm{an}}_K$ which becomes a finite disjoint union of open (resp. closed) discs (resp. annuli whose orientations are preserved by $\mathrm{Gal}(\widehat{K^{\mathrm{alg}}}/K)$) after base change to $\widehat{K^{\mathrm{alg}}}$. The *skeleton* of a virtual annulus C is the set Γ_C of points without neighborhoods in C isomorphic to a virtual disc. These points form an interval.

For $c \in K$, and $\rho \geqslant 0$ we set $x_{c,\rho}(f) := \sup_{n\geqslant 0} |\frac{f^{(n)}(c)}{n!}|_K \cdot \rho^n$, for all $f \in K[T]$. For all $x \in \mathbb{A}^{1,\mathrm{an}}_K$ there exists a complete valued field extension Ω/K such that $x = \pi_{\Omega/K}(x_{t,\rho})$, where $\pi_{\Omega/K} : \mathbb{A}^{1,\mathrm{an}}_\Omega \to \mathbb{A}^{1,\mathrm{an}}_K$ is the canonical projection, and $t \in \Omega$. By an abuse, we write $x = x_{t,\rho}$ if no confusion is possible. The choice of $t \in \Omega$ and $\rho \geqslant 0$ is not unique (cf. [Pul15] for more details).

1.1. Quasi-smooth Berkovich curves

Let X be a quasi-smooth Berkovich curve in the sense of [Duc]. This means that Ω^1_X is locally free of rank one, and it corresponds to a rig-smooth K-analytic curve in the rigid analytic terminology. A *weak triangulation* of X is a locally finite subset $S \subset X$ such that each connected component of $X - S$ is a virtual open annulus, or a virtual open disc. We denote by Γ_S the union of S with all the skeletons of the virtual open annuli that are connected components of $X - S$. We call Γ_S the *skeleton* of S.

The *analytic skeleton* Γ_X of X is the set of points x without neighborhoods in X isomorphic to a virtual open disc. It is not always the skeleton of a weak triangulation (e.g. if $X = \mathbb{P}^{1,\mathrm{an}}_K$, then $\Gamma_X = \emptyset$). The analytic skeleton Γ_X is contained in the skeletons of all weak triangulations.

Assume that X is connected, and that S is a weak triangulation of X. Then

i) If S is the empty set, then X is either a virtual open disc or annulus;

ii) If Γ_S is the empty set, then X is a virtual open disc;

iii) The curve $X - \Gamma_S$ is a disjoint union of virtual open discs. If for all connected component Y of X, $\Gamma_S \cap Y \neq \emptyset$, those discs are all relatively compact in X. In this case the map

$$\tau_S \;:\; X \;\longrightarrow\; \Gamma_S \tag{1.3}$$

which is the identity on Γ_S, and which sends the connected components of $X - \Gamma_S$ into their boundary in Γ_S is a continuous open retraction.

Theorem 1.1.1 ([Duc]). *Each quasi-smooth K-analytic curve admits a weak triangulation.*

1.1.1. Log-linearity, directions, slopes. From the existence of weak triangulations, one deduces that every point of X has a neighborhood that is uniquely arcwise connected. On such a subset, it makes sense to speak of the segment $[x, y]$ joining two given points x and y, hence of convex subsets (see also [BR10, Section 2.5]).

A subset Γ of X is said to be a *finite (resp. locally finite) subgraph* of X if there exists a finite (resp. locally finite) family $\mathscr{V}$ of affinoid domains of X that covers Γ and such that, for every element V of $\mathscr{V}$, V is uniquely arcwise connected, and $\Gamma \cap V$ is the convex hull of a finite number of points. By Theorem 1.1.1, Γ_S is a locally finite graph.

We now want to define a notion of log-linearity. To do so, we first need to explain how to measure distances. Let C be a closed virtual annulus over K. Its preimage over $\widehat{K^{\mathrm{alg}}}$ is a finite union of closed annuli. If $C(c, [R_1, R_2])$ is one of them, we set $\mathrm{Mod}(C) = R_2/R_1$. This is well defined up to the choice of an orientation of C (e.g. the inversion of the variable $T \mapsto T^{-1}$ changes the sign of the modulus), and it is invariant by isomorphisms of annuli preserving the orientation (cf. [Duc]).

Let $I = [x, y]$ be a closed segment in $\mathbb{A}^{1,\mathrm{an}}_K$ containing only points of type 2 or 3. Then I is the skeleton of a virtual closed annulus $C \subseteq \mathbb{A}^{1,\mathrm{an}}_K$, and we set $\ell(I) = \log(\mathrm{Mod}(C))$. Pushing these ideas further, one can show that it is possible to define a *canonical length* ℓ for any closed segment inside a curve that contains only points of type 2 or 3 (see [Duc, Proposition 4.5.7]). The definition may actually be extended to any curve, see [Duc, Corollaire 4.5.8]).

Definition 1.1.2. *Let $X_{[2,3]}$ be the set of points of X that are of type 2 or 3. Let J be an open segment inside $X_{[2,3]}$ and identify it with a real interval. A map $f\colon J \to \mathbb{R}_{\geqslant 0}$ is said to be* log-linear *if there exists $\gamma \in \mathbb{R}$ such that, for every $a < b \in J$, we have*

$$\log(f(b)) - \log(f(a)) \;=\; \gamma \cdot \ell([a, b])\;. \tag{1.4}$$

If J is oriented as from a to b (resp. from b to a), we set $\partial_J f := \gamma$ (resp. $\partial_J f := -\gamma$), and we call this number the slope of f along the oriented segment J.

We define an equivalence class on the segments out of a point x. We say that the open segment $]x, y[$ is equivalent to $]x, z[$ if there exists a third non empty open segment $]x, t[$ contained in $]x, y[\cap]x, z[$. We say that a class of germ of segments out of x is a *direction* out of x, or equivalently a *germ of segment* out of x, or again a *branch* out of x. A *section of a branch* b out of x is any open connected subset U of X having x at its boundary, such that $U \cup \{x\}$ is topologically a tree (no loops), and $b \subset U$. By Theorem 1.1.1, every branch out of x admits a section isomorphic to an open annulus. There are well-defined notions of direct and inverse images of branches that correspond to the intuitive ones. Let $\varphi\colon X\to Y$ be a morphism of curves and let $x\in X$ be a point such that $\varphi^{-1}(\varphi(x))$ is finite. Then the image of a branch out of x is a branch out of $\varphi(x)$ and the inverse image of a branch out of $\varphi(x)$ is a union of branches out of some point $y\in\varphi^{-1}(\varphi(x))$. In particular for all germs of segments b out of $x\in X$ we denote by $\deg(b)$ the number of germs of segments in $X_{\widehat{K^{\mathrm{alg}}}}$ over b.

If $f:X\to\mathbb{R}$ is a function which is log-linear along a direction b out of x, we denote its slope by

$$\partial_b f(x)\;, \tag{1.5}$$

where by convention b is always oriented as out of the point x.

Definition 1.1.3. *Assume that f is* log*-linear along all directions out of x, and that $\partial_b f(x)=0$ for all, but a finite number of directions. The Laplacian of f is by definition the sum*

$$dd^cf(x)\;:=\;\sum_b \deg(b)\cdot\partial_b f(x) \tag{1.6}$$

of all the slopes for all germ of segment b out of x. We say that f is harmonic (resp. super-harmonic, sub-harmonic) at x if $dd^cf(x)=0$ (resp. $dd^cf(x)\leqslant 0$, $dd^cf(x)\geqslant 0$).

1.1.2. Open boundary. Assume that K is algebraically closed. Let O be a connected component of $X-S$. Then O is either an open disc or annulus.

Assume that $O=D^-(0,R)$ is an open disc. If O is not relatively compact in X, then it is a connected component of it.

Assume that $O=C^-(0;]R_1,R[)$ is an annulus. If O is not relatively compact in X, then it is either a connected component of it, or its closure $\overline{O}$ in X is of the form $\overline{O}=O\cup\{x\}$, where $x\in\overline{O}-O$. So $x=\lim_{\rho\to R_1^+}x_{0,\rho}$ or $x=\lim_{\rho\to R^-}x_{0,\rho}$. Assume that x lies on the R_1's side.

In both cases (disc or annulus), for all $0<\varepsilon<R-R_1$, S is again a weak triangulation of $X-C'$, where $C':=C^-(0;[R-\varepsilon,R[)$. Moreover, if Y is the connected component of X containing O, then $Y-C'$ is connected too. We call *germ of segment at the open boundary of* X any unspecified germ of segment $]x_{0,R-\varepsilon},x_{0,R}[$ which is the skeleton of a non relatively compact annulus $C^-(0;]R-\varepsilon,R[)$ placed as above inside the curve X.

If K is general we define the open boundary of X as the image of the open boundary of $X_{\widehat{K^{\mathrm{alg}}}}$.

As an example if X is a virtual open disc, then its open boundary counts one element, if it is a virtual open annulus, then its open boundary counts two elements.

1.1.3. Γ_S-coverings. Without loss of generality, we can assume that X is connected.

If Γ_S is empty, then X is a virtual open disc with empty weak triangulation. In this case the unique Γ_S-covering of X is by definition the trivial one given by the whole disc $\{X\}$.

We now assume $\Gamma_S\neq\emptyset$. In this case, since X is connected, we have a retraction (1.3) of X onto Γ_S. For all $x\in\Gamma_S$ we consider a star-shaped open neighborhood Λ_x of x in Γ_S.[2] Its inverse image

$$Y_x\;:=\;\tau_S^{-1}(\Lambda_x) \tag{1.7}$$

by the retraction (1.3) is open analytic domain in X such that $\Gamma_S\cap Y_x=\Lambda_x$.

If, for all $x\in\Gamma_S$, we consider an analytic neighborhood Y_x as above, then $X=\cup_{x\in\Gamma_S}Y_x$.

[2] We mean that $\Lambda_x\subseteq\Gamma_S$ is a simply connected neighborhood of x in Γ_S (no loops), and moreover that $\Lambda_x-\{x\}$ is a finite disjoint union of segments $[x,y[$ out of x, all incident upon x, such that $]x,y[$ is the skeleton of a virtual open annulus in X (this is possible since Γ_S is locally finite).

Definition 1.1.4 (Γ_S-covering)**.** *Assume X connected, and $\Gamma_S \neq \emptyset$. A Γ_S-covering of X is a covering formed by the family of all connected components C of $X - S$, together with an open neighborhood of each point $x \in S$ of the form $Y_x = \tau_S^{-1}(\Lambda_x)$. We assume moreover that the intersection of three distinct elements of the covering is always empty. In the sequel C (resp. Y_x) will be endowed with the empty weak triangulation $S_C = \emptyset$ (resp. $S_{Y_x} = \{x\}$), we then have $\Gamma_{S_C} = \Gamma_C$ (resp. $\Gamma_{S_{Y_x}} = \Lambda_x$).*

If $x \in S$ then Y_x is not necessarily a quasi-Stein[3] in the sense of Kiehl [Kie67]. However, each point $x \in S$ admits a quasi-Stein open neighborhood which is obtained from some $Y_x = \tau_S^{-1}(\Lambda_x)$ by removing a finite number of virtual discs $D_1, \ldots, D_n$ with boundary $\{x\}$, and replacing them by some virtual open annuli at the open boundaries of $D_1, \ldots, D_n$ (cf. Section 1.1.2).

From a Γ_S-covering we can always obtain a quasi-Stein covering of X by shrinking the neighborhoods Y_x in this way, and adding the remaining discs $D_1, \ldots, D_n$ to the covering.

Definition 1.1.5 (quasi Γ_S-covering)**.** *We call* quasi Γ_S-covering *a covering of X formed by quasi-Stein opens, which is obtained from a Γ_S-covering as above.*

1.1.4. Extension of scalars. Following [PP15b], we now quickly recall how to extend canonically weak triangulations by base change of K. More details can be found in [PP15b].

Let S be a weak triangulation of X. The inverse image $S_{\widehat{K^{\mathrm{alg}}}}$ of S in $X_{\widehat{K^{\mathrm{alg}}}}$ is easily seen to be a weak triangulation of $X_{\widehat{K^{\mathrm{alg}}}}$.

Assume now that K is algebraically closed, and let L/K be a complete valued field extension. Denote the canonical projection by $\pi_{L/K} : X_L \to X$. By [PP15b], the fiber $\pi_{L/K}^{-1}(x)$ has a canonical point $\sigma_L(x)$ of type 2 such that $\pi_{L/K}^{-1}(x) - \{\sigma_L(x)\}$ is a disjoint union of open discs in X_L all having $\sigma_L(x)$ at their boundary. Moreover the set $S_L := \{\sigma_L(x),\ x \in S\}$ is a weak triangulation of X_L, and the projection $\pi_{L/K}$ induces an isomorphism between Γ_{S_L} and Γ_S.

If L is spherically complete and algebraically closed, the group $\mathrm{Gal}^{\mathrm{cont}}(L/K)$, of continuous automorphisms of L over K, fixes each point of Γ_{S_L}, and permutes transitively the discs of $\pi_{L/K}^{-1}(x) - \{\sigma_L(x)\}$, and also the set of L-rational points of $\pi_{L/K}^{-1}(x)$. In particular these discs are all isomorphic.

Lemma 1.1.6 ([PP13a, Prop. 2.1.7])**.** *There exists a complete valued field Ω/K containing isometrically all fields $\mathscr{H}(x)$, $x \in X$. In other words all the points of X are Ω-rational.* □

Notation 1.1.7. *We now fix, once for all, a field Ω containing isometrically $\mathscr{H}(x)$, for all $x \in X$. Moreover we assume that Ω is algebraically closed, spherically complete, and that $|\Omega| = \mathbb{R}_{\geqslant 0}$.*

Notation 1.1.7 is not strictly necessary, but it simplifies the exposition. We notice that a quasi-smooth curve over Ω has no point of type 3 nor 4.

Remark 1.1.8. *We recall that by a result of Q.Liu [Liu87] every curve with finite genus (see [PP13b] for the definition of genus) over Ω is either projective, or quasi-Stein (in the sense of [Kie67]). In particular, all analytic domains X of $\mathbb{P}^{1,\mathrm{an}}_{\Omega}$ distincts from it are quasi-Stein of genus zero.*

Definition 1.1.9 (Maximal and generic discs)**.** *For all $x \in X$ we fix an $\mathscr{H}(x)$-rational point $t_x \in X_{\mathscr{H}(x)}$, lifting $x \in X$. We call the connected component of $X_\Omega - \Gamma_{S_\Omega}$ containing t_x the* maximal disc

[3] An example is given by an elliptic curve X with good reduction. In this case a weak triangulation of X is given by an individual point $S = \{x\}$ which is the unique point of X without neighborhoods isomorphic to an analytic domain of the affine line. In this case $\Gamma_S = S = \{x\} = \Lambda_x$, and the unique open of the Γ_S-covering is $Y_x = X$. The same happens for $\mathbb{P}^{1,\mathrm{an}}_K$ with a triangulation $S = \{x\}$, with x of type 2 or 3.

centered at x. We denote it by $D(x,S)$. We call the disc $D(x)$ which is the connected component of $\pi^{-1}_{\Omega/K}(x) - \{\sigma_\Omega(x)\}$ containing t_x the generic disc *of x. By definition one has $D(x) \subseteq D(x,S)$.*

Concretely, if $x \notin \Gamma_S$ and if D is the connected component of $X - \Gamma_S$ containing x (which is necessarily a virtual open disk), then $D(x,S) := D_\Omega$, and if $x \in \Gamma_S$ then $D(x,S) := D(x)$.

1.2. Controlling graphs

Let $\mathcal{T}$ be a set, and let $f : X \to \mathcal{T}$ be any function. Following [Pul15] and [PP13a] we now introduce a graph inside X that controls the locus outside Γ_S where f is locally constant.

Definition 1.2.1. *Let $\Gamma_S(f)$ be the set of points $x \in X$ such that there is no open discs D satisfying: (i) $x \in D$; (ii) $D \cap \Gamma_S = \emptyset$; (iii) f is constant on D. We call $\Gamma_S(f)$ the controlling graph of f.*

By definition we have $\Gamma_S \subseteq \Gamma_S(f)$. It is also easily seen that if $x \in \Gamma_S(f) - \Gamma_S$, then the segment connecting x to Γ_S is contained in $\Gamma_S(f)$. Indeed if a disc D verifying (i), (ii), (iii) contains one of the points of the segment, then it also contains x. This shows that if X is connected, $\Gamma_S(f)$ is a connected sub-graph of X, containing Γ_S, and that $X - \Gamma_S(f)$ is a disjoint union of virtual open discs (on which f is constant). In fact $\Gamma_S(f)$ is also characterized by the fact that it is the smallest connected graph containing Γ_S such that f is locally constant outside $\Gamma_S(f)$.

2. Linear differential equations

Classical complex differential equations over Riemann surfaces have the nice property that their restriction to any disk is trivial. In other words if $G(z)$ is a $n \times n$ matrix whose entries are analytic functions over a disk $D = \{z \in \mathbb{C}$, such that $|z| < r\}$, then the differential system $\frac{d}{dz}(Y(z)) = G(z)Y(z)$ admits a full basis of analytic solutions converging on D. Equivalently the radius of convergence of the Taylor expansion of its solutions at 0 is r (i.e. as large as possible).

This property is not verified over an ultrametric field K as showed by the example of the equation $y' = y$ (cf. introduction) which is defined on the whole affine line and whose solution $\exp(T)$ does not converges on the whole line.

In the ultrametric context one of the major invariants associated to $\mathscr{F}$ is the radius of convergence of its solutions, which is a function on X constructed from the default of convergence of Taylor solutions of $\mathscr{F}$. It will play an important role on this article. We now recall its definition from [Bal10], [PP15a], [PP15b]. We consider those papers as general references.

2.1. Radius of convergence function

Let X be a quasi-smooth K-analytic Berkovich curve. Recall that this is a locally ringed topological space with a structural sheaf $\mathcal{O}_X$ of analytic functions on X (cf. [Ber90, Section 3.1]) the topology of such a curve is described in detail in the book [Duc].[4]

By *differential equation* we mean a locally free $\mathcal{O}_X$-module of finite rank $\mathscr{F}$, together with a connection $\nabla : \mathscr{F} \to \Omega^1_X \widehat{\otimes} \mathscr{F}$ i.e. a map satisfying the Leibniz rule $\nabla(f \cdot m) = d(f) \otimes m + f\nabla(m)$ for all $f \in \mathcal{O}_X(U)$, $m \in \mathscr{F}(U)$, and for all open $U \subset X$ of the Berkovich topology.

Morphisms of differential equations are $\mathcal{O}_X$-linear maps commuting with the connections.

We say that $\mathscr{F}$ is trivial if it is isomorphic to a direct sum of copies of the equation $d : \mathcal{O}_X \to \Omega^1$.

[4] In this article we do not consider the G-topology (cf. [Ber90, Section 3.3]). Opens of X are subsets of X that are open with respect to the Berkovich topology, coverings are collections of opens of X whose union is X, and sheafs on X are genuine sheafs over this topological space X.

Consider now a point $x \in X$. By Remark 1.1.8 the restriction of $\mathscr{F}_\Omega = \mathscr{F} \otimes_K \Omega$ to the disk $D(x,S)$ is a quasi-Stein space, so the locally free $\mathcal{O}_{D(x,S)}$-module $(\mathscr{F}_\Omega)_{|D(x,S)}$ corresponds to a finitely generated projective module over the ring $\mathcal{O}(D(x,S)) = \{f := \sum_{n\geqslant 0} a_n(T-t_x)^n\ ,\ f \text{ converges on } D(x,S)\}$ (cf. (1.2)). Moreover by a result of Lazard (cf. [Laz62]), the restriction of $(\mathscr{F}_\Omega)_{|D(x,S)}$ to $D(x,S) := D(t_x, S_\Omega)$ is free, since Ω is spherically complete (see also [Chr11, Théorème 4.40]).

Now, we denote by

$$D(x,\mathscr{F}) \tag{2.1}$$

the largest disc centered at t_x, contained in $D(x,S)$, on which $\mathscr{F}$ is trivial. Such a disc is not empty by the Cauchy existence theorem [DGS94, Appendix III]. Let T be a coordinate on $D(x,S)$. We denote by $\mathcal{R}^{\mathscr{F}}(x) > 0$ the radius of $D(x,\mathscr{F})$ in the coordinate T. If $\rho_{S,T}(x)$ is the radius of $D(x,S)$, the ratio

$$\mathcal{R}_{S,1}(x,\mathscr{F}) \ :=\ \mathcal{R}^{\mathscr{F}}(x)/\rho_{S,T}(x) \tag{2.2}$$

is independent of T by the following

Lemma 2.1.1. *Let $R_1, R_2 > 0$. Up to a translation, any K-isomorphism $\alpha : D^-(0,R_1) \xrightarrow{\sim} D^-(0,R_2)$ is given by a power series of the form $f(T) = \sum_{i\geqslant 1} a_i T^i \in K[[T]]$, with*

$$|a_1| \ =\ R_2/R_1\ , \qquad |a_i| \leqslant R_2/R_1\ , \quad \forall\, i \geqslant 2\ . \tag{2.3}$$

In particular, it multiplies distances by the constant factor R_2/R_1: for any complete valued extension L of K, and for all $t_1, t_2 \in D^-(0,R_1)(L)$ we have $|\alpha(t_1) - \alpha(t_2)| = \frac{R_2}{R_1} \cdot |t_1 - t_2|$.

As a consequence, such an isomorphism may only exist when $R_2/R_1 \in |K^*|$. □

Definition 2.1.2 (Radius of convergence)**.** *We call $\mathcal{R}_{S,1}(x,\mathscr{F})$ the radius of convergence of $(\mathscr{F},\nabla)$ at $x \in X$. Following [Pul15] and [PP13a] its controlling graph (cf. Section 1.2) will be denoted by*

$$\Gamma_{S,1}(\mathscr{F}) \ :=\ \Gamma_S(\mathcal{R}_{S,1}(-,\mathscr{F}))\ . \tag{2.4}$$

Theorem 2.1.3 ([Pul15],[PP15b])**.** *The function $x \mapsto \mathcal{R}_{S,1}(x,\mathscr{F})$ enjoys the following properties:*

i) *The controlling graph $\Gamma_{S,1}(\mathscr{F})$ of $\mathcal{R}_S(-,\mathscr{F})$ is locally finite;*
ii) *$\mathcal{R}_{S,1}(-,\mathscr{F})$ is a continuous function on X, which is independent of the choice of t_x, and of Ω/K. It is moreover piecewise* log*-linear along each segment in X, and its slopes belong to $\frac{1}{r!}\mathbb{Z}$, where r is the local rank of $\mathscr{F}$;*
iii) *Let D be a virtual open disc which is a connected component of $X - \Gamma_S$. Let C be any open annulus in D, and let $I := \Gamma_C$ be its skeleton. If I is oriented as out of D, then the function $y \mapsto \mathcal{R}_{S,1}(y,\mathscr{F})$ is* log*-decreasing and* log*-concave along I;*
iv) *Let C be a virtual open annulus which is a connected component of $X - S$. Let I be its skeleton. Then $y \mapsto \mathcal{R}_{S,1}(y,\mathscr{F})$ is* log*-concave along I.* □

Remark 2.1.4. *If X is an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_K$, and let T be a global coordinate on X. Let $\mathcal{R}^{\mathscr{F}}(x)$ and $\rho_{S,T}(x)$ be the radii of $D(x,\mathscr{F})$ and $D(x,S)$ respectively in that coordinate. Then $\mathcal{R}_{S,1}(x,\mathscr{F}) = \mathcal{R}^{\mathscr{F}}(x)/\rho_{S,T}(x)$. The function $x \mapsto \rho_{S,T}(x)$ can be easily described: it is continuous, constant on each disc of $X - \Gamma_S$, and if $I \subseteq \Gamma_S$ is a segment, its slope along I is 1, if I is oriented toward the point $\infty = \mathbb{P}^{1,\mathrm{an}}_K - \mathbb{A}^{1,\mathrm{an}}_K$. In this case the function $\mathcal{R}^{\mathscr{F}}$ also enjoys the properties of Theorem 2.1.3.*

A morphism between differential equations is an $\mathcal{O}_X$-linear map commuting with the connections. We recall that its Kernel and Cokernel are locally free $\mathcal{O}_X$-modules on X (cf. [PP13a, 1.0.2]). The category of differential equations is hence abelian. We denote by $\mathrm{Hom}^\nabla(\mathscr{F},\mathscr{F}')$ the group of morphisms.

Lemma 2.1.5. *Let $\mathscr{F}, \mathscr{F}'$ be differential equations over X. Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be an $\mathcal{O}_X$-linear morphism. The following conditions are equivalent*

i) *α commutes with the connection;*

ii) *For all connected component X' of X, there exists a point $x \in X'$ of type 2, 3, or 4, such that $\alpha(x) : \mathscr{F}(x) \to \mathscr{F}'(x)$ commutes with the connections over $\mathscr{H}(x)$.*

Proof. We can assume X connected. i) $\Rightarrow$ ii) is evident. Assume that ii) holds. Consider a quasi Γ_S-covering $\{U_i\}_i$ of X formed by quasi-Stein domains on which $\mathscr{F}$, $\mathscr{F}'$, and $\Omega^1_{X/K}$ are all free. So α commutes with the connections if and only if so does each $\alpha_{|U_i}$. Assume that the claim is proved for quasi-Stein curves, then it holds for the opens U_i containing x. Now if U_j is another open of the covering such that $U_i \cap U_j$ is not empty, the intersection always contains a point of type 2, so $\alpha_{|U_j}$ commutes with the connection. Since X is connected this proves that ii) $\Rightarrow$ i).

Hence we can assume X quasi-Stein, and that $\mathscr{F}$, $\mathscr{F}'$, and Ω^1_X are all free. Let $d : \mathcal{O}(X) \to \mathcal{O}(X)$ be a derivation generating Ω^1_X. In some bases, α is given by a matrix H, and we have to prove that it is solution of the differential equation $d(H) = GH$, associated to the differential module $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F}')$. Here H is seen as a vector with entries in $\mathcal{O}(X)$ and G as a square matrix with entries in $\mathcal{O}(X)$. We know that its specialization $H(x)$ over $\mathscr{H}(x)$ is solution of $d(H(x)) = G(x)H(x)$. Since $\mathcal{O}_{X,x} \subset \mathscr{H}(x)$ is injective, the equality $d(H) = GH$ holds over some affinoid neighborhood of x. Hence it also holds over X, by analytic continuation (cf. [Ber90, 3.3.21]). □

2.2. Stratifications.

It follows from [Gro68], [Ber74], [Ill72], [Ber96], (and others), that the category of differential equations over X is equivalent to that of so called *stratifications*. We here quickly recall the definitions.

2.2.1. Let $\Delta : X \to X \times X$ be the diagonal closed immersion. Let $\mathscr{I} \subset \mathcal{O}_{X\times X}$ be the ideal corresponding to Δ. Set $\mathscr{P}^n_{X/K} := \mathcal{O}_{X\times X}/\mathscr{I}^{n+1}$, and $\mathscr{P}^\infty_{X/K} := \varprojlim_n \mathscr{P}^n_{X/K}$. We will say that the elements of $\mathscr{P}^n_{X/K}$ are convergent functions on the *n-th infinitesimal neighborhood* of the diagonal, while those in $\mathscr{P}^\infty_{X/K}$ corresponds to the *formal neighborhood* of the diagonal.

We now trivialize these notions locally on X. Let $x \in X$ and let U be an analytic neighborhood of x. Up to shrinking U we may assume that U is quasi-Stein, that Ω^1_U is free, and that there is an étale map $T : U \to \mathbb{A}^{1,\mathrm{an}}_K$. Let $p_1, p_2 : X \times X \to X$ be the canonical projections. Denote by $T_i := p_i^*(T) = T \circ p_i \in \mathcal{O}_{U\times U}$. The image of $T_1 - T_2 \in \mathscr{I}$ in $\Omega^1_X = \mathscr{I}/\mathscr{I}^2$ is the generator dT of Ω^1_X. Consider now $\mathcal{O}_{U\times U} = \mathcal{O}_U \widehat{\otimes}_K \mathcal{O}_U$ as an $\mathcal{O}_U$-ring via $p_2^* : \mathcal{O}_U \to \mathcal{O}_U \widehat{\otimes}_K \mathcal{O}_U$, $p_2^*(g) = 1 \otimes g$. Since X is quasi-smooth, a classical computation shows that we have a (non canonical) isomorphism $\mathscr{P}^n_{U/K} \xrightarrow{\sim} \mathcal{O}_U[T_1 - T_2]/(T_1 - T_2)^{n+1}$ associating to $f \otimes g$ the Taylor expansion $(1 \otimes g) \cdot \sum_{k=0}^n f^{(k)}(T_2) \frac{(T_1 - T_2)^k}{k!}$, where $f^{(k)}(T_2)$ means $\Delta^*\Big(\big((\frac{d}{dT})^k f\big) \otimes 1\Big) \in \mathcal{O}_U$. It follows that $\mathscr{P}^\infty_{U/K} \xrightarrow{\sim} \mathcal{O}_U[[T_1 - T_2]]$.

In this situation, we call *tubular neighborhood of the diagonal of U* a Weierstrass domain of $U \times U$ of the form $\mathcal{T}(U, T, R) := \{|T_1 - T_2| \leqslant R\}$.

The ring $\mathscr{P}^\infty_{U/K}$, is a natural place where searching solutions of differential equations. In fact all differential solutions are trivialized by $\mathscr{P}^\infty_{U/K}$. We now recall quickly this fact.

By the above local description of $\mathscr{P}^\infty_{U/K}$, the diagram

$$\begin{array}{ccc} \mathcal{O}_U & \xrightarrow{p_1^*} & \mathscr{P}^\infty_{U/K} \\ \uparrow & & \uparrow{\scriptstyle p_2^*} \\ K & \longrightarrow & \mathcal{O}_U \end{array} \qquad (2.5)$$

provides a natural identification $(\Omega^1_{U/K}\widehat{\otimes}_{\mathcal{O}_U,p_1^*}\mathscr{P}^\infty_{U/K}) \xrightarrow{\sim} \Omega^1_{\mathscr{P}^\infty_{U/K}/\mathcal{O}_U}$, where in the tensor product $\mathscr{P}^\infty_{U/K}$ is considered as an $\mathcal{O}_U$-ring with respect to p_1^*, while the module of differentials $\Omega^1_{\mathscr{P}^\infty_{U/K}/\mathcal{O}_U}$ represents the $\mathcal{O}_U$-linear derivations of $\mathscr{P}^\infty_{U/K}$ with respect to the $\mathcal{O}_U$-ring structure given by p_2^* (not p_1^*). By this identification the derivation $d/dT:\mathcal{O}_U\to\mathcal{O}_U$ corresponds to a $\mathcal{O}_U$-linear derivation of $\mathscr{P}^\infty_{U/K}\xrightarrow{\sim}\mathcal{O}_U[[T_1-T_2]]$ which acts as d/dT_1.

Now consider $\mathscr{P}^\infty_{U/K}$ as an $\mathcal{O}_U$-ring via p_1^*. The above identification of differentials permits to consider the *scalar extension* to $\mathscr{P}^\infty_{U/K}$ of any differential equation $\mathscr{F}$ over U. In fact all differential equation become trivial over $\mathscr{P}^\infty_{U/K}$. More precisely there is an $\mathscr{P}^\infty_{U/K}$-linear isomorphism

$$\chi \;:\; \mathscr{F}\widehat{\otimes}_{\mathcal{O}_U,p_1^*}\mathscr{P}^\infty_{U/K} \xrightarrow{\sim} \mathscr{F}\widehat{\otimes}_{\mathcal{O}_U,p_2^*}\mathscr{P}^\infty_{U/K} \tag{2.6}$$

which commutes with structure of trivial differential equation over $\mathscr{P}^\infty_{U/K}$ of the right hand side. Loosely speaking this means that $\mathscr{F}$ has a basis of solutions over $\mathscr{P}^\infty_{U/K}$. Namely, up to shrinking U, $\mathscr{F}$ is free and we can consider an isomorphism $\mathscr{F}\xrightarrow{\sim}\mathcal{O}_U^r$, i.e. a basis of $\mathscr{F}$. So the connection of $\mathscr{F}$ corresponds to a K-linear endomorphism $\nabla:\mathscr{F}\to\mathscr{F}$. We associate to it a matrix $G\in M_{r\times r}(\mathcal{O}_U)$ whose columns are the images of the chosen basis of $\mathscr{F}$. With these choices we have the following explicit expression of the matrix of χ

$$Y_\chi \;:=\; \sum_{n\geqslant 0} G_n(T_2)\frac{(T_1-T_2)^n}{n!} \;\in\; GL_{r\times r}(\mathscr{P}^\infty_{U/K})\;, \tag{2.7}$$

where $G_n(T_2)\in M_{r\times r}(\mathcal{O}_U)$ is inductively defined by the relations $G_0=\mathrm{Id}$, $G_1=G$, $G_{n+1}=\frac{d}{dT}(G_n)+G_n\cdot G$. One verifies easily that $\frac{d}{dT_1}(Y_\chi)=G(T_1)\cdot Y_\chi$, because $G(T_1) = \sum_{k\geqslant 0}G^{(k)}(T_2)(T_1-T_2)^k/k!$, and $G_{n+1}=\sum_{k=0}^n\binom{n}{k}\cdot G_1^{(k)}\cdot G_{n-k}$.

2.2.2. If U is an affinoid domain, an induction gives $\|G_n\|_U\leqslant\max(\|d/dT\|_U,\|G\|_U)^n$, where $\|\cdot\|_U$ is the sup-norm on U (the norm of a matrix is by definition the sup of the norms of its entries), and $\|d/dT\|_U=\sup\{\frac{\|d/dT(f)\|_U}{\|f\|_U}, f\neq 0\}$ is the norm operator of $d/dT:\mathcal{O}_U\to\mathcal{O}_U$. From this we deduce that Y_χ belongs to $\mathcal{T}(U,T,R)$ for all $R<\frac{\omega}{\max(\|d/dT\|_U,\|G\|_U)}$, where $\omega:=\lim_n|n!|^{1/n}$. If the valuation of K is trivial on $\mathbb{Z}$, then $\omega=1$, otherwise $\omega=|p|^{\frac{1}{p-1}}$ where p is the characteristic of $\widetilde{K}$.

2.2.3. We now come back to the global curve X. We say that an open neighborhood $\mathcal{T}$ of the diagonal of $X\times X$ is *admissible* if, for all $x\in X$, there exists an affinoid neighborhood U of x in X as above, and a neighborhood $\mathcal{T}(U,T,R)$ of the diagonal of $U\times U$ such that $\mathcal{T}(U,T,R)\subseteq\mathcal{T}$. A *stratification* over X is a locally free $\mathcal{O}_X$-module of finite rank $\mathscr{F}$ together with an $\mathcal{O}_{\mathcal{T}}$-linear isomorphism

$$\chi \;:\; (p_2^*\mathscr{F})_{|\mathcal{T}} \xrightarrow{\sim} (p_1^*\mathscr{F})_{|\mathcal{T}} \tag{2.8}$$

for some unspecified admissible neighborhood of the diagonal $\mathcal{T}$. The isomorphism χ is moreover subjected to the cocycle conditions:

i) If $p_{i,j}:X\times X\times X\longrightarrow X\times X$ is the projection on the (i,j)-factor, then over $p_{1,2}^{-1}(\mathcal{T})\cap p_{2,3}^{-1}(\mathcal{T})\cap p_{1,3}^{-1}(\mathcal{T})$ one has $p_{1,2}^*(\chi)\circ p_{2,3}^*(\chi)=p_{1,3}^*(\chi)$.

ii) $\Delta^*(\chi)=\mathrm{Id}_{\mathscr{F}}$ (here we canonically identify $\Delta^*p_i^*\mathscr{F}$ with $\mathscr{F}$).

A morphism $\alpha:(\mathscr{F}_1,\chi_1)\to(\mathscr{F}_2,\chi_2)$ between stratifications is an $\mathcal{O}_X$-linear morphism $\alpha:\mathscr{F}_1\to\mathscr{F}_2$ such that $p_1^*\alpha\circ\chi_1=\chi_2\circ p_2^*\alpha$. The following result is classical (e.g. see [LS07, Section 4.1.3]).

Theorem 2.2.1 ([Ber96]). *The category of differential equations over X is equivalent to the category of stratifications over X.*

The above equivalence roughly goes as follows. If $(\mathscr{F},\nabla)$ is a differential equation, the corresponding stratification consists in the same $\mathcal{O}_X$-module $\mathscr{F}$ together with the stratification whose local expression is given by (2.6).

If $\alpha:(\mathscr{F}_1,\nabla_1)\to(\mathscr{F}_2,\nabla_2)$ is a morphism of differential equations, then α commutes also with the corresponding stratifications, so that the equivalence is the identity on the morphisms.

We now want to recover the connection ∇ from the stratification χ. This can be done by showing that the matrix $G:=\frac{d}{dT_1}(Y_\chi)\cdot Y_\chi^{-1}$ actually lies in $\mathcal{O}_U$. Or we can consider consider the reduction of χ in $\mathscr{P}^2_{U/K}\cong\mathcal{O}_U\oplus\mathscr{I}_U/\mathscr{I}_U^2$, and consider its retraction onto $\mathscr{I}_U/\mathscr{I}_U^2$.

3. S-infinitesimal automorphisms

Let X be a quasi-smooth curve, and let S be a weak triangulation of X.

Definition 3.0.2. *Let $\sigma: X\xrightarrow{\sim}X$ be a K-isomorphism. We say that σ is an S-infinitesimal automorphism of X if $\sigma_\Omega: X_\Omega\xrightarrow{\sim}X_\Omega$ induces an automorphism of each maximal disc $D(x,S)\subseteq X_\Omega$, for all $x\in X$. We often say infinitesimal instead of S-infinitesimal if no confusion is possible.*

Here and below σ_Ω means $\sigma\widehat{\otimes}\mathrm{Id}_\Omega$. By definition, an S-infinitesimal automorphism fixes all the points of Γ_S. We denote by $\mathfrak{S}(X,S)$ the group of S-infinitesimal automorphisms of X.

An S-infinitesimal automorphism is often not S'-infinitesimal with respect to another weak triangulation S'. However if $\Gamma_S=\Gamma_{S'}$, then σ is S-infinitesimal if and only if it is S'-infinitesimal. This is because $D(x,S)=D(x,S')$ for all $x\in X$. A similar consideration gives the following

Lemma 3.0.3. *Let σ be an S-infinitesimal of X. Then:*

i) *If $Y\subseteq X$ is a connected analytic domain admitting a* non empty *weak triangulation S_Y such that $\Gamma_S\cap Y=\Gamma_{S_Y}$, then σ induces an S_Y-infinitesimal automorphism of Y.*
ii) *If Y is a connected component of $X-S$ or $X-\Gamma_S$ (necessarily a virtual open disc or annulus) together with the empty weak-triangulation $S_Y=\emptyset$, then σ induces an S_Y-infinitesimal automorphism of Y.* □

Remark 3.0.4. *Lemma 3.0.3 applies to all opens of a Γ_S-covering of X (cf. Def. 1.1.4).*

Remark 3.0.5. *An automorphism σ is S-infinitesimal if and only if $\sigma_{\widehat{K^{\mathrm{alg}}}}$ is $S_{\widehat{K^{\mathrm{alg}}}}$-infinitesimal.*

3.1. The function $\mathcal{R}_S(-,\sigma)$ and its controlling graph.

We now define a function $\mathcal{R}_S(-,\sigma): X\to\mathbb{R}_{\geqslant 0}$ that controls how an infinitesimal automorphism σ is close to the identity. For this we need the following straightforward consequence of Lemma 2.1.1:

Lemma 3.1.1. *Let σ be any automorphism of an open disc D. Let $t\in D$ be a rational point. Then*

i) *There exists a smallest closed disc[5] $D^+(t,\sigma)\subset D$ centered at t which is globally stable by σ. Moreover $D^+(t,\sigma)$ is the smallest closed disc containing t and $\sigma(t)$;*
ii) *Each (open or closed) disc D' satisfying $D^+(t,\sigma)\subseteq D'\subseteq D$ is globally stable by σ.*
iii) *For all disc D' as in ii), the annulus $C:=D-D'$ is globally stable under σ.* □

Remark 3.1.2. *Let C be an annulus as in Lemma 3.1.1. Then σ does not necessarily induce a S-infinitesimal automorphism on C with respect to the empty weak triangulation. As an example,*

[5] Notice that $D^+(t,\sigma)$ is allowed to be equal to the individual point $\{t\}$.

if σ is the multiplication by $q \in K$ with $|q-1| = 1$ acting on the open unit disc D. Then σ is infinitesimal with respect to the empty weak triangulation of D because $D(x,S) = D$ for all $x \in D$. In this case $D^+(0,\sigma)$ is reduced to $\{0\}$ and no discs in $C \subset D - \{0\}$ are stable by σ.

Let X be a quasi smooth curve. For all $x \in X$, we denote by

$$D^+_S(x,\sigma) \tag{3.1}$$

the smallest closed disc in $D(x,S)$ containing t_x and $\sigma_\Omega(t_x)$. Let T be a coordinate on $D(x,S) \subseteq X_\Omega$. Denote by $\mathcal{R}^\sigma(x) \geqslant 0$ the radius of $D^+_S(x,\sigma)$ in the coordinate T. If $\rho_{S,T}(x)$ is the radius of $D(x,S)$ with respect to the same coordinate, the ratio

$$\mathcal{R}_S(x,\sigma) \ := \ \mathcal{R}^\sigma(x)/\rho_{S,T}(x) \tag{3.2}$$

is independent of T by Lemma 2.1.1. Moreover, by section 1.1.4, the action of $\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$ on $\pi^{-1}_{\Omega/K}(x)$ is transitive on the Ω-rational points, and it preserves the modulus of the discs. Since it commutes with σ_Ω, the function $\mathcal{R}_S(-,\sigma)$ is independent of the choice of t_x and Ω.

Definition 3.1.3. *We call $\mathcal{R}_S(-,\sigma)$ the radius of σ. Its controlling graph will be denoted (with an abuse) by*

$$\Gamma_S(\sigma) \ := \ \Gamma_S(\mathcal{R}_S(-,\sigma)) \ . \tag{3.3}$$

Remark 3.1.4. *If $x \in \Gamma_S$, then $\sigma(x) = x$. Nevertheless σ_Ω is not necessarily the identity on $D(x,S)$, so the function $\mathcal{R}_S(-,\sigma)$ is not necessarily equal to 0 on the points of Γ_S.*

3.2. Compatibility with the restriction to an analytic domain.

Let Y be an analytic domain of X, together with a weak triangulation S_Y. Assume that σ induces by restriction an automorphism $\sigma_{|Y}$ of Y. Remark 3.1.2 shows that $\sigma_{|Y}$ is not necessarily S_Y-infinitesimal. It may also arises that $\sigma_{|Y}$ is S_Y-infinitesimal, but the restriction of the function $\mathcal{R}_S(-,\sigma)$ to the set Y does not coincide with the function $\mathcal{R}_{S_Y}(-,\sigma_{|Y})$. The following remark is an example of this phenomenon.

Remark 3.2.1. *Let $X = D^-(0,1)$ be the open unit disc together with the empty weak-triangulation S_Y, and let σ_q be the multiplication by $q \in K$ with $|q-1| < 1$. Then $\mathcal{R}_S(x,\sigma_q) = |q-1||T|(x)$ for all $x \in X$. If $Y = D^-(0,\rho)$ with $\rho < 1$, together with the empty weak-triangulation $S_Y = \emptyset$, then σ_q restricts to Y and it induces an S_Y-infinitesimal automorphism $(\sigma_q)_{|Y}$ of Y. But for all $x \in Y$ we have $\mathcal{R}_{S_Y}(x,(\sigma_q)_{|Y}) = \mathcal{R}_S(x,\sigma_q)/\rho$.*

Proposition 3.2.2. *Let (Y,S_Y) be a pair as in point i) or ii) of Lemma 3.0.3, then σ induces an S_Y-infinitesimal automorphism of Y, and*

$$\mathcal{R}_S(y,\sigma)_{|Y} \ = \ \mathcal{R}_{S_Y}(y,\sigma_{|Y}) \ . \tag{3.4}$$

Moreover

$$\Gamma_S(\sigma) \cap Y \ = \ \Gamma_{S_Y}(\sigma_{|Y}) \ . \tag{3.5}$$

Proof. For all $y \in Y$ we have $D(y,S_Y) = D(y,S)$. This together with Def. 1.2.1 imply (3.5). □

Remark 3.2.3. *Proposition 3.2.2 applies in particular to all opens of a Γ_S-covering of X (cf. Def. 1.1.4), but not to quasi Γ_S-coverings. Indeed point iii) of Lemma 3.1.1 ensures the existence of a quasi Γ_S-covering of X stable by σ. Notice however that the action of σ on the opens of such a covering is not necessarily S-infinitesimal as showed in the Remark 3.1.2.*

3.3. A finiteness result

The aim of this section is to prove the following analogue of Theorem 2.1.3 :

Theorem 3.3.1. *The function $x \mapsto \mathcal{R}_S(x,\sigma)$ enjoys the following properties:*

i) *$\mathcal{R}_S(x,\sigma)$ is a continuous function on X. It is moreover piecewise* log*-linear along each segment in X, and its slopes belong to $\mathbb{Z}$;*

ii) *If C is a connected component of $X - S$ (either a virtual open disc or annulus) there exist an analytic function $f_\sigma \in \mathcal{O}(C)$, and a real number $\alpha \in \mathbb{R}_{\geqslant 0}$, such that*

$$\mathcal{R}_S(x,\sigma) \;=\; \alpha \cdot |f_\sigma|(x)\;. \tag{3.6}$$

for all $x \in C$. In particular $\mathcal{R}_S(x,\sigma)$ is harmonic outside S.

iii) *Its controlling graph $\Gamma_S(\sigma)$ is locally finite. Moreover, if $\sigma \neq \mathrm{Id}_X$, the end points of $\Gamma_S(\sigma)$ that do not belong to Γ_S are exactly the rigid points of X that are fixed by σ.*

Remark 3.3.2. *In analogy with Theorem 2.1.3 one has the following immediate consequences:*

v) *Let D be a virtual open disc which is a connected component of $X - \Gamma_S$. Let C be any open annulus in D, and let $I := \Gamma_C$ be its skeleton. If I is oriented as out of D, then the function $y \mapsto \mathcal{R}_S(y,\sigma)$ is* log*-increasing and* log*-convex along I;*

vi) *Let C be a virtual open annulus which is a connected component of $X - S$. Let $I := \Gamma_C$ be its skeleton. Then $y \mapsto \mathcal{R}_S(y,\sigma)$ is* log*-convex along I.*

Proof of Theorem 3.3.1. The claims hold for X if and only if they hold for $X_{\widehat{K^{\mathrm{alg}}}}$ (cf. Remark 3.0.5). So, without loss of generality, we may assume that $K = \widehat{K^{\mathrm{alg}}}$, and that X is connected.

Lemma 3.3.3. *Theorem 3.3.1 holds if X is an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_K$.*

Proof. Let $T : X \hookrightarrow \mathbb{A}^{1,\mathrm{an}}_K$ be a global coordinate on X. Set $\delta_{\sigma,T} := T \circ \sigma - T \in \mathcal{O}(X)$. The value of $(\delta_{\sigma,T})_\Omega$ at t_x is $T(\sigma_\Omega(t_x)) - T(t_x)$. For all $f \in \mathcal{O}_X$ we have $|f|(x) = |f_\Omega|(t_x)$, so the norm $|\delta_{\sigma,T}|(x)$ equals the distance $|\sigma(t_x) - t_x|_\Omega$ in the coordinate T. By Lemma 3.1.1, we then have $\mathcal{R}^\sigma(x) = |\delta_{\sigma,T}|(x)$. If $\rho_{S,T}(x)$ denotes the radius of $D(x,S)$ in the coordinate T, the claim follows from the properties of $x \mapsto \rho_{S,T}(x)$ (cf. Remark 2.1.4) since $\mathcal{R}_S(x,\sigma) = |\delta_{\sigma,T}|(x)/\rho_{S,T}(x)$. □

Let now X be a general quasi-smooth curve. We consider a Γ_S-covering of X. If O is an open of the covering we call S_O a weak triangulation of O as in Definition 1.1.4. By Proposition 3.2.2 we are reduced to proving the claim for an individual open O of the covering.

If O is a connected component of $X - S$ (which is necessarily either an open annulus or disc) Theorem 3.3.1 is a consequence of Lemma 3.3.3. Claims ii) and iii) are then clear. Since a germ of segment always belongs to a connected component of $X - S$, the claims about the log-linearity and the slopes are also consequence of Lemma 3.3.3.

It remains to prove the continuity and local finiteness of $\Gamma_S(\sigma)$ at a point $x \in S$. We have to find a neighborhood of x in X of the form $Y_x = \tau_S^{-1}(\Lambda_x)$ (cf. Def. 1.1.4) on which the claims hold. We can exclude points of type 3 since, by [Duc, Thm.4.3.5], Y_x can be chosen either as a closed disc containing x, or as a closed annulus containing x in its skeleton Γ_C.

We now study the behavior of $\mathcal{R}_S(-,\sigma)$ in the neighborhood of a point of type 2 of S. To continue our proof we need the following results:

Theorem 3.3.4 ([PP15b])**.** *Assume $K = \widehat{K^{\mathrm{alg}}}$. Let x be a point of X of type 2. Let $b_1,\dots,b_n,c$ be distinct directions out of x. Let N be a positive integer. There exists an affinoid neighborhood Z*

of x in X, a quasi-smooth affinoid curve Y, an affinoid domain W of $\mathbb{P}^{1,an}_K$ and a finite étale map $\psi\colon Y \to W$ such that

i) *Z is isomorphic to an affinoid domain of Y and x lies in the interior of Y;*
ii) *the degree of ψ is prime to N;*
iii) *$\psi^{-1}(\psi(x)) = \{x\}$;*
iv) *almost every connected component of $Y \setminus \{x\}$ is an open unit disc with boundary $\{x\}$;*
v) *almost every connected component of $W - \{\psi(x)\}$ is an open unit disc with boundary $\{\psi(x)\}$;*
vi) *for almost every connected component C of $Y - \{x\}$, the induced morphism $C \to \psi(C)$ is an isomorphism;*
vii) *for every $i \in \{1, \dots, n\}$, the morphism ψ induces an isomorphism between a section of b_i and a section of $\psi(b_i)$ and we have $\psi^{-1}(\psi(b_i)) \subseteq Z$;*
viii) *$\psi^{-1}(\psi(c)) = \{c\}$.* □

Lemma 3.3.5 ([PP15b])**.** *Let K be an algebraically closed field. Let Z be a quasi-smooth K-analytic curve. Let $\psi : X \to Z$ be a finite morphism. Let $x \in X$ be a point of type 2 or 3. Assume that $d = [\mathscr{H}(x) : \mathscr{H}(\psi(x))]$ is prime to p.[6] Then every connected component of $\pi^{-1}_{\Omega/K}(\psi(x)) \setminus \{\sigma_\Omega(\psi(x))\}$ is a disc and the morphism ψ_Ω induces a trivial cover of degree d over it.* □

Let x be a point of S of type 2. By Theorem 3.3.4 there is an affinoid domain V of X, containing x, and an affinoid domain W' of $\mathbb{A}^{1,\mathrm{an}}_K$, together with a finite étale map $\psi : V \to W'$ such that

(a) $V - \{x\}$ and $W' - \{\psi(x)\}$ are both disjoint union of open discs;
(b) $V \cap \Gamma_S = \{x\}$;
(c) ψ induces an isomorphism $D \xrightarrow{\sim} \psi(D)$ on each disc D in $V - \{x\}$.

We endow V with the weak triangulation $S_V := \{x\}$. We can localize on V as in Proposition 3.2.2.

We may also assume that the degree $d := [\mathscr{H}(x) : \mathscr{H}(\psi(x))]$ is prime to p, so that the map ψ induces an isomorphism on the generic discs $\psi_\Omega : D(x) \xrightarrow{\sim} D(\psi(x))$, by Lemma 3.3.5.

Since $x \in S$ we have $D(x) = D(x, S)$. Moreover by choosing on W' the weak triangulation given by $S' := \{\psi(x)\}$ we also have $D(\psi(x)) = D(\psi(x), S')$. So ψ induces an isomorphism on the maximal discs $\psi_\Omega : D(x, S) \xrightarrow{\sim} D(\psi(x), S')$. In the other cases the maximal disc is just the connected component of $V - \{x\}$ (resp. $W' - \{\psi(x)\}$) containing the point. So, by point (c), we also have an isomorphism for all point $y \in V$:

$$\psi_\Omega \ : \ D(y, S) \xrightarrow{\ \sim\ } D(\psi(y), S')\ . \tag{3.7}$$

Now we consider ψ_Ω as a simultaneous coordinate on each $D(y, S)$, for all $y \in V$, and we define

$$\delta_{\sigma,\psi} \ := \ \psi \circ \sigma - \psi \ \in \ \mathcal{O}(V)\ . \tag{3.8}$$

The map $y \mapsto |\delta_{\sigma,\psi}|(y)$ is continuous at x, it is locally constant outside a finite sub-graph of V, and it controls the distance $|\sigma(t_y) - t_y|_\Omega$ measured with the coordinate ψ_Ω. Let $\rho_{S,\psi}(y)$ be the radius of $D(y, S)$ measured with the same coordinate on it. Then by Lemma 2.1.1 we have

$$\mathcal{R}_S(y, \sigma) \ = \ |\delta_{\sigma,\psi}|(y)/\rho_{S,\psi}(y)\ . \tag{3.9}$$

The function $y \mapsto \rho_{S,\psi}(y)$ is constant on V since the radius of $D(y, S) = D(y, S_V)$ with respect to ψ coincides by definition with the radius of $D(\psi(y), S')$. This proves the continuity of

[6] If the residual field $\widetilde{K}$ has characteristic 0, then $p = 1$ and this condition is always satisfied.

$\mathcal{R}_S(-,\sigma)_{|V} = \mathcal{R}_{S_V}(-,\sigma_{|V})$ on V, and the finiteness of $\Gamma_S(\sigma)\cap V = \Gamma_{S_V}(\sigma_{|V})$.

There is a finite number of branches $b_1,\ldots,b_n$ out of $x\in S$ that do not belong to V. Let b be one of them, O_b be the connected component of $X-S$ (either an open disc or annulus) containing b. We already know that $\mathcal{R}_S(-,\sigma)_{|O_b} = \mathcal{R}_\emptyset(-,\sigma_{|O_b})$ is continuous on O_b, and that $\Gamma_S(\sigma)\cap O_b = \Gamma_\emptyset(\sigma_{|O_b})$ is locally finite. Theorem 3.3.1 then follows from Proposition 3.3.6 below. □

Proposition 3.3.6. *$\mathcal{R}_S(-,\sigma)$ is continuous on the closure $\overline{O}_b$ of O_b in X, and $\Gamma_S(\sigma)\cap O_b$ is finite around x, i.e. there exists a neighborhood U of x in $\overline{O}_b$ such that $\Gamma_S(\sigma)\cap U$ is finite.*

Proof. By Theorem 3.3.4 we may find an affinoid neighborhood Z of x and an étale map $\psi : Z\to \mathbb{A}^{1,\mathrm{an}}_K$ of degree prime to p, verifying point vii) of Theorem 3.3.4 at the branch b. More precisely we may find sections C_b and $C_{\psi(b)}$ of b and $\psi(b)$ respectively that are open annuli, and such that ψ induces an isomorphism between annuli:

$$\psi \;:\; C_b \xrightarrow{\sim} C_{\psi(b)}\;. \tag{3.10}$$

Up to shrinking Z we may assume that its closure verifies $\overline{C}_b\cap S = \{x\}$, and that $\Gamma_S\cap C_b$ is either empty or equal to Γ_{C_b}. Now, define as above $\delta_{\sigma,\psi} := \psi\circ\sigma-\psi\in\mathcal{O}(Z)$.

We now distinguish two situations $C_b\cap\Gamma_S = \Gamma_{C_b}$ and $C_b\cap\Gamma_S=\emptyset$.

If $C_b\cap\Gamma_S=\Gamma_{C_b}$, we can localize to C_b as in Proposition 3.2.2. We proceed then similarly as in (3.9) to show, by Lemma 2.1.1, that for all $y\in C_b\cup\{x\}$ one has $\mathcal{R}_S(y,\sigma) := |\delta_{\sigma,\psi}|(y)/\rho_{S,\psi}(y)$. So we are done in this case.

Assume now that $C_b\cap\Gamma_S=\emptyset$. Since $X-\Gamma_S$ is a disjoint union of discs, O_b is one of them, so the localization to C_b affects $\mathcal{R}_S(-,\sigma)$ (cf. Remark 3.1.2). In order to describe the link between $\mathcal{R}_S(-,\sigma)$ and the norm of $\delta_{\sigma,\psi}$ we need the following Lemma which is deduced from [BGR84, 9.7.1/2]:

Lemma 3.3.7. *Let T be a coordinate on $\mathbb{A}^{1,\mathrm{an}}_K$. Let $D^-(0,1)$ be the open unit disc, and let*

$$C \;:=\; C^-(0;]R,1[) \;=\; \{x \text{ such that } R<|T(x)|<1\}\;,\quad 0<R<1\;. \tag{3.11}$$

Let $\psi : C^-(0;]R,1[)\xrightarrow{\sim} C^-(0;]R,1[)$ be an isomorphism. Then

i) *ψ permutes the set of maximal discs in C (i.e. if D is a connected component of $C-\Gamma_C$, then ψ induces an isomorphism of D with another connected component D').*

ii) *ψ is either the identity on the skeleton Γ_C or it is the map sending $x_{0,\rho}$ into $x_{0,\rho^{-1}R}$.*

Moreover if ψ induces the identity on Γ_C, then it is also isometric : for all L/K and all L-rational points $t_1,t_2\in C(0;]R,1[)$ one has $|\psi(t_1)-\psi(t_2)|_L = |t_1-t_2|_L$. □

Let us come back to our situation: x is a point of type 2, so $O_b\cong D^-(0,1)$, and $C_b\cong C^-(0;]R,1[)$ for some $0<R<1$, as in Lemma 3.3.7. We can assume that $\psi(x)=x_{0,1}$, and that $C_{\psi(b)}$ is an annulus with $x_{0,1}$ in its boundary. By translating, rescaling, and possibly considering the inversion $x\mapsto x^{-1}$ of $\mathbb{G}^{\mathrm{an}}_{m,K}$ we can assume that $\psi(C_b) = C_{\psi(b)} = C^-(0;]R,1[)$ with the same R of C_b.

So by Lemma 3.3.7, ψ is isometric. With these normalizations, by the above arguments, for all $y\in C_b\cup\{x\}$ we have $\mathcal{R}_S(y,\sigma) = |\delta_{\sigma,\psi}|(y)$. This concludes the proof of Proposition 3.3.6. □

4. Deformation

In this section we show how to deform a differential equation into a so called σ-difference equation.

4.1. σ-difference equations.

Let $\sigma : X \xrightarrow{\sim} X$ be an automorphism of X. A σ-difference equation over X is a locally free $\mathcal{O}_X$-module $\mathscr{F}$ of finite rank, together with an $\mathcal{O}_X$-linear isomorphism

$$\sigma \;:\; \mathscr{F} \;\xrightarrow{\;\sim\;}\; \sigma^*\mathscr{F}\;. \tag{4.1}$$

A morphism between σ-difference equations $\alpha : (\mathscr{F},\sigma) \to (\mathscr{F}',\sigma')$ is an $\mathcal{O}_X$-linear map $\alpha : \mathscr{F} \to \mathscr{F}'$ such that $\sigma'\circ\alpha = \sigma^*(\alpha)\circ\sigma$. Usual operations of linear algebra exist. The category admits an internal tensor product, and a unit object which is $(\mathcal{O}_X, \mathrm{Id}_{\mathcal{O}_X} : \mathcal{O}_X \xrightarrow{\sim} \sigma^*(\mathcal{O}_X))$. It not possible to localize such a structure to an analytic domain Y in X, simply because Y is possibly not stable under σ.

If $\mathscr{F}$ is free and X is quasi-Stein, then (4.1) corresponds to a σ-semi-linear automorphism $\sigma : \mathscr{F}(X) \xrightarrow{\sim} \mathscr{F}(X)$ (i.e. satisfying $\sigma(af) = \sigma(a)\sigma(f)$, for all $a \in \mathcal{O}(X)$, $f \in \mathscr{F}(X)$). If a basis of $\mathscr{F}$ is chosen, this corresponds to a system of the form:

$$\sigma(f_1,\ldots,f_r)^t = A_\sigma\cdot(\sigma(f_1),\ldots,\sigma(f_r))^t\;, \qquad A_\sigma \in GL_r(\mathcal{O}(X))\;. \tag{4.2}$$

Definition 4.1.1. *Let $\Sigma \subseteq \mathfrak{S}(X,S)$ be a family of S-infinitesimal automorphisms of X. A Σ-module is a locally free $\mathcal{O}_X$-module $\mathscr{F}$ of finite type, together with a structure of σ-difference equation for all $\sigma \in \Sigma$. A morphism of Σ-modules is an $\mathcal{O}_X$-linear morphism commuting with the action of all $\sigma \in \Sigma$. We denote by $\mathrm{Hom}^\Sigma(\mathscr{F},\mathscr{F}')$ the group of morphisms.*

4.2. σ-compatibility

Let $\sigma \subseteq \mathfrak{S}(X,S)$ be an S-infinitesimal automorphism of X. Let $\mathscr{F}$ be a differential equation over X. We say that $\mathscr{F}$ is *σ-compatible* if for all $x \in X$ we have $D^+(x,\sigma) \subset D(x,\mathscr{F})$ (cf. (3.1) and (2.1)). This is equivalent to the condition that for all $x \in X$ the condition

$$\mathcal{R}_S(x,\sigma) \;<\; \mathcal{R}_{S,1}(x,\mathscr{F})\;. \tag{4.3}$$

Lemma 4.2.1. *Assume that X is connected. The following conditions are equivalent:*

i) *(4.3) holds for all $x \in X$;*

ii) *(4.3) holds for all $x \in S$, and over all germs of segments at the open boundary of X.*[7]

Proof. This follows by the concavity properties of $\mathcal{R}_{S,1}(-,\mathscr{F})$ (cf. properties iii) and iv) of Theorem 2.1.3), and by the convexity properties of $\mathcal{R}_S(-,\sigma)$ (cf. properties v) and vi) of Remark 3.3.2). □

4.3. Deformation

We recall that all finite locally free $\mathcal{O}_{D(x,\mathscr{F})}$-module is free since Ω is spherically complete.

Our main result is the following:

Theorem 4.3.1. *Let σ be an S-infinitesimal automorphism of X, and let $\mathscr{F}$ be a σ-compatible differential equation. Then there exists on $\mathscr{F}$ a canonical structure of σ-difference equation characterized by the fact that, for all $x \in X$, all solutions $(f_1,\ldots,f_r)^t$ of $\mathscr{F}_{|D(x,\mathscr{F})}$ in a given basis, is also a solution of (4.2) with respect to the same basis.*

If $\alpha : \mathscr{F} \to \mathscr{F}'$ is an $\mathcal{O}_X$-linear map between σ-compatible differential equations commuting with the connections, then α also commutes with the corresponding action of σ on $\mathscr{F}$ and $\mathscr{F}'$.

Proof. We consider the map $\Delta_\sigma : X \to X \times X$ defined by $\Delta_\sigma = (\sigma, \mathrm{Id})$. By Lemma 4.3.3 below

[7] As an example, if $X = C^-(0;]R_1,R_2[)$ is an open annulus with empty weak triangulation, then condition ii) asks that there exist unspecified $\varepsilon_1,\varepsilon_2 > 0$ such that (4.3) holds for all $x \in]x_{0,R_2-\varepsilon_2}, x_{0,R_2}[$ and all $x \in]x_{0,R_1}, x_{0,R_1+\varepsilon_1}[$.

there is an admissible open neighborhood $\mathcal{T}$ of the diagonal such that Δ_σ factorizes as

$$\Delta_\sigma \ : \ X \ \to \ \mathcal{T} \ \subseteq \ X \times X \tag{4.4}$$

and that the stratification χ of (2.8), associated to $\mathscr{F}$, is defined over $\mathcal{T}$. We can consider the pull back of χ by Δ_σ. Since $\Delta_\sigma^* \circ p_1^*(\mathscr{F}) = \sigma^*\mathscr{F}$ and $\Delta_\sigma^* \circ p_2^*(\mathscr{F}) = \mathscr{F}$ we find an isomorphism

$$\sigma^{\mathscr{F}} := \Delta_\sigma^*(\chi) \ : \ \mathscr{F} \xrightarrow{\sim} \sigma^*\mathscr{F} \tag{4.5}$$

that satisfies the required properties. □

Remark 4.3.2. *Assume that X is quasi-Stein, and that Ω^1_X and $\mathscr{F}$ are both free over X. Let $Y' = G \cdot Y$, $G \in M_r(\mathcal{O}(X))$, be the differential equation attached to $\mathscr{F}$ in some basis. Then the action of σ on $\mathscr{F}$ is given, in the same basis, by the equation $\sigma(Y) = A_\sigma \cdot Y$, where*

$$A_\sigma \ := \ \Delta_\sigma(Y_\chi) \in GL_r(\mathcal{O}(X)) \ , \tag{4.6}$$

and $Y_\chi \in GL_r(\mathcal{O}(\mathcal{T}))$ is the matrix of χ in the same basis of $\mathscr{F}$.

Lemma 4.3.3. *If $\mathscr{F}$ is σ-compatible, there exists an admissible open subset $\mathcal{T} \subset X \times X$, containing the image of Δ_σ and the diagonal, on which the stratification χ associated to $\mathscr{F}$ converges.*

Proof. Let $x \in X - \Gamma_S$ and let D_x be the connected component (necessarily a virtual open disc) containing x. Let T_x be a coordinate on D_x, and let R_{D_x} be the radius of D_x in that coordinate. If $\delta_{\sigma,T_x} := T_x \circ \sigma - T_x$, then for all $y \in D_x$ we have $\mathcal{R}_S(y,\sigma)_{|D_x} = \mathcal{R}_\emptyset(y, \sigma_{|D_x}) = |\delta_{\sigma,T_x}|(y)/R_{D_x}$.

The radius of convergence of $\mathscr{F}$ enjoys the same properties and for all $y \in D_x$ we have $\mathcal{R}_S(y,\mathscr{F})_{|D_x} = \mathcal{R}_\emptyset(y,\mathscr{F}_{|D_x}) = \mathcal{R}^{\mathscr{F}}(y)/R_{D_x}$, where $\mathcal{R}^{\mathscr{F}}(y)$ is the radius of the largest open disc in $(D_x)_\Omega$ containing t_y on which $\mathscr{F}_\Omega$ is trivial. The function $y \mapsto |\delta_{\sigma,T_x}|(y)$ is increasing on the segments in D_x (oriented as towards $+\infty$), and $y \mapsto \mathcal{R}^{\mathscr{F}}(y)$ is decreasing (cf. Theorem 2.1.3, and Remark 3.3.2). So, by (4.3), there exists $R_x \geqslant 0$ such that for all $y \in D_x$ we have

$$|\delta_{\sigma,T_x}|(y) < R_x < \mathcal{R}^{\mathscr{F}}(y) \ . \tag{4.7}$$

With the notations of Section 2.2.1 we consider the following admissible open of $D_x \times D_x$

$$\mathcal{T}(D_x, T_x, R_x^-) \ := \ \{|T_1 - T_2| < R_x\} \ \subseteq \ D_x \times D_x \ . \tag{4.8}$$

It is then easy to check that the stratification χ defined by $\mathscr{F}_{|D_x}$ converges on $\mathcal{T}(D_x, T_x, R_x^-)$ and

$$\Delta_\sigma(D_x) \ \subseteq \ \mathcal{T}(D_x, T_x, R_x^-) \ . \tag{4.9}$$

Indeed the algebra of functions over $\mathcal{T}(D_x, T_x, R_x^-)$ is formed by the formal power series $f(T_1,T_2) = \sum_{n\geqslant 0} f_n(T_2)(T_1 - T_2)^n$, with $f_n \in \mathcal{O}(D_x)$, such that for all virtual closed sub-disc $D \subseteq D_x$ one has $\lim_n \|f_n\|_D \cdot \rho^n = 0$, for all $\rho < R_{D_x}$. We see that

$$\Delta_\sigma(f(T_1,T_2)) \ = \ f(\sigma \circ T, T) \ = \ \sum_{n\geqslant 0} f_n(T) \cdot \delta_{\sigma,T}(T) \ . \tag{4.10}$$

Condition (4.7) implies that $\delta_{\sigma,T}$ is bounded on D_x and $\|\delta_{\sigma,T}\|_{D_x} \leqslant R_x \leqslant R_{D_x}$, so (4.10) converges as a series of functions in $\mathcal{O}(D_x)$.

On the other hand, with the notations of (2.7), the matrix of χ is given in some basis by

$$Y_\chi(T_1,T_2) \ := \ \sum_{n\geqslant 0} G_n(T_2)(T_1 - T_2)^n/n! \ . \tag{4.11}$$

Its convergence locus is related to $\mathcal{R}^{\mathscr{F}}(-)$ by the relation

$$\mathcal{R}^{\mathscr{F}}(y) \ := \ \min\Big(\ R_{D_x} \ , \ \liminf_n \frac{1}{\sqrt[n]{|G_n|(y)/n!}} \ \Big) \ , \qquad \text{for all } y \in D_x \ . \tag{4.12}$$

So (4.7) implies that Y_χ lies in $M_{r\times r}(\mathcal{O}(\mathcal{T}(D_x,T_x,R_x^-)))$.

To conclude the proof we proceed as follows. We consider the open covering $\{U_x\}_{x\in X}$ of X formed by

i) for all $x\in X-\Gamma_S$ we consider the connected component $U_x:=D_x$ of $X-\Gamma_S$ containing x, together with the triplet (D_x,T_x,R_x^-) that we have just obtained;

ii) for all $x\in\Gamma_S$ we consider an arbitrary open neighborhood $U_x:=Y_x$ of x, of the form $Y_x=\tau_S^{-1}(\Lambda_x)$, with $\Lambda_x=\Gamma_S\cap Y_x$, together with an arbitrary triplet (Y_x,T_x,R_x^-), where $R_x>0$ is such that χ converges on $\mathcal{T}(Y_x,T_x,R_x^-)$ (cf. Section 2.2.2).

Notice that, for all $x\in X$, $U_x\times U_x$ is open in $X\times X$, so $\mathcal{T}(U_x,T_x,R_x^-)$ is open in $X\times X$. We now consider the following open neighborhood of the diagonal of $X\times X$:

$$\mathcal{T}\;:=\;\bigcup_{x\in X}\mathcal{T}(U_x,T_x,R_x^-)\;=\;\Big(\bigcup_{x\in X-\Gamma_S}\mathcal{T}(D_x,T_x,R_x^-)\Big)\bigcup\Big(\bigcup_{x\in\Gamma_S}\mathcal{T}(Y_x,T_x,R_x^-)\Big)\tag{4.13}$$

By construction χ converges on $\mathcal{T}$. On the other hand, for all $x\in X$, we have $\Delta_\sigma(x)\in\mathcal{T}$, indeed if $x\in X-\Gamma_S$ this follows from (4.9), and if $x\in\Gamma_S$ this is evident since $\sigma(x)=x$, so $\Delta_\sigma(x)$ lies in the diagonal. □

Definition 4.3.4. *Let $\Sigma\subseteq\mathfrak{S}(X,S)$ be a family of S-infinitesimal automorphisms. We say that a differential equation $\mathscr{F}$ is Σ-compatible if it is σ-compatible, for all $\sigma\in\Sigma$.*

The structure of Σ-module on $\mathscr{F}$ obtained by Theorem 4.3.1 is called the deformation *of the differential equation, and it will be indicated as $\mathrm{Def}_{S,\Sigma}(\mathscr{F})$ or simply $\mathrm{Def}_\Sigma(\mathscr{F})$.*

Theorem 4.3.1 gives a functor called (S,Σ)-*deformation*

$$\mathrm{Def}_{S,\Sigma}\;:\;\{\Sigma-\text{compatible differential equations}/X\}\;\longrightarrow\;\{\Sigma-\text{Modules}/X\}\,.\tag{4.14}$$

Definition 4.3.5 (*Stratified Σ-modules*)**.** *We call the essential image of* $\mathrm{Def}_{S,\Sigma}$ stratified Σ-modules*.*

The functor $\mathrm{Def}_{S,\Sigma}$ is additive, exact, it commutes with tensor products, and it is faithful since it is the identity on the morphisms. But it is not necessarily fully faithful as shown by the following example.

Example 4.3.6. *Let q be a root of unity satisfying $|q-1|<1$. Let $X=C^-(0;]R_1,R_2[)$ be an annulus with empty weak triangulation, and let $\sigma:=\sigma_q$ be the automorphisms sending $T\mapsto qT$. The deformation functor sends the unit object $\mathbb{I}=(d:\mathcal{O}(X)\to\mathcal{O}(X))$ into the unit object $\mathbb{I}=(\sigma_q:\mathcal{O}(X)\xrightarrow{\sim}\mathcal{O}(X))$. The endomorphisms of $(\mathcal{O}(X),\sigma_q)$ are naturally in bijection with the elements of $\mathcal{O}(X)^{\sigma_q=1}=\{f\in\mathcal{O}(X),$ such that $\sigma_q(f)=f\}$. Since q is a root of unity there are several non constant function of this type. Hence the inclusion $\mathrm{Def}_\sigma:\mathrm{End}(\mathbb{I})\to\mathrm{End}(\mathrm{Def}_\sigma(\mathbb{I}))$ is strict.*

4.4. Fully faithfulness and non degeneracy

In this section we provide a condition, called non degeneracy, on the family $\Sigma\subseteq\mathfrak{S}(X,S)$ that guarantee the fully faithfulness of Def_Σ.

It is clear that if a differential equation $\mathscr{F}$ is σ-compatible, it is also σ^n-compatible for all $n\in\mathbb{Z}$. More generally if Σ is a family of S-infinitesimal automorphisms, and if $\mathscr{F}$ is σ-compatible for all $\sigma\in\Sigma$, then it is σ-compatible for all σ lying in the subgroup $\langle\Sigma\rangle$ of $\mathfrak{S}(X,S)$ generated by Σ.

We are then naturally induced to work with groups of S-infinitesimal automorphisms.

Definition 4.4.1 (Non degeneracy)**.** *Let $\Sigma\subseteq\mathfrak{S}(X,S)$ be a family of S-infinitesimal automorphisms. We say that the action of Σ is non degenerate if for all connected component X' of X there is a* *point $x \in X'$ such that for all open disc D such that $\bigcup_{\sigma\in\Sigma} D^+(x,\sigma) \subseteq D \subseteq D(x,S)$ one has*[8]

$$\mathcal{O}(D)^{\Sigma_\Omega=1} \;=\; \Omega\,. \tag{4.15}$$

Proposition 4.4.2 (Criterion of non degeneracy)**.** *Let $x \in X$, and let $\Sigma \subseteq \mathfrak{S}(X,S)$. Assume that we have a sequence of elements $\{\sigma_n\}_n$ in the group $\langle\Sigma\rangle$ such that*

i) $\bigcap_n D^+(x,\sigma_n) = \{t_x\}$*;*

ii) *For infinitely many n the disc $D^+(x,\sigma_n)$ is not reduced to $\{t_x\}$.*

Then the action of Σ is non degenerate.

Proof. Let T be a coordinate on $D(x,S)$. Let D be an open disc as in Definition 4.4.1. Let $f \in \mathcal{O}(D)$ be a function stable under Σ, then $g(T) := f(T) - f(t_x)$ is also stable under Σ. Now we have a sequence $n \mapsto \sigma_n(t_x)$ of zeros of g accumulating to t_x. So $g = 0$ and $f = f(t_x)$ is constant. □

Remark 4.4.3. *Let $T_x : U \to \mathbb{A}^{1,\mathrm{an}}_K$ be a local coordinate on some neighborhood U of x verifying Lemma 3.3.5. As usual, let $\delta_{\sigma,T_x} := T_x\circ\sigma - T_x$. Then conditions i) and ii) of Proposition 4.4.2 are equivalent to the condition that the closure in $\mathbb{R}$ of the set $\{|\delta_{\sigma_n,T_x}|(x)\ ,\ n \geqslant 0\} - \{0\}$ contains 0.*

Proposition 4.4.4. *If the action of Σ is non degenerate, then $\mathrm{Def}_{S,\Sigma}$ is fully faithful.*

Proof. The functor is the identity on the morphisms, so it is enough to show that if a morphism $\alpha : \mathscr{F} \to \mathscr{F}'$ of $\mathcal{O}_X$-modules commutes with Σ, then it also commutes with ∇. This is true if and only if it is true for $\alpha_\Omega : \mathscr{F}_\Omega \to \mathscr{F}'_\Omega$. Now, by Lemma 2.1.5, it is enough to show that α_Ω commutes with ∇ over the disc D of Definition 4.4.1. The ring $\mathcal{O}(D)^\Sigma$ contains Ω, and the space $\mathrm{Hom}^\Sigma(\mathscr{F}_{|D},\mathscr{F}_{|D})$ is an $\mathcal{O}(D)^\Sigma$-module containing $\mathrm{Hom}^\nabla(\mathscr{F}_{|D},\mathscr{F}_{|D})$ as a sub-Ω-vector space. Since $\mathcal{O}(D)^{\Sigma_\Omega} = \Omega$, $\mathrm{Hom}^\Sigma(\mathscr{F}_{|D},\mathscr{F}_{|D})$ is an Ω-vector space too.

Now choose D so that the differential equations $\mathscr{F}_{|D}$ and $(\mathscr{F}')_{|D}$ are trivial on D (this is possible since $\mathscr{F}$ and $\mathscr{F}'$ are both Σ-compatible). Their deformations over D are hence trivial too, and the deformation commutes with the localization to D. Hence the space $\mathrm{Hom}^\Sigma(\mathscr{F}_{|D},\mathscr{F}'_{|D})$ is an Ω-vector space of dimension $\mathrm{rank}(\mathscr{F})\cdot\mathrm{rank}(\mathscr{F}')$. The dimension of $\mathrm{Hom}^\nabla(\mathscr{F}_{|D},\mathscr{F}'_{|D})$ is the same, so they coincide. Hence α_Ω commutes with ∇ over D, and the claim is proved. □

4.5. Analyticity of the action of Σ.

Until now we have studied the action of a family of automorphisms on X. We now consider the action of a K-analytic group G on it. In this section we prove that the deformation of a differential equation produces an *analytic* (semi-linear) action of G on $\mathscr{F}$ lifting the action of G on X. This kind of object is commonly know as *G-equivariant sheaf on X* (cf. [Mum08, Section 12] and [MFK94]).

The action of a K-analytic group G on X is a morphism

$$\mu \;:\; G\times X \;\longrightarrow\; X \tag{4.16}$$

satisfying the natural conditions of [Mum08, Section 12], and [Ber90, Section 5.1] for the details about the setting. In particular, for all complete valued field extension L/K and all L-rational point $g : \mathscr{M}(L) \to G_L$ we have an automorphism $\sigma_g := \mu_L\circ(g\times\mathrm{Id}_{X_L})$ of X_L

$$\sigma_g \;:\; X_L \;\xrightarrow{\sim}\; X_L\,. \tag{4.17}$$

This family verifies $\sigma_g\circ\sigma_h = \sigma_{g\cdot h}$ for all $g,h \in G(L)$. So for all L/K we have a group morphism

$$G(L) \;\longrightarrow\; \mathrm{Aut}(X_L) \tag{4.18}$$

[8] Here $\Sigma_\Omega := \{\sigma_\Omega\}_{\sigma\in\Sigma}$, where as usual $\sigma_\Omega = \sigma\widehat{\otimes}\mathrm{Id}_\Omega$.

Let un call Σ_L the image of $G(L)$ in $\mathrm{Aut}(X_L)$.

Definition 4.5.1. *We say that the action of G on X is S-infinitesimal if for all L/K and for all $g \in G(L)$ the map σ_g is an S_L-infinitesimal automorphism of X_L.*

We say that the action of G is non degenerate if there exists L/K such that the action of $G(L)$ on X_L is non degenerate.

We say that a differential equation $\mathscr{F}$ on X is G-compatible, if for all L/K and all $g \in G(L)$ the equation $\mathscr{F}_L$ is σ_g-compatible.[9]

4.5.1. Analytic semi-linear G-modules. We shall now introduce (somehow informally) the notion of analytic semi-linear G-module (cf. Definition 4.5.3). In order to do that, for all L/K we interpret the family of automorphism $\{\sigma_g : X_L \xrightarrow{\sim} X_L\}_{g\in G(L)}$ as a *covering* of X_L. A *semi-linear $G(L)$-module* is then a *gluing datum* on a family $\{\mathscr{F}_g\}_g$ of locally free $\mathcal{O}_{X_L}$-modules on the covering, where $\mathscr{F}_g = \mathscr{F}$ for all $g \in G(L)$. Concretely this amounts to give a locally $\mathcal{O}_{X_L}$-module $\mathscr{F}_L$ together with a family of $\mathcal{O}_{X_L}$-linear isomorphism

$$\{\ \sigma_g^{\mathscr{F}_L}\ :\ \mathscr{F}_L \xrightarrow{\ \sim\ } \sigma_g^*(\mathscr{F}_L)\ \}_{g\in G(L)}\ , \tag{4.19}$$

subjected to the cocycle condition

$$\sigma_{gh}^{\mathscr{F}_L}\ =\ \sigma_h^*(\sigma_g^{\mathscr{F}_L}) \circ \sigma_h^{\mathscr{F}_L}\ . \tag{4.20}$$

We may give the following informal definition:

Definition 4.5.2. *A semi-linear G-module is a locally free $\mathcal{O}_X$-module $\mathscr{F}$ of finite type together with a semi-linear $G(L)$-module structure on $\mathscr{F}_L$ for all L/K, which satisfies the evident compatibilities for all base change of the ground field K.*

More generally we can perform the same construction for all base change S/K, i.e we regard the objects as functors on the category of K-analytic spaces. So by Yoneda's Lemma we obtain the following definition.

Let $p_X : G \times X \to X$ be the second projection. Consider the simplicial object

$$G \times G \times X \ \underset{d_2}{\overset{d_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{d_1}}}}}\ G \times X \ \underset{\mu}{\overset{p_X}{\rightrightarrows}}\ X\ . \tag{4.21}$$

where $d_0 = p_2 \times p_3$, $d_1 = m_G \times \mathrm{Id}_X$, and $d_2 = \mathrm{Id}_G \times \mu$, and p_i is the i-th projection of $G \times G \times X$.

Definition 4.5.3. *An analytic semi-linear G-module is a locally free $\mathcal{O}_X$-module $\mathscr{F}$ together with an isomorphism*

$$\sigma_G\ :\ p_X^*(\mathscr{F}) \xrightarrow{\ \sim\ } \mu^*(\mathscr{F})\ , \tag{4.22}$$

satisfying the cocycle condition. This means that the following diagram commutes

$$\begin{array}{ccc} p_3^*(\mathscr{F}) & \xrightarrow{\ d_0^*(\sigma_G)\ } & (\mu \circ d_0)^*(\mathscr{F}) \\ & {\scriptstyle d_1^*(\sigma_G)} \searrow \quad \swarrow {\scriptstyle d_2^*(\sigma_G)} & \\ & \eta^*(\mathscr{F}) & \end{array} \tag{4.23}$$

where $\eta := \mu \circ d_1 = \mu \circ d_2$.

For all L/K and all $g \in G(L)$ we can pull-back σ_G by the map $g \times \mathrm{Id}_{X_L} : X_L \to G_L \times X_L$ and we obtain the family of maps (4.19), and the diagram (4.23) gives the cocycle relation (4.20).

[9]This is equivalent to saying that for all point $g : \mathscr{H}(g) \to G$ the equation $\mathscr{F}_{\mathscr{H}(g)}$ is σ_g-compatible.

Remark 4.5.4. *If X is quasi-Stein and if $\mathscr{F}$ is free, a semi-linear $G(L)$-module corresponds to a family of difference equations of the form $\sigma_g(\vec{f}) = A_{\sigma_g}\cdot\vec{f}$ over X_L (cf. (4.2)). The action of G is analytic if and only if A_{σ_g} is an analytic function on $G\times X$, i.e. also with respect to the variable g.*

Theorem 4.5.5. *Assume that the action of G on X is S-infinitesimal. There exists a functor*

$$\{G\text{-compatible diff. eq.}\}/X \;\longrightarrow\; \{\text{Analytic semi-linear } G\text{-modules}\}/X\;. \tag{4.24}$$

The functor associates to a differential equation $\mathscr{F}$ over X the same $\mathcal{O}_X$-module $\mathscr{F}$ together with a semi-linear action of G. If Σ_L is the image of $G(L)$ in $\mathrm{Aut}(X_L)$, the action of G is characterized by the fact that for all L/K the action of Σ_L on $\mathscr{F}_L$ so obtained coincides with that of Theorem 4.3.1.

The functor is the identity on the morphisms, in particular it is faithful. If the action of G on X is non degenerate, it is also fully faithful.

Proof. We consider the map $\Delta_G := \mu\times p_X$

$$\Delta_G \;:\; G\times X \;\longrightarrow\; X\times X\;. \tag{4.25}$$

By the Lemma 4.5.6 below, the image of the map Δ_G is contained in some admissible neighborhood $\mathcal{T}$ of the diagonal over which the stratification $\chi : p_2^*(\mathscr{F})_{\mathcal{T}} \xrightarrow{\sim} p_1^*(\mathscr{F})_{\mathcal{T}}$ associated to the differential equation $\mathscr{F}$ converges. So by pull-back we have an isomorphism $\sigma_G := \Delta_G^*(\chi)$ as in (4.22). It is clear that it gives a structure of analytic semi-linear G-module on $\mathscr{F}$ with the required properties. In particular the cocycle condition (4.23) follows from the cocycle condition i) of Section 2.2.3 verified by χ. □

Lemma 4.5.6. *There exists an admissible neighborhood of the diagonal $\mathcal{T}$ of $X\times X$ on which the stratification associated with $\mathscr{F}$ converges, such that the image of the morphism*

$$\Delta_G \;:\; G\times X \;\longrightarrow\; X\times X \tag{4.26}$$

is contained in $\mathcal{T}$.

Proof. We can assume $K=\widehat{K^{\mathrm{alg}}}$. If $g\in G(L)$ for some L/K, we have a commutative diagram:

$$\begin{array}{ccc} G_L\times X_L & \xrightarrow{(\Delta_G)_L} & X_L\times X_L \\ {\scriptstyle g\times\mathrm{Id}_{X_L}}\uparrow & \nearrow{\scriptstyle \Delta_{\sigma_g}} & \\ X_L & & \end{array} \tag{4.27}$$

We consider a large field Ω/K such that the vertical maps of the following diagram are all surjective

$$\begin{array}{ccccc} G(\Omega)\times X(\Omega) & \xrightarrow{\Delta_G(\Omega)} & X(\Omega)\times X(\Omega) & \supset & \mathcal{T}(\Omega) \\ \downarrow & & \downarrow & & \downarrow \\ G\times X & \xrightarrow{\Delta_G} & X\times X & \supset & \mathcal{T} \end{array} \tag{4.28}$$

This is possible thanks to [PP13a, Prop. 2.1.7]. It is then enough to prove that

$$\Delta_G(\Omega)(\,G(\Omega)\times X(\Omega)\,) \;\subseteq\; \mathcal{T}(\Omega)\;. \tag{4.29}$$

We now show that this is automatic. Let $\{U_x\}_{x\in X}$ be the open covering of X that we have obtained in the proof of Lemma 4.3.3. Then $\{(U_x)_\Omega\}_{x\in X}$ is again an open covering of X_Ω since the projection $\pi_{\Omega/K}: X_\Omega\to X$ gives an isomorphism $\Gamma_{S_\Omega}\xrightarrow{\sim}\Gamma_S$. As in the proof of 4.3.3 define $\mathcal{T}$ as the union (4.13) of local neighborhoods of the diagonal defined by some conditions of type $|(T_x)_1-(T_x)_2|<R$, where $T_x: U_x\to\mathbb{A}_K^{1,\mathrm{an}}$ is a local étale coordinate on some open U_x. This implies that $\mathcal{T}_\Omega$ is defined by the same conditions, with $(T_x)_\Omega:(U_x)_\Omega\to\mathbb{A}_\Omega^{1,\mathrm{an}}$, and the same R. Now the radii $\mathcal{R}_S(-,\sigma)$ and

$\mathcal{R}_{S,1}(-,\mathscr{F})$ are stable by scalar extension to Ω, so the proof of Lemma 4.3.3 shows that for all $g \in \Sigma_\Omega = G(\Omega)$ we have $\Delta_{\sigma_g}(X_\Omega) \subseteq \mathcal{T}_\Omega$. This implies (4.29) by the diagram (4.27). □

5. Deformation and quasi unipotence over the Robba ring

Several situations requires an analysis along a germ of segment along a Berkovich curve. This corresponds to the study of differential equations over the Robba ring $\mathfrak{R}$. So we now restrict our attention to this case, which is the case studied in [ADV04] in the context of q-difference equations.

In sections 5.1 we state the Deformation over $\mathfrak{R}$ (that needs some ad-hoc definitions). In sections 5.2, 5.3, 5.4, 5.5, we will assume the following

Hypothesis 5.0.7. *K is discretely valued, of mixed characteristic, and with perfect residual field.*

As a consequence every locally free $\mathfrak{R}$-module of finite type will be free by [Chr11, Thm. 4.40].

Definition 5.0.8. *We set $C_\varepsilon := C^-(0;]1-\varepsilon,1[)$. We always consider on C_ε the empty weak triangulation. The Robba ring is defined as*

$$\mathfrak{R} \;:=\; \bigcup_{\varepsilon>0} \mathcal{O}(C_\varepsilon) \;=\; \Big\{\sum_{n\in\mathbb{Z}} a_n T^n \mid \exists \varepsilon>0 \text{ such that } \lim_{n\to\pm\infty} |a_n|\rho^n = 0\;,\; \forall \rho \in]1-\varepsilon,1[\Big\}\;. \tag{5.1}$$

5.1. Deformation

Since we are working over a *germ* of annulus, some definition have to be adapted.

If σ is an infinitesimal automorphism of C_ε, then it is so also for all $C_{\varepsilon'}$ with $\varepsilon' < \varepsilon$ (cf. Lemma 3.0.3). The function $x \mapsto \mathcal{R}_\emptyset(x,\sigma)$ over C_ε commutes with the restriction to $C_{\varepsilon'}$, $\varepsilon' < \varepsilon$. An automorphism of $\mathfrak{R}$ is called *infinitesimal* if it comes from an infinitesimal automorphism of C_ε for some $\varepsilon > 0$ (with respect to the empty weak triangulation). If Σ is a family of infinitesimal automorphisms of $\mathfrak{R}$ we denote by $\Sigma_\varepsilon \subseteq \Sigma$ the subfamily of Σ of those automorphisms that are defined over C_ε, and are infinitesimal on it.

A differential equation over $\mathfrak{R}$ is, by definition, a locally free $\mathcal{O}_{C_\varepsilon}$-module together with a connection, for some unspecified $\varepsilon > 0$.

On the other hand a Σ-module $\mathscr{F}$ over $\mathfrak{R}$ does not necessarily come from a Σ_ε-module $\mathscr{F}_\varepsilon$ over C_ε. Indeed we lose the action of each $\sigma \in \Sigma - \Sigma_\varepsilon$. The definitions are then the following

Definition 5.1.1. *Let $\mathscr{F}$ be a differential equation over $\mathfrak{R}$, defined over $C_{\varepsilon_{\mathscr{F}}}$ for some $\varepsilon_{\mathscr{F}} > 0$.*

For all $\varepsilon < \varepsilon_{\mathscr{F}}$ let $\Sigma_\varepsilon(\mathscr{F}) \subseteq \Sigma_\varepsilon$ be the subset formed by those $\sigma \in \Sigma_\varepsilon$ such that $\mathscr{F}_\varepsilon$ is σ-compatible over C_ε.[10] *We say that $\mathscr{F}$ is Σ-compatible if for all $\varepsilon > 0$ we have*

$$\bigcup_{0<\varepsilon'<\varepsilon} \Sigma_{\varepsilon'}(\mathscr{F}) \;=\; \Sigma\,. \tag{5.2}$$

Theorem 4.3.1 furnishes an faithful functor associating to a Σ-compatible differential equation, a Σ-module over $\mathfrak{R}$. As usual we call the essential image of Def_Σ *stratified* Σ-modules.

Definition 5.1.2. *Let Σ be a family of infinitesimal automorphisms of $\mathfrak{R}$. We say that Σ is non degenerate if for all $\varepsilon > 0$ there exists $0 < \varepsilon' < \varepsilon$ such that $\Sigma_{\varepsilon'}$ is non degenerate over $C_{\varepsilon'}$.*

Since the family Σ_ε is not necessarily contained in $\Sigma_\varepsilon(\mathscr{F})$, we then proceed as follows:

[10] We recall that this happens if and only if $\mathcal{R}_\emptyset(x_{0,\rho},\sigma) < \mathcal{R}_{\emptyset,1}(x_{0,\rho},\mathscr{F})$, for all $\rho \in]1-\varepsilon,1[$.

Proposition 5.1.3 (Fully faithfulness)**.** *Assume that Σ is a non degenerate family of automorphisms of $\mathfrak{R}$. Let $\{\varepsilon_n\}_{n\geqslant 0}$ be a strictly decreasing sequence of positive non zero real numbers. For all n we consider a sub-family $\Sigma_n\subseteq\Sigma_{\varepsilon_n}$ which is non degenerate over C_{ε_n}.*

We say that a differential equation is $(\Sigma_n)_n$-compatible if for all n one has $\Sigma_n\subseteq\Sigma_{\varepsilon_n}(\mathscr{F})$.

Let $\mathcal{C}$ be the category formed by differential equations $\mathscr{F}$ that are $(\Sigma_n)_n$-compatible.[11] *Then the restriction of the deformation functor to $\mathcal{C}$ is fully faithful.* □

Remark 5.1.4. *This equivalence does not require any additional assumption on the objects as solvability, Frobenius structure, or non Liouville exponents.*

5.2. Solvability and Frobenius structure

We now assume hypothesis 5.0.7. For $\varepsilon>0$ we set $A_\varepsilon:=\{1-\varepsilon<|T|<1+\varepsilon\}$, and $A_0:=\{|T|=1\}$. Similarly as in the case of Robba ring we set $\mathcal{O}^\dagger(A_0):=\cup_{\varepsilon>0}\mathcal{O}(A_\varepsilon)$. Its elements are power series $f=\sum_{i\in\mathbb{Z}}a_iT^i$, $a_i\in K$, such that there is $\varepsilon>0$ such that $\lim_{n\to\pm\infty}|a_i|(1+\varepsilon)^n=0$, and $\lim_{n\to\pm\infty}|a_i|(1-\varepsilon)^n=0$. A differential equation $\mathscr{F}$ over $\mathcal{O}^\dagger(A_0)$ or $\mathfrak{R}$ is called solvable if

$$\lim_{\rho\to 1^-}\mathcal{R}_{\emptyset,1}(x_{0,\rho},\mathscr{F}) \;=\; 1\,. \tag{5.3}$$

We now focus on the Frobenius structure. Let $\phi_K:K\to K$ be a lifting of the p-th power map $x\mapsto x^p$ of the residual field $\widetilde{K}$ of K. Let A be one of the rings $\mathcal{O}^\dagger(A_0)$ or $\mathfrak{R}$. Let $\phi(T)\in A$ be a function such that $x_{0,\rho}(\phi(T)-T^p)<\rho$ for all ρ close to 1. The setting $\sum a_iT^i\mapsto\sum\phi_K(a_i)\phi(T)^i$ is a ring endomorphism of A called a *Frobenius*. We say that a differential equation $\mathscr{F}$ over A has a Frobenius structure of order $n>0$ if there is an isomorphism of differential modules $(\phi^n)^*(\mathscr{F})\xrightarrow{\sim}\mathscr{F}$. A differential equation admitting an unspecified Frobenius structure is solvable, and enjoy several nice properties, one of them is the quasi unipotence that we will explain in the next sections.

5.3. Special extensions

By a result of Katz (cf. [Kat86]), finite separable Galois extensions of $\widetilde{K}(\!(t)\!)$ correspond to the so called *special coverings* of $\mathbb{G}_{m,\widetilde{K}}=\mathrm{Spec}(\widetilde{K}[t,t^{-1}])$. We now recall the definitions of [Mat02].

A finite étale coverings $\widetilde{V}\to\mathbb{G}_{m,\widetilde{K}}$ is *special* if it is tame at 0 and if its geometric monodromy group has a unique p-Sylow subgroup (cf. [Kat86]). One proves that $\widetilde{V}$ is affine. If $\widetilde{\mathrm{B}}$ is its algebra, we say that $\widetilde{\mathrm{B}}/\widetilde{K}[t,t^{-1}]$ is a *special extension*. By the theory of Monsky-Washnitzer (cf. [MW68]), *special extensions* of $\widetilde{K}[t,t^{-1}]$ can be lifted (preserving the Galois group) to the so called Special extensions of $K^\circ[T,T^{-1}]^\dagger$, where

$$K^\circ[T,T^{-1}]^\dagger \;=\; \{f=\sum_{i\in\mathbb{Z}}a_iT^i,\ a_i\in K^\circ,\ \exists\varepsilon>0\ \lim_{n\to\pm\infty}|a_i|(1+\varepsilon)^n=0\ ,\ \lim_{n\to\pm\infty}|a_i|(1-\varepsilon)^n=0\} \tag{5.4}$$

is the Monsky-Washnitzer's weak completion of $K^\circ[T,T^{-1}]$. Special extensions of $K^\circ[T,T^{-1}]^\dagger$ produce (by scalar extension) the so called *Special extensions* of $\mathcal{O}^\dagger(A_0)=K^\circ[T,T^{-1}]^\dagger\otimes_{K^\circ}K$.

The $\mathfrak{R}$-algebras obtained by scalar extension from *Special extensions* of $\mathcal{O}^\dagger(A_0)$ will be called *étale extensions* of $\mathfrak{R}$. We need to introduce the sub-ring of bounded functions in $\mathfrak{R}$:

$$\mathcal{E}^\dagger \;:=\; \{\ f\in\mathfrak{R}\mid \lim_{\rho\to 1^-}x_{0,\rho}(f)<+\infty\ \}\,. \tag{5.5}$$

The ring $\mathcal{E}^\dagger$ has two topologies. The first one arises by restriction from that of $\mathfrak{R}$ (which is a $\mathcal{LF}$ space as inductive limit of the Frechet spaces $\mathcal{O}(C_\varepsilon)$). For this topology $\mathcal{E}^\dagger$ is dense in $\mathfrak{R}$. The second topology on $\mathcal{E}^\dagger$ is given by the Gauss norm $x_{0,1}$, for which $\mathcal{E}^\dagger$ is *not complete*. By the fact that the

[11] It is a full subcategory of the category of all differential equations.

valuation of K is discrete one proves that $(\mathcal{E}^\dagger, x_{0,1})$ is a *Henselian* field with residual field $\widetilde{K}(\!(t)\!)$. One has the following inclusions

$$\mathcal{O}^\dagger(A_0) \subset \mathcal{E}^\dagger \subset \mathfrak{R}\,. \tag{5.6}$$

We have introduced $\mathcal{E}^\dagger$ because it is a field, and because it is an intermediate object between $\mathcal{O}^\dagger(A_0)$ and $\mathfrak{R}$. Special extensions of $\mathcal{O}^\dagger(A_0)$ and $\mathfrak{R}$ correspond bijectively to unramified extensions of $\mathcal{E}^\dagger$. The situation is resumed in the following diagram (for more details we refer to [ADV04], [Mat02]):

$$\begin{array}{ccccc}
\left\{\begin{array}{l}\text{Special}\\ \text{extensions of } \mathcal{O}^\dagger(A_0)\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathcal{E}^\dagger} & \left\{\begin{array}{l}\text{Finite unramified}\\ \text{extensions of } \mathcal{E}^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathfrak{R}} & \left\{\begin{array}{l}\text{Special}\\ \text{extensions of } \mathfrak{R}\end{array}\right\} \\
{\scriptstyle -\otimes K}\uparrow\wr & \odot & \wr\uparrow{\scriptstyle -\otimes K} & & \\
\left\{\begin{array}{l}\text{Special extensions}\\ \text{of } K^\circ[T,T^{-1}]^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes(\mathcal{E}^\dagger)^\circ} & \left\{\begin{array}{l}\text{Finite unramified}\\ \text{extensions of } (\mathcal{E}^\dagger)^\circ\end{array}\right\} & & \\
{\scriptstyle -\otimes\widetilde{K}}\downarrow\wr & \odot & \wr\downarrow{\scriptstyle -\otimes\widetilde{K}} & & \\
\left\{\begin{array}{l}\text{Special}\\ \text{coverings of } \widetilde{K}[t,t^{-1}]\end{array}\right\} & \xrightarrow[\sim]{\text{Pull-back}} & \left\{\begin{array}{l}\text{Finite separable}\\ \text{extensions of } \widetilde{K}(\!(t)\!)\end{array}\right\}. & &
\end{array} \tag{5.7}$$

Hypothesis 5.3.1. *From now on we fix an algebraic closure* $\mathrm{Frac}(\mathfrak{R})^{\mathrm{alg}}$ *of* $\mathrm{Frac}(\mathfrak{R})$ *and we consider only Special (resp. unramified, étale) extension of* $\mathcal{O}^\dagger(A_0)$*, (resp.* $\mathcal{E}^\dagger$*,* $\mathfrak{R}$*) inside it.*

Remark 5.3.2. *An unramified extension of* $\mathcal{E}^\dagger$ *is again a ring of power series of the same type as those in* $\mathcal{E}^\dagger$ *with respect to another variable, and another base field* L *which is a finite unramified extension of* K *(cf. [Mat02]). The same holds for special extensions of* $\mathfrak{R}$*.*

The results we are going to use hold after replacing the base field K by a finite unspecified extension L/K. For any K-algebra $\mathcal{B}$ and all finite field extension L/K we set

$$\mathcal{B}_L := \mathcal{B}\otimes_K L\,, \qquad \mathcal{B}_{K^{\mathrm{alg}}} := \bigcup_{L/K \text{ finite}} \mathcal{B}_L\,. \tag{5.8}$$

If $\mathcal{B}$ is one of the above differential rings $\mathcal{O}^\dagger(A_0)$, $\mathcal{E}^\dagger$, or $\mathfrak{R}$, then a differential module $\mathscr{F}$ over $\mathcal{B}_{K^{\mathrm{alg}}}$ comes by scalar extension from a differential module over $\mathcal{B}_L$ for some finite extension L/K. So, by deformation, the same holds for *stratified* Σ-modules.

5.3.1. Deformation over $\mathcal{O}^\dagger(A_0)$.

Below we work with differential equations and Σ-modules over the rings $\mathfrak{R}$, $\mathcal{E}^\dagger$, $\mathcal{O}^\dagger(A_0)$. Since we need to interchange the base ring, moving along the first line of (5.7), we fix once for all a family of infinitesimal automorphisms of $\mathcal{O}^\dagger(A_0)$. The definition of infinitesimal automorphisms of $\mathcal{O}^\dagger(A_0)$, and related ones, are obtained imitating the definitions of Section 5.1, by replacing everywhere C_ε by A_ε. We only notice that a differential equation $\mathscr{F}$ over $\mathcal{O}^\dagger(A_0)$ is Σ-compatible if and only if for all $\sigma\in\Sigma$ we have $\mathcal{R}_\emptyset(x_{0,1},\sigma) < \mathcal{R}_{\emptyset,1}(x_{0,1},\mathscr{F})$. Indeed by continuity the inequality remains true over some unspecified segment $]x_{0,1-\varepsilon}, x_{0,1+\varepsilon}[$ of A_ε.

Remark 5.3.3. *An infinitesimal automorphism of* $\mathfrak{R}$ *naturally acts on* $\mathcal{E}^\dagger$*. Indeed it induces an automorphism of* C_ε*, so the composition of* σ *with a bounded functions on* C_ε *remains bounded.*

If an automorphism σ *of* $\mathcal{O}^\dagger(A_0)$ *is infinitesimal, then it is also an infinitesimal automorphism of* $\mathfrak{R}$ *(cf. Lemma 3.0.3), and hence of* $\mathcal{E}^\dagger$*.*

If $\Sigma \subseteq \mathrm{Aut}(\mathcal{O}^\dagger(A_0))$ *is non degenerate as a family of automorphisms of* $\mathfrak{R}$*, then it is so also as a family of automorphisms of* $\mathcal{O}^\dagger(A_0)$*. The converse is unclear. We pay attention to the fact that it* *does not seem automatic that* non degeneracy *translates as well from* $\mathcal{O}^\dagger(A_0)$ *to* $\mathfrak{R}$.

5.3.2. Extension of Σ to Special extensions. Let $\mathcal{B}$ be one of the rings $\mathcal{O}^\dagger(A_0)$, $\mathcal{E}^\dagger$, $\mathfrak{R}$. Let σ be an infinitesimal automorphism of $\mathcal{O}^\dagger(A_0)$. We will need to apply σ to the formal symbol $\log(T)$. For this we write $\sigma(T)/T = 1 + \frac{\sigma(T)-T}{T}$. Since σ is infinitesimal $x_{1,1}((\sigma(T)-T)/T) = x_{1,1}(\delta_\sigma(T)/T) < 1$. Then $\sigma(T)/T$ takes values in the disc $\mathrm{D}^-(1,1)$, and the composite function $\log(\sigma(T)/T)$ converges in the annulus of definition of $\sigma(T)$. We are then allowed to define it as

$$\log(\sigma(T)) \;:=\; \log(T) + \log(\sigma(T)/T)\;, \quad \text{with } \log(\sigma(T)/T) \in \mathcal{B}\;. \tag{5.9}$$

The action of Σ on Special extensions is described by the following

Lemma 5.3.4. *Let σ be an infinitesimal automorphism of $\mathcal{O}^\dagger(A_0)$, let $\widetilde{\mathrm{B}}/\widetilde{K}[t,t^{-1}]$ be a Galois Special covering, and let $\mathrm{B}/\mathcal{O}^\dagger(A_0)$ (resp. $\mathrm{B}\otimes_{\mathcal{O}^\dagger(A_0)}\mathcal{E}^\dagger/\mathcal{E}^\dagger$) be the corresponding Special (resp. unramified) extension.*

Then σ extends uniquely, up to composition with Galois automorphisms of $\mathrm{Gal}(\mathrm{B}/\mathcal{O}^\dagger(A_0)) \xrightarrow{\sim} \mathrm{Gal}(\widetilde{\mathrm{B}}/\widetilde{K}[t,t^{-1}])$, to a continuous automorphism of $\mathrm{B}/\mathcal{O}^\dagger(A_0)$ and to $\mathrm{B}[\log(T)]$ as in (5.9).

In particular there exists a unique extension of σ inducing the identity on the residual ring $\widetilde{\mathrm{B}}$ of $\mathcal{O}^\dagger(A_0)$. By uniqueness σ commutes with the action of the Galois group.

The same holds for $(\mathrm{B}\otimes_{\mathcal{O}^\dagger(A_0)}\mathcal{E}^\dagger)/\mathcal{E}^\dagger$.

Proof. It follows from the formal properties of the Henselian couples [Ray70]. □

5.4. Katz-Matsuda's canonical extension

As above we assume hypothesis 5.0.7. Now we show how to obtain the analogues of the results of [Mat02] about the canonical extension by Σ-deformation.

Notation 5.4.1. *For any ring with derivation $d : \mathcal{B} \to \mathcal{B}$, we denote by $d - \mathrm{Mod}(\mathcal{B})$ the category of locally free $\mathcal{B}$-modules of finite type $\mathscr{F}$ together with a connection $\nabla : \mathscr{F} \to \mathscr{F}$ satisfying the Leibnitz rule with respect to d.*

If a Frobenius $\phi : \mathcal{B} \to \mathcal{B}$ is given, we denote by $d - \mathrm{Mod}(\mathcal{B})^{(\phi)}$ the full subcategory formed by those $\mathscr{F}$ admitting an isomorphism $\phi^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ commuting with the connections (morphisms are not supposed to commute with the Frobenius).

If $\mathcal{C}(\mathcal{B})$ is one of the above categories, by $\mathcal{C}(\mathcal{B}_{K^{\mathrm{alg}}})$ we mean the inductive limit $\mathcal{C}(\mathcal{B}_{K^{\mathrm{alg}}}) := \varinjlim_{L/K}\mathcal{C}(\mathcal{B}_L)$, where L/K runs in the set of finite extensions of K.

We quickly recall the context. In [Kat86] and [Kat87] Katz proved that, if F is an arbitrary field of characteristic 0, each formal differential equation over $F(\!(T)\!)$ comes by scalar extension from a differential equation over $F[T,T^{-1}]$. More precisely there exists a full sub-category of $d/dT - \mathrm{Mod}(F[T,T^{-1}])$, formed by the so called Special objects, which is equivalent to $d/dT - \mathrm{Mod}(F(\!(T)\!))$ via the scalar extension functor $d/dT - \mathrm{Mod}(F[T,T^{-1}]) \to d/dT - \mathrm{Mod}(F(\!(T)\!))$. The section of the scalar extension functor so obtained is called *canonical extension*:

$$\mathrm{Can}\;:\; d/dT - \mathrm{Mod}(F(\!(T)\!)) \;\longrightarrow\; d/dT - \mathrm{Mod}(F[T,T^{-1}])\;. \tag{5.10}$$

Now the rings $\mathcal{O}^\dagger(A_0)$ and $\mathfrak{R}$ are considered as natural liftings in characteristic 0 of $\widetilde{K}[t,t^{-1}]$ and $\widetilde{K}(\!(t)\!)$ respectively. Differential equations over $\mathcal{O}^\dagger(A_0)$ and $\mathfrak{R}$ with (unspecified) Frobenoius structure are considered as p-adic analogues of the Katz's context. Along this analogy S.Matsuda (cf. [Mat02]) shows the p-adic analogue of the above result for quasi-unipotent differential modules.

Definition 5.4.2. *Let $\mathcal{B}$ be one of the rings $\mathcal{O}^\dagger(A_0)$, $\mathcal{E}^\dagger$, $\mathfrak{R}$.*

We denote by U_n the free $\mathcal{B}$-module of rank n with connection $\nabla^{U_n} : U_n \to U_n$ given in the basis $e_1, \ldots, e_n$ by $\nabla^{U_n}(e_i) = T^{-1}e_{i+1}$ for all $i = 1, \ldots, n$ and $\nabla^{U_n}(e_n) = 0$. A differential module $\mathscr{F}$ over $\mathcal{B}$ is called unipotent *if it is isomorphic to a direct sum of modules of type U_n.*

$\mathscr{F}$ is called quasi-unipotent *if its scalar extension to an unspecified special extension* B *of $\mathcal{B}$ is isomorphic to $U \otimes_{\mathcal{B}} \mathrm{B}$, where U is unipotent over $\mathcal{B}$.*

The result of S. Matsuda states the analogous of (5.10) for quasi-unipotent differential modules with unspecified Frobenius structure. By the p-adic local monodromy theorem (cf. [And02, Section 7.3], [Ked04], [Meb02]) each differential module in $d/dT - \mathrm{Mod}(\mathfrak{R})^{(\phi)}$ becomes quasi-unipotent after an unspecified scalar extension of the ground field K. Putting these two results together we obtain that, the scalar extension functor $d/dT - \mathrm{Mod}(\mathcal{O}^\dagger(A_0)_{K^{\mathrm{alg}}})^{(\phi)} \to d/dT - \mathrm{Mod}(\mathfrak{R}_{K^{\mathrm{alg}}})^{(\phi)}$ admits a section called *canonical extension*

$$\mathrm{Can} \quad : \quad d/dT - \mathrm{Mod}(\mathfrak{R}_{K^{\mathrm{alg}}})^{(\phi)} \ \to \ d/dT - \mathrm{Mod}(\mathcal{O}^\dagger(A_0)_{K^{\mathrm{alg}}})^{(\phi)} \ , \tag{5.11}$$

which is an equivalence of categories with its essential image (cf. [And02, 7.3]). In particular, this implies that after base change to some finite extension L/K all differential module with Frobenius structure over $\mathfrak{R}$ admits a basis in which the matrix of the connection lies in $M_n(\mathcal{O}^\dagger(A_0))$.

5.4.1. Deformation of the canonical extension. We now resume the straightforward consequence of the above results.

Let Σ be a family of infinitesimal operators of $\mathcal{O}^\dagger(A_0)$. For $\mathcal{B} = \mathcal{O}^\dagger(A_0)$, or $\mathcal{B} = \mathfrak{R}$ let $d/dT - \mathrm{Mod}(\mathcal{B})^{(\phi),\mathrm{comp}(\Sigma)}$ be the full sub-category of $d/dT - \mathrm{Mod}(\mathcal{B})^{(\phi)}$ formed by Σ-compatible differential equations over $\mathcal{B}$ (cf. (5.2)). This is also a full-sub-category of the category $d/dT - \mathrm{Mod}(\mathcal{B})^{\mathrm{comp}(\Sigma)}$ of Σ-compatible differential equations, hence the scalar extension functor $d/dT - \mathrm{Mod}(\mathcal{O}^\dagger(A_0))^{(\phi),\mathrm{comp}(\Sigma)} \to d/dT - \mathrm{Mod}(\mathfrak{R})^{(\phi),\mathrm{comp}(\Sigma)}$ commutes with the deformation functors. As a consequence the canonical extension functor also commutes with the deformations as soon as the deformations are equivalences. Namely assume that Σ is a family of infinitesimal automorphisms of $\mathcal{O}^\dagger(A_0)$ which is non degenerate as a family of automorphisms of $\mathfrak{R}$. Fix a sequence $(\Sigma_n)_n$, as in Proposition 5.1.3, where Σ_n is non degenerate over C_{ε_n}. We have a commutative diagram:

$$\begin{array}{ccc} d/dT - \mathrm{Mod}(\mathcal{O}^\dagger(A_0))^{(\phi),\mathrm{comp}(\Sigma_n)_n} & \xrightarrow{\ \mathrm{Can}\ } & d/dT - \mathrm{Mod}(\mathfrak{R})^{(\phi),\mathrm{comp}(\Sigma_n)_n} \\ \mathrm{Def}_\Sigma \downarrow \wr & & \wr \downarrow \mathrm{Def}_\Sigma \\ (\Sigma_n)_n - \mathrm{Mod}(\mathcal{O}^\dagger(A_0))^{(\phi),\mathrm{strat}} & \dashrightarrow^{\mathrm{Can}} & (\Sigma_n)_n - \mathrm{Mod}(\mathfrak{R})^{(\phi),\mathrm{strat}} \end{array} \tag{5.12}$$

where, for $\mathcal{B} = \mathcal{O}^\dagger(A_0), \mathfrak{R}$, we denote by $d/dT - \mathrm{Mod}(\mathcal{B})^{(\phi),\mathrm{comp}(\Sigma_n)_n}$ the full subcategory formed by $(\Sigma_n)_n$-compatible differential equations, and by $(\Sigma_n)_n - \mathrm{Mod}(\mathcal{B})^{(\phi),\mathrm{strat}}$ we denote the corresponding category of stratified $(\Sigma_n)_n$-modules (i.e. its essential image by Deformation).

5.5. p-adic local monodromy Theorem

We maintain the assumption 5.0.7. In this section we prove the p-adic local monodromy theorem for *stratified* $(\Sigma_n)_n$-modules over $\mathfrak{R}$.

Setting 5.5.1. *Let ϕ be a Frobenius of $\mathcal{O}^\dagger(A_0)$, and let Σ be a non degenerate family of infinitesimal automorphisms of $\mathcal{O}^\dagger(A_0)$ which is non degenerate as a family of infinitesimal automorphisms of $\mathfrak{R}$. Fix a sequence $(\Sigma_n)_n$, as in Proposition 5.1.3, where Σ_n is non degenerate over C_{ε_n}.*

Theorem 5.5.2 (p-adic local monodromy theorem for stratified Σ-modules)**.** *Each stratified $(\Sigma_n)_n$-module over $\mathfrak{R}$ admitting an (unspecified) Frobenius structure becomes quasi-unipotent after a base* *change to $\mathfrak{R}_L$, for some unspecified finite extension L/K.*

Proof. The claim says that each object $(\mathscr{F},\Sigma)$ in $(\Sigma_n)_n - \mathrm{Mod}(\mathfrak{R}_{K^{\mathrm{alg}}})^{(\phi),\mathrm{strat}}$ is trivialized by $\mathfrak{R}'[\log(T)]$, where $\mathfrak{R}'/\mathfrak{R}_L$ is some Special extension of $\mathfrak{R}_L$. This means that $(\mathscr{F},\Sigma)$ has a complete basis of solutions in $\mathfrak{R}'[\log(T)]$. We know that deformation preserves Taylor solutions (cf. Remark 4.3.2), the strategy is to prove that deformation also preserves *étale* solutions, i.e. solutions in some $\mathfrak{R}'[\log(T)]$.

By [And02, Cor.7.1.6], up to enlargements of K, every differential module $\mathscr{F}$ with Frobenius structure over $\mathfrak{R}$ is a direct sum of sub-modules of the form $\mathrm{N}\otimes\mathrm{U}_m$ where N is trivialized by an étale extension $\mathfrak{R}'/\mathfrak{R}$ (without logarithm) of $\mathfrak{R}$, and $(\mathrm{U}_m,\nabla^{\mathrm{U}_m})$ is the m-dimensional unipotent differential module over $\mathfrak{R}$ (cf. Def. 5.4.2). Since the deformation equivalence preserves this decomposition, we can assume $\mathscr{F}=\mathrm{N}$ or $\mathscr{F}=\mathrm{U}_m$. If we are in the first case $\mathscr{F}=\mathrm{N}$, we will say that $\mathscr{F}$ has *finite local monodromy.*

We first prove the result for U_m. It is well known that U_m is trivialized by $\mathcal{O}^\dagger(A_0)[\log(T)]$, where $\log(T)$ is an indeterminate (i.e. merely a symbol). We now consider $\log(T)$ as a function over the disc $D^-(1,1)$. It is not algebraic over $\mathcal{O}^\dagger(A_0)$, so the restriction map

$$\mathcal{O}^\dagger(A_0)[\log(T)] \;\longrightarrow\; \mathcal{O}(D^-(1,1)) \tag{5.13}$$

is an injective ring morphism commuting with d/dT, and Σ. This map identifies Taylor solutions of U_m at $T=1$ with "*abstract*" solutions of U_m in $\mathcal{O}^\dagger(A_0)[\log(T)]$. Since on the right hand side these solutions are simultaneously solutions of the differential equation U_m and of its deformation $\mathrm{Def}_\Sigma(\mathrm{U}_m)$ (cf. Remark 4.3.2), the same holds on the left hand side. Hence $\mathrm{Def}_\Sigma(\mathrm{U}_m)$ is trivialized by $\mathcal{O}^\dagger(A_0)[\log(T)]$.

We now focus on differential modules N with finite local monodromy. As for U_m we now embed the Special extension trivializing N into $\mathcal{O}(D^-(1,1))$ and we will compare Taylor solutions with étale solutions as above.

Up to enlarge K, we may assume that (N,∇) is an n-dimensional differential module over $\mathfrak{R}$ trivialized by some étale extension $\mathfrak{R}'/\mathfrak{R}$. Let $\mathrm{B}/\mathcal{O}^\dagger(A_0)$ be the corresponding Special extension of $\mathcal{O}^\dagger(A_0)$. By canonical extension (cf. [Mat02, Cor. 5.12]) N comes, by scalar extension, from a differential module $\mathrm{N}_0 := \mathrm{Can}(\mathrm{N})$ over $\mathcal{O}^\dagger(A_0)$ which is trivialized by B. Let $Y_{\mathrm{B}}\in GL_n(\mathrm{B})$ be a complete basis of solutions of N_0 with values in B.

For all $\sigma\in\Sigma$, let $\sigma_{\mathrm{B}}:\mathrm{B}\to\mathrm{B}$ be the corresponding endo-morphism of B (cf. Lemma 5.3.4). We define the matrix $A_{\sigma_{\mathrm{B}}}$ of the action of σ_{B} by

$$\sigma_{\mathrm{B}}(Y_{\mathrm{B}}) \;=\; A_{\sigma_{\mathrm{B}}}\cdot Y_{\mathrm{B}}\;. \tag{5.14}$$

Since Y_{B} is invertible, so does $A_{\sigma_{\mathrm{B}}}=\sigma_{\mathrm{B}}(Y_{\mathrm{B}})\cdot Y_{\mathrm{B}}^{-1}$. Moreover $A_{\sigma_{\mathrm{B}}}\in GL_n(\mathcal{O}^\dagger(A_0))$ because the Galois group commutes with the unique extension of d/dT to B, so it acts on Y_{B} by right multiplication by matrices in $GL_n(K)$. Hence, since the Galois group also commutes with σ_{B}, $A_{\sigma_{\mathrm{B}}}$ is stable by Galois.

Now we denote by $A_\sigma\in GL_n(\mathcal{O}^\dagger(A_0))$ the matrix of the action of σ on N_0 obtained by deformation of ∇. Namely let $x\in A_0$ be any rational point, and let $D\subseteq A_0$ be the open disc with radius 1, centered at x. Since N_0 is Σ-compatible, (N_0,∇) is trivial on D. If $Y\in GL_n(D)$ is a Taylor solution matrix of N_0, then A_σ is defined by

$$\sigma(Y) \;=\; A_\sigma\cdot Y\;. \tag{5.15}$$

Consider now the reduction $\widetilde{x}$ of x in $\mathbb{G}_{m,\widetilde{K}}$. If $\widetilde{V}\to\mathbb{G}_{m,\widetilde{K}}$ is the Special covering corresponding to B, then its fiber at $\widetilde{x}$ is a finite étale covering of $\widetilde{x}$. Up to replacing K by a finite extension, this is given by a trivial covering of finite degree d. Let $\widetilde{y}_1,\ldots,\widetilde{y}_d$ be the points of $\widetilde{V}$ over $\widetilde{x}$.

By [BL85, Prop. 2.2, (i)] each $\widetilde{y}_i \in \widetilde{V}$ lifts into an open disc D_i contained in the dagger affinoid $V^\dagger$ corresponding to B. Since the morphisms $\psi : V^\dagger \to \mathbb{G}^\dagger_{m,K}$ is finite with same degree d, it induces a trivial covering over D. In particular ψ induces an isomorphism $\psi : D_i \xrightarrow{\sim} D$ for all D.

Now σ_{B} induces an automorphism of each D_i because the reduction of σ_{B} is the identity on $\widetilde{\mathrm{B}}$. Since σ_{B} lifts σ, then $\psi : D_i \xrightarrow{\sim} D$ commutes with σ_{B} and σ. Moreover if we identify in this way D_i with D, then $Y = Y_{\mathrm{B}} \cdot C$, for some $C \in GL_n(K)$, by the uniqueness of Taylor solutions. Hence

$$A_{\sigma_{\mathrm{B}}} \;=\; A_\sigma \;. \tag{5.16}$$

Now the Taylor solutions of a differential equation are also solutions of its Σ-deformation by Remark 4.3.2. The base change by the matrix Y in $(\mathrm{N}_0)_{|D}$ trivializes the differential equation over D, and hence simultaneously all the actions of σ obtained by deformation. In the new basis we find $A_\sigma = \mathrm{Id}$ for all $\sigma \in \Sigma$.

We now look at $\psi^*\mathrm{N}_0 := \mathrm{N}_0 \otimes_{\mathcal{O}^\dagger(A_0)} \mathrm{B}$ over $V^\dagger$. The entries of Y_{B} are global sections on $V^\dagger$ that coincide with Y over D_i. Since $A_{\sigma_{\mathrm{B}}} = A_\sigma$, the base change by Y_{B} (which trivializes the differential equation) gives a new matrix $A'_{\sigma_{\mathrm{B}}}$ which is the identity over D. By analytic continuation, the matrix $A'_{\sigma_{\mathrm{B}}}$ is the identity everywhere over $V^\dagger$, so this base change trivializes the entire action of Σ over $V^\dagger$. □

The proof shows in particular the following result:

Corollary 5.5.3. *Let* M *be a differential module over* $\mathcal{O}^\dagger(A_0)$ *with unspecified Frobenius structure. Assume that* M *is trivialized by some Special extension of* $\mathcal{O}^\dagger(A_0)_L$*, for some finite extension* L/K*. Then* M *has bounded Taylor solutions on each disc* $D \subset A_0$*.*

Proof. Indeed Y_{B} is the restriction of a global solution over $V^\dagger$, so it is bounded on each D_i. □

Such differential modules are unit-root by [Mat02]. The fact that a unit root differential modules has bounded solutions is a well known result, at least since [Kat73] (see also [CT09], [CT11]).

The proof also gives the following nice result, that could be helpful to work explicitly with Special extensions of $\mathcal{O}^\dagger(A_0)$:

Proposition 5.5.4. *If* B *is a Special extension of* $\mathcal{O}^\dagger(A_0)$*, there exists an injective ring morphism*

$$\mathrm{Tay}_1 \;:\; \mathrm{B}_{K^{\mathrm{alg}}} \;\longrightarrow\; \mathcal{O}(D^-(1,1))_{K^{\mathrm{alg}}} \tag{5.17}$$

commuting with d/dT*, the Frobenius, and* Σ*.*

Proof. With the notations of the proof of Theorem 5.5.2, if $D = D^-(1,1)$, we first consider the restriction from B to $\mathcal{O}(D_i)$, and then we apply the pull-back by $\psi^{-1} : D \xrightarrow{\sim} D_i$. □

As a last result, we give the following converse of Theorem 5.5.2, which is a characterization of the category of stratified $(\Sigma_n)_n$-modules:

Corollary 5.5.5. *We preserve the assumption 5.5.1. Let* $\mathscr{F}$ *be a* $(\Sigma_n)_n$*-compatible differential equation over* $\mathfrak{R}$*, together with an action of* Σ*. Assume that there exists a finite extension* L/K *and an étale extension* $\mathfrak{R}'/\mathfrak{R}_L$ *such that* $\mathscr{F} \otimes_{\mathfrak{R}} \mathfrak{R}'[\log(T)]$ *has a basis on which the connection and the action of* Σ *are both trivial. Then the action of* Σ *coincides with that obtained by deformation from* ∇*.*

Proof. By Theorem 5.5.2, the action of Σ obtained by deformation becomes trivial in the same basis trivializing the connection. So the two actions of Σ coincide after base change, hence they were equal before base change. □

6. Difference equations over the affine line.

We now investigate a particular class of automorphism, those of the form $\sigma_{q,h}(T) = qT + h$, for some $q, h \in K$. If $q \neq 0$, this is an automorphism of $\mathbb{A}_K^{1,\mathrm{an}}$ with inverse $\sigma_{q^{-1},-q^{-1}h}$.

If $q = 1$ we say that $\sigma_{1,h}$ is a finite difference operator, and if $h = 0$ we say that $\sigma_{q,0}$ is a q-difference operator. In general we say that $\sigma_{q,h}$ is a difference operator.

6.1. S-infinitesimality of $\sigma_{q,h}$.

In general, for $q_1, q_2 \neq 0$, we have

$$\sigma_{q_1,h_1} \circ \sigma_{q_2,h_2} \;=\; \sigma_{q_1q_2,q_2h_1+h_2}\ . \tag{6.1}$$

We can define a group operation on $\mathcal{G} := \mathbb{G}_m^{\mathrm{an}} \times \mathbb{A}_K^{1,\mathrm{an}}$ by $(q_1,h_1)(q_2,h_2) := (q_1q_2, q_2h_1+h_2)$. Since the operation are given by polynomials, this is a K-analytic group. Also the action $\mathcal{G}\times\mathbb{A}_K^{1,\mathrm{an}} \to \mathbb{A}_K^{1,\mathrm{an}}$ given by $((q,h);T) \mapsto qT+h$ is given by polynomials, so it is a morphism of K-analytic spaces as in Section 4.5. If $q \neq 1$, $\sigma_{q,h}$ has a unique fixed rigid point which is

$$a \;:=\; -h/(q-1)\ . \tag{6.2}$$

Moreover by a translation sending a into 0, $\sigma_{q,h}$ become just the multiplication by q: $\sigma_{q,h}(T-a) = q(T-a)$. This often permits to reduce to the case where $h = 0$. We then deduce the following

Lemma 6.1.1. *$\sigma_{q,h}$ extends to an automorphism of $\mathbb{P}_K^{1,\mathrm{an}}$. If $q = 1$, then $+\infty$ is its unique fixed rational point in $\mathbb{P}_K^{1,\mathrm{an}}(K)$. If $q \neq 1$, then a and $+\infty$ are its unique fixed points in $\mathbb{P}_K^{1,\mathrm{an}}(K)$.* □

Lemma 6.1.2 (Disks that are stable under $\sigma_{q,h}$)**.** *The following hold:*

- i) *If $|q| \neq 1$, then the restriction of $\sigma_{q,h}$ to a disc in $\mathbb{P}_\Omega^{1,\mathrm{an}}$ is never an automorphism of the disc. In particular $\sigma_{q,h}$ is not isometric.*
- ii) *If $|q| = 1$, and if $|q-1| = 1$, then the family formed by the open/closed discs $D^{\pm}(a,\rho)$, $\rho \geqslant 0$, $a = -h/(q-1)$, and by their complements in $\mathbb{P}_\Omega^{1,\mathrm{an}}$, is the unique family of (open or closed) discs in $\mathbb{P}_\Omega^{1,\mathrm{an}}$ on which $\sigma_{q,h}$ induces an automorphism.*
- iii) *If $|q-1| < 1$, the family of discs in $\mathbb{P}_\Omega^{1,\mathrm{an}}$ on which $\sigma_{q,h}$ induces an automorphism is formed by the discs $D^-(c,\rho) \subseteq \mathbb{A}_\Omega^{1,\mathrm{an}}$ (resp. $D^+(c,\rho) \subseteq \mathbb{A}_\Omega^{1,\mathrm{an}}$) satisfying*
$$|\sigma_{q,h}(c) - c| \;<\; \rho \qquad (\textit{resp. } |\sigma_{q,h}(c) - c| \;\leqslant\; \rho)\ , \tag{6.3}$$
and by their complements in $\mathbb{P}_\Omega^{1,\mathrm{an}}$.
- iv) *In particular, if D is a virtual open (resp. closed) disc with boundary x which is stable under $\sigma_{q,h}$, then each virtual open disc with boundary in $[x,+\infty[$ (resp. $]x,+\infty[$) is stable by $\sigma_{q,h}$.* □

In the situation of point ii) of the Lemma 6.1.2, the unique differential equation which is $\sigma_{q,h}$-compatible over a discs centered at a, or over its complement, is the trivial one. So this case is not interesting from the point of view of this paper.

Hypothesis 6.1.3. *From now on we assume $|q-1| < 1$. In particular, if $q \neq 1$, the absolute value of K is not trivial.*

Proposition 6.1.4. *Let $X \subseteq \mathbb{P}_K^{1,\mathrm{an}}$ be a connected analytic domain distinct from $\mathbb{P}_K^{1,\mathrm{an}}$, and $\mathbb{P}_K^{1,\mathrm{an}} - \{t\}$ for any point $t \in \mathbb{P}_K^{1,\mathrm{an}}$ of type 1 or 4. Then:*

- i) *The analytic skeleton Γ_X of X is the skeleton of a (not unique) weak triangulation (cf. Section 1.1). Each other weak triangulation S of X verifies $\Gamma_X \subseteq \Gamma_S$.*

ii) *Let Y be a connected component of X_Ω. Then each connected component of the complement of Y in $\mathbb{P}^{1,\mathrm{an}}_\Omega$ is either an open or closed disc, or it is reduced to a point x such that $x \in \overline{\Gamma_Y} - \Gamma_Y$, where $\overline{\Gamma_Y}$ is the closure of Γ_Y in $\mathbb{P}^{1,\mathrm{an}}_K$. In particular x is an end point of $\overline{\Gamma_X}$ of type 1 or 4.*

iii) *$\sigma_{q,h}$ induces an automorphism of X if and only if it induces an automorphism of the complement of X_Ω in $\mathbb{P}^{1,\mathrm{an}}_\Omega$. Moreover, if S is a weak triangulation of X, the action of $\sigma_{q,h}$ on X is S-infinitesimal if and only if for all connected component Y of X_Ω the following hold*

(a) *$\sigma_{q,h}$ induces an automorphism of each connected component of $\mathbb{P}^{1,\mathrm{an}}_\Omega - Y$;*

(b) *If D is a virtual open disc in X such that $S \cap D \neq \emptyset$, then for each point $x \in S_\Omega \cap D_\Omega$ which is an end-point of Γ_{S_Ω} there exists an open disc $D_x \subset D_\Omega$ with boundary x which is globally fixed by $(\sigma_{q,h})_\Omega$.* □

6.2. Non degeneracy and fully faithfulness

6.2.1. q-Taylor expansion. For all natural number $n \geqslant 1$ we set

$$[n]_q \ := \ 1 + q + q^2 + \cdots + q^{n-1} \ , \qquad [n]_q^! \ := \ [1]_q[2]_q[3]_q \cdots [n]_q \ . \tag{6.4}$$

For all $c \in \Omega$ we set $(T-c)^{[0]}_{q,h} = 1$ and, for all $n \geqslant 1$ we set

$$(T-c)^{[n]}_{q,h} \ := \ (T-c)(T-\sigma_{q,h}(c))(T-\sigma^2_{q,h}(c)) \cdots (T-\sigma^{n-1}_{q,h}(c)) \ . \tag{6.5}$$

We define the twisted (q,h)-derivative as $d_{q,h}(f) := \frac{f\circ\sigma_{q,h}-f}{(q-1)T+h}$. In particular $d_{q,h}$ is a K-linear map satisfying $d_{q,h}((T-c)^{[n]}_{q,h}) = [n]_q(T-c)^{[n-1]}_{q,h}$, for all $n \geqslant 1$. The denominator of $d_{q,h}$ has a zero. The following Proposition shows, in particular, that $d_{q,h}$ is well defined around that zero.

Proposition 6.2.1. *Let $D^-(c,R) \subset \mathbb{A}^{1,\mathrm{an}}_\Omega$ be a disc invariant under $\sigma_{q,h}$. Let $f(T) := \sum_{n\geqslant 0} a_i(T-c)^i \in \mathcal{O}(D^-(c,R))$. Then*

i) *$f(T)$ can be uniquely written as $f(T) = \sum_{n\geqslant 0} \widetilde{a}_n (T-c)^{[n]}_{q,h} \in \mathcal{O}(D^-(c,R))$. In particular, if $q \neq 1$, $d_{q,h}$ is defined around $a = -h/(q-1)$;*

ii) *For all ρ satisfying $|(q-1)c+h| < \rho < R$ one has $|f|(x_{c,\rho}) := \sup_{n\geqslant 0}|a_n|\rho^n = \sup_{n\geqslant 0}|\widetilde{a}_n|\rho^n$;*

iii) *The radius of convergence of f at c is given by the formula:*

$$\mathit{Radius}(f(T)) \ := \ \liminf_n |a_n|^{-1/n} \ = \ \liminf_n |\widetilde{a}_n|^{-1/n} \ . \tag{6.6}$$

iv) *If $q = 1$ assume that $h \neq 0$, if $q \neq 1$ assume that q is not a root of unity. Then one has the (q,h)-Taylor expansion formula $f(T) = \sum_{n\geqslant 0} d^n_{q,h}(f)(c) \cdot \frac{(T-c)^{[n]}_{q,h}}{[n]^!_q}$.*

Proof. The proof follows closely [Pul08, Lemma 5.3], we omit it for expository reasons. □

Corollary 6.2.2 (Non degeneracy of $\sigma_{q,h}$)**.** *Let X be an analytic domain of $\mathbb{P}^{1,\mathrm{an}}_K$, together with a weak triangulation S, such that $\sigma_{q,h}$ acts S-infinitesimally on X. Assume that X is not an open disc centered at ∞ with empty weak triangulation. If $q = 1$ assume that $h \neq 0$, and if $q \neq 1$ assume that q is not a root of unity. Then the action of $\sigma_{q,h}$ is non degenerate on X.*

Under the same assumptions, $\sigma_{q,h}$ is non degenerate if we replace X by the Robba ring.

Proof. By the assumption there exists a point $x \in X$ such that $\infty \notin D(x,S)$. Let $D \subseteq X_\Omega$ be an open disc such that $D^+(x,\sigma) \subset D \subseteq D(x,S)$. We can write each $f \in \mathcal{O}(D)$ as $f(T) = \sum_{n\geqslant 0} d^n_{q,h}(f)(c) \cdot \frac{(T-c)^{[n]}_{q,h}}{[n]^!_q}$. Now $\sigma_{q,h}(f) = f$ means $d_{q,h}(f) = 0$, which holds if and only if f is constant. □

Remark 6.2.3. *The action of the group $\boldsymbol{\mu}_{p^n}$ of p^n-th roots of unity, and also of $\boldsymbol{\mu}_{p^\infty} = \cup_n \boldsymbol{\mu}_{p^n}$, are always degenerate. This is due to the existence of the function $\ell_x(T) := \log(T/t_x)$, for all $x \in \mathbb{A}^{1,\mathrm{an}}_K$, that verifies $\ell_x(qT) = \ell_x(T)$ for all $q \in \boldsymbol{\mu}_{p^\infty}$. So (4.15) can not occur.*

6.3. Confluence.

In this section we give a characterization of the essential image of the deformation functor $\mathrm{Def}_{\sigma_{q,h}}$, and we define a quasi inverse functor *confluence*. For this we need to fix a global coordinate on X, so we are induced to make the following assumption:

Hypothesis 6.3.1. *From now on we assume that X is an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_K$.*

6.3.1. (q,h)-Taylor solution. Let X be an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_K$, with a weak triangulation S, and an S-infinitesimal action of $\sigma_{q,h}$. We now give a criterion for a $\sigma_{q,h}$-difference module $(\mathscr{F}, \sigma_{q,h})$ to be the deformation of a differential equation ∇, under the assumptions of non degeneracy of Corollary 6.2.2. We will need to use the action of $d_{q,h}$ on $\mathscr{F}$ (cf. (6.7)), this introduces a pole at $a = -h/(q-1)$ (cf. (6.2)), so we assume that $a \notin X$ (i.e. that $\sigma_{q,h}$ has no fixed points in X).

In this situation we can always find a Γ_S-covering of X formed by quasi-Stein analytic domains of X on which $\mathscr{F}$ and Ω^1_X are free. By Corollary 6.2.2 if $\sigma_{q,h}$ is non degenerate on X, it is so on each open of the covering. So, by fully-faithfulness, the differential equation will be unique on the intersections. So the local pieces glue to a global differential equation over X.[12]

Hypothesis 6.3.2. *We assume that $\mathscr{F}$ and Ω^1_X are free, that X is quasi-Stein[13], and that $a \notin X$.*

With this assumption the action of $\sigma_{q,h}$ corresponds (in a basis $\mathbf{e}$ of $\mathscr{F}$) to an equation $\sigma_{q,h}(Y) = A(q,h;T) \cdot Y$, with $A(q,h;T) \in GL_n(\mathcal{O}(X))$. Equivalently we have an equation of type

$$d_{q,h}(Y) \;=\; G_{[1]}(q,h;T) \cdot Y \;, \tag{6.7}$$

where $G_{[1]} := \frac{A - \mathrm{Id}}{(q-1)T+h} \in M_n(\mathcal{O}(X))$.[14] If (6.7) admits a complete basis of solutions $Y_D \in GL_n(\mathcal{O}(D))$, over some $\sigma_{q,h}$-invariant open disc D, then we can express it as $Y_D(T) = \sum_{n\geqslant 0} d^n_{q,h}(Y_D)(c) \cdot \frac{(T-c)^{[n]}_{q,h}}{[n]^!_q}$. Now, iterating (6.7), we find $d^n_{q,h}(Y_D) = G_{[n]}(q,h;T) \cdot Y_D$, where $G_{[n]}$ are inductively defined by the relations $G_{[0]} = \mathrm{Id}$, and $G_{[n+1]} = \sigma_{q,h}(G_{[n]}) \cdot G_{[1]} + d_{q,h}(G_{[n]})$.

Assume for a moment that $(\mathscr{F}, \sigma_{q,h}) = \mathrm{Def}_{\sigma_{q,h}}(\mathscr{F}, \nabla)$ is obtained by $\sigma_{q,h}$-deformation from a differential equation. Then the matrix of the stratification χ associated to ∇ can be written as

$$Y_\chi \;=\; \sum_{n\geqslant 0} G_{[n]}(q,h;T_2) \frac{(T_1 - T_2)^{[n]}_{q,h}}{[n]^!_q} \;\in\; GL_n(\mathcal{O}(\mathcal{T}))\;, \tag{6.8}$$

where $\mathcal{T}$ is an admissible open neighborhood of the diagonal containing $\Delta_{\sigma_{q,h}}(X)$.

We now come back to the general case of a possibly not stratified equation (6.7). In this case we consider (6.8) as a (possibly divergent) series of functions over some unspecified admissible open neighborhood $\mathcal{T}$ of the diagonal. We now investigate whether (6.8) converges to the matrix of a

[12] Namely the intersections are quasi-Stein, and the matrix of ∇ is given by $G = d/dT_1(Y_\chi) \cdot Y_\chi^{-1}$ (cf. (2.7)). Over an intersection the two matrices of the stratifications differs by multiplication by a matrix killed by d/dT_1, so they furnishes the same G.

[13] Conjecturally every connected analytic domain of $\mathbb{A}^{1,\mathrm{an}}_K$ is quasi-Stein.

[14] Notice that $G_{[1]}$, and also $G_{[n]}$, has a denominator. It belong however to $M_n(\mathcal{O}(X))$ because $a \notin X$.

stratification corresponding to a $\sigma_{q,h}$-compatible differential equation. We define for all $x \in X$

$$\mathcal{R}^{\mathscr{F},\sigma_{q,h}}(x,\mathbf{e}) \ := \ \min\Big(\ \rho_{S,T}(x) \ , \ \liminf_n \Big(|G_{[n]}(q,h;T)|(x)/[n]^!_q \Big)^{-1/n} \Big) \ , \tag{6.9}$$

where $\rho_{S,T}(x)$ in the function of Remark 2.1.4.

Corollary 6.3.3. *Assume that, for all $x \in X$, we have (cf. (3.2))*

$$\mathcal{R}^{\mathscr{F},\sigma_{q,h}}(x,\mathbf{e}) \ > \ \mathcal{R}^{\sigma_{q,h}}(x) \ .^{15} \tag{6.10}$$

Then (6.8) converges in $M_n(\mathcal{O}(\mathcal{T}))$, for some admissible neighborhood of the diagonal containing $\Delta_{\sigma_{q,h}}(X)$, and it lies in $GL_n(\mathcal{O}(\mathcal{T}))$. Moreover it is the matrix of a stratification corresponding to a $\sigma_{q,h}$-compatible differential equation $(\mathscr{F},\nabla)$ whose $\sigma_{q,h}$-deformation is $(\mathscr{F},\sigma_{q,h})$, and one has

$$\mathcal{R}^{(\mathscr{F},\nabla)}(x) \ = \ \mathcal{R}^{\mathscr{F},\sigma_{q,h}}(x,\mathbf{e}) \ . \tag{6.11}$$

Proof. We begin by the following

Lemma 6.3.4. *Assume that X is an affinoid domain, and that there exists R such that*

$$\max_{x\in X} \mathcal{R}_S(x,\sigma_{q,h}) \cdot \rho_{S,T}(x) \ < \ R \ < \ \min_{x\in X}\{\textit{Radius of } (D(x,S))\} \ .^{16} \tag{6.12}$$

Then the following are equivalent:

i) *For all $x \in X$ we have $\mathcal{R}^{\mathscr{F},\sigma_{q,h}}(x) > R$;*

ii) *(6.8) converges in $M_n(\mathcal{O}(\mathcal{T}))$, with $\mathcal{T} := \mathcal{T}(X,T,R)$, and it lies in $GL_n(\mathcal{O}(\mathcal{T}))$. Moreover it is the matrix of a stratification over X corresponding to a $\sigma_{q,h}$-compatible differential equation $(\mathscr{F},\nabla)$ whose $\sigma_{q,h}$-deformation is $(\mathscr{F},\sigma_{q,h})$.*

Proof. If $q \neq 1$, by a translation we can assume that $h = 0$ (cf. (6.2)). In this case the Proposition is proved in [Pul08, Lemma 5.16]. If $q = 1$ and $h \neq 0$, the proof follows similarly. □

The proof of Corollary 6.3.3 then goes as follows. We find a covering of X on which Lemma 6.3.4 applies. More precisely, if $x \in \Gamma_S$, we consider a star-shaped affinoid neighborhood of x in X of the form $Y_x = \tau_S^{-1}(\Lambda_x)$, similarly as in Definition 1.1.4. By construction Y_x is stable by $\sigma_{q,h}$, and for all $y \in Y_x$ we have $\rho_{S,T}(y) = \rho_{S_{Y_x},T}(y)$, $\mathcal{R}^{\mathscr{F},\sigma_{q,h}}(y) = \mathcal{R}^{\mathscr{F}_{|Y_x},\sigma_{q,h}}(y)$, and also $\mathcal{R}_{S_{Y_x}}(y,\sigma_{q,h}) = \mathcal{R}_S(y,\sigma_{q,h})$. Up to shrinking Λ_x, by continuity we have (6.12), and i). This proves the existence of a good differential equation over Y_x. Since we are in the affine line, this is actually a covering of X as soon as X is not an open disc with empty weak triangulation (i.e. $\Gamma_S = \emptyset$). And one sees that we can assume that the intersection of three affinoids of the covering is empty, so the local data glue over X.

If $\Gamma_S = \emptyset$, by Lemmas 6.1.2 and 3.1.1, $\sigma_{q,h}$ is actually S'-infinitesimal with respect to a convenient weak triangulation given by a point $x \in X$. If x is close enough to the open boundary of the disc X, replacing $S = \emptyset$ by $S' = \{x\}$ doesn't cause any trouble. And we can apply the above proof. □

Corollary 6.3.5. *Under the assumptions of Corollary 6.3.3, the matrix of ∇ is given by the formula*

$$G(T) \ := \ \lim_{n\to+\infty} \frac{A(q^{p^n},[p^n]_q \cdot h;T) - \mathrm{Id}}{(q^{p^n}-1)T + [p^n]_q \cdot h} \ . \tag{6.13}$$

[15] Recall that $\mathcal{R}^{\sigma_{q,h}}(x) = \mathcal{R}_S(x,\sigma_{q,h}) \cdot \rho_{S,T}(x) = |(q-1)t_x + h|(x)$, as in the proof of Lemma 3.3.3.

[16] $X_{\widehat{K^{\mathrm{alg}}}}$ is a disjoint union of affinoid domains of the type $Y = D^+(c_0,R_0) - \cup_{i=1}^s D^-(c_i,R_i)$, for which $\min_{x\in Y}\{\text{Radius of } (D(x,S))\} = \min(R_0,R_1,\dots,R_s)$.

Proof. If $Y_\chi(T_1,T_2)$ is the matrix of the stratification, we have $G(T) = \frac{d}{dT_1}(Y_\chi)\cdot Y_\chi^{-1}$ and $A(q,h;T) = Y_\chi(qT+h,T)$. The claim follows from the fact that $\lim_{\sigma\to 1}\frac{\sigma-1}{\sigma(T)-T} = \frac{d}{dT}$, and recalling that $\sigma_{q,h}^{p^n} = \sigma_{q^{p^n},[p^n]_q\cdot h}$ tends to 1 as $n\to+\infty$. □

Remark 6.3.6. *If $D\subset(X-\Gamma_S)$ is a disc, and if I is a segment in D oriented as outside D, then $\mathcal{R}^{\mathscr{F},\sigma_{q,h}}(x,\mathbf{e})$ is logarithmically not increasing along I since each function $x\mapsto|G_{[n]}|(x)$ is not* log*-decreasing, and $\rho_{S,T}$ is locally constant outside Γ_S. If X is connected, and if $\Gamma_S\neq\emptyset$, this shows that (6.10) holds for all $x\in X$ if and only if it holds for all $x\in\Gamma_S$. If $\Gamma_S=\emptyset$, then X is a virtual open disc and it is enough to test (6.10) at the open boundary of the disc.*

6.3.2. Let $\mathcal{G}$ be the group structure on $\mathbb{G}_m^{\mathrm{an}}\times\mathbb{A}_m^{1,\mathrm{an}}$ defined in section 6.1. For all $0<\tau<1$ and $\nu>0$ the product of discs $\mathcal{G}_{\tau,\nu} := D^-(1,\tau)\times D^-(0,\nu)$ is a K-analytic subgroup. The results of section 4.5 apply to $\mathcal{G}_{\tau,\nu}$. If its action is S-infinitesimal, by Corollary 6.2.2 it is also non degenerate.

The following proposition shows how to recover the differential equation from its $\mathcal{G}_{\tau,\nu}$-deformation.

Proposition 6.3.7. *Assume that $\mathcal{G}_{\tau,\nu}$ acts S-infinitesimally on X, let $Y'=G(T)Y$ be a $\mathcal{G}_{\tau,\nu}$-compatible differential equation, and let $\{\sigma_{q,h}(Y)=A(q,h;T)Y\}_{(q,h)\in\mathcal{G}_{\tau,\nu}}$ be the corresponding $\mathcal{G}_{\tau,\nu}$-deformation. Then for all $(a,b)\in K^2-\{(0,0)\}$ one has*

$$G(T) \;=\; (aT+b)^{-1}\cdot\Big[\,a\cdot\frac{\partial}{\partial q}A(q,h;T)\Big|_{q=1,h=0} \;+\; b\cdot\frac{\partial}{\partial h}A(q,h;T)\Big|_{q=1,h=0}\Big]\,. \tag{6.14}$$

In particular $G(T) = T^{-1}[\frac{d}{dq}A(q,h;T)]_{|q=1,h=0} = [\frac{d}{dh}A(q,h;T)]_{|q=1,h=0}$.

Proof. As in the proof of Proposition 4.4.4, we can assume that X is equal to the open disc D centered at t_x as in Definition 4.4.1. If $Y_\chi(T_1,T_2)$ is the matrix of the stratification, then $A(q,h;T) = Y_\chi(qT+h,T)$. If $\gamma_{(a,b)}: D^-(0,\varepsilon)\to\mathcal{G}_{\tau,\nu}$ is the path $r\mapsto(1+ar,br)$, then the limit $\frac{d}{dr}(\sigma_{\gamma_{(a,b)}(r)}) := \lim_{r\to 0}\frac{\sigma_{1+ar,br}-\mathrm{Id}}{r}$ converges to $(aT+b)\frac{d}{dT}$. The claim then follows from $G=\frac{d}{dT_1}(Y_\chi)\cdot Y_\chi^{-1}$. □

Remark 6.3.8. *If an individual action of $\sigma_{q,h}$ satisfying Corollaries 6.2.2 and 6.3.3 is given, this produces a $\sigma_{q,h}$-compatible differential equation $(\mathscr{F},\nabla)$. If now X is an affinoid domain in $\mathbb{A}_K^{1,\mathrm{an}}$ as in [Pul08] (though this work also over a more general class of analytic domains), the differential equation so obtained is $\mathcal{G}_{\tau,\nu}$-compatible for some pair (τ,ν), and so, by $\mathcal{G}_{\tau,\nu}$-deformation of ∇, the original action of $\sigma_{q,h}$ extends to an action of $\mathcal{G}_{\tau,\nu}$.*

6.4. An example on the Tate curve

A Tate curve X_a is obtained as a quotient of $\mathbb{G}_{m,K}^{\mathrm{an}}$ by the action of $a^{\mathbb{Z}}$, where $a\in K$ has norm $|a|<1$. The analytic skeleton of X_a is a loop, and it is the skeleton of a weak triangulation S. Now, $\mathbb{G}_{m,K}^{\mathrm{an}}\subseteq\mathbb{A}_K^{1,\mathrm{an}}$ is stable under the action of $\mathcal{G}_{1,0}=D^-(1,1)$. That action commutes with the multiplication by a, and it defines an S-infinitesimal non degenerate action on X_a.

It has been shown in [PP13a] that all differential equation $\mathscr{F}$ over X_a has constant radius $\mathcal{R}_{S,1}(-,\mathscr{F})$. On the other hand it is easy to see that for all $q\in D^-(1,1)$, one has $\mathcal{R}_S(-,\sigma_{q,0})=|q-1|$. The action of $\sigma_{q,0}$ is visibly non degenerate as soon as q is not a root of unity. This permits to describe all differential equations of X_a as semi-linear analytic $G_{\tau,0}$-modules for some $\tau>0$.

7. Morita's p-adic Gamma function and Kubota-Leopolodt's L-functions

In this section we apply the previous theory to a particular difference equation satisfied by the Morita's p-adic Γ-function Γ_p. We firstly prove some useful results in section 7.1.

7.1. Small radius for rank one (q,h)-difference equations

In section 2.2.2 we have defined the number $\omega = \lim_n |n!|^{1/n}$. For all $q \in K^\times$, $|q-1|<1$, we now set $\omega_q := \lim_{n\to\infty} |[n]^!_q|^{1/n}$. One verifies (cf. [DV04, 3.5]) that, if $\kappa \geqslant 1$ is the smallest integer such that $|q^\kappa - 1| < \omega$, then $\omega_q = \omega$ if $\kappa = 1$ and $\omega_q = \big([\kappa]_q \cdot \omega\big)^{1/\kappa}$ if $\kappa \geqslant 2$. In particular $\omega_1 = \omega$.

Let X be an affinoid domain of $\mathbb{A}^{1,\mathrm{an}}_K$, with a weak triangulation S, and an S-infinitesimal non degenerate action of $\sigma_{q,h}$ (cf. Corollary 6.2.2). Let $x \in X$ be a point such that $D(x)$ is fixed by $(\sigma_{q,h})_\Omega$.[17] Since $\sigma_{q,h}$ is an automorphism of $D(x)$, its composite with a bounded function remains bounded. So $d_{q,h}$ acts on the ring $\mathcal{B}(D(x))$ of bounded functions over D. We have an isometric inclusion $\mathscr{H}(x) \to \mathcal{B}(D(x))$ given by the Taylor expansion at t_x: $f \mapsto \sum_{n\geqslant 0} f^{(n)}(t_x)(T-t_x)^n/n!$.

Let d be $d_{q,h}$ or d/dT. We set $\|d\|_{\mathcal{B}(D(x))} := \max_{f\in\mathcal{B}(D(x)),f\neq 0} \|d(f)\|_{D(x)}/\|f\|_{D(x)}$, where $\|.\|_{D(x)}$ is the sup-norm on $D(x)$. It follows from Proposition 6.2.1 that $\|d\|_{\mathcal{B}(x)} = r(x)^{-1}$.

Lemma 7.1.1 (explicit small radius)**.** *Let $\mathscr{F}$ be a $\sigma_{q,h}$-module. With the notation of (6.9) we have* $\liminf_n (|G_{[n]}|(x)/|[n]^!_q|)^{-1/n} \geqslant \frac{\omega_q}{\max(|G_{[1]}|(x), \|d_{q,h}\|_{\mathcal{B}(D(x))})}$. *Moreover if* $\mathrm{rank}(\mathscr{F}) = 1$, *then* $|G_{[1]}|(x) > \|d_{q,h}\|_{\mathcal{B}(D(x))}$ *if and only if* $\liminf_n(|G_{[n]}|(x)/|[n]^!_q|)^{-1/n} < \omega_q \cdot \|d_{q,h}\|^{-1}_{\mathcal{B}(D(x))}$. *In this case*

$$\liminf_n \Big(|G_{[n]}|(x)/|[n]^!_q|\Big)^{-1/n} \;=\; \frac{\omega_q}{|G_{[1]}|(x)}\;. \tag{7.1}$$

The same statements hold for rank one differential equations replacing $d_{q,h}, \omega_q, [n]^!_q, G_{[1]}$ with $d/dT, \omega, n!, G_1$, where G_1 is the matrix of (2.7).

Proof. By $G_{[n+1]} = d_{q,h}(G_{[n]}) + \sigma_q(G_{[n]})G_{[1]}$ we inductively have $|G_{[n]}|(x) \leqslant \max(|G_{[1]}|(x), \|d_{q,h}\|_{\mathcal{B}(D(x))})^n$, and equality holds if $|G_{[1]}|(x) > \|d_{q,h}\|_{\mathcal{B}(D(x))}$ and if $\mathscr{F}$ has rank one. Now, since the sequence $|[n]^!_q|^{1/n}$ is convergent to ω_q, one has $\liminf_n(|G_{[n]}|(x)/|[n]^!_q|)^{-1/n} = \omega_q \cdot \liminf_n |G_{[n]}|(x)^{-1/n}$. □

7.2. Morita's p-adic Gamma function as solution of a difference equation

In this section $K = \mathbb{Q}_p$. Assume that $p \neq 2$ is a prime number. The Morita's p-adic Gamma function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$, is the unique continuous function on $\mathbb{Z}_p$ verifying $\Gamma_p(0) = 1$, and the functional equation $\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } |x|=1 \\ -\Gamma_p(x) & \text{if } |x|<1\,, \end{cases}$. Its values on natural numbers $n \geqslant 1$ are given by $\Gamma_p(n) = (-1)^n \cdot \prod_{i=1,(i,p)=1}^{n-1} i$. It is known since [Mor75] that $\Gamma_p(T)$ is locally analytic with local radius greater than $|p|$. Subsequently Dwork [Dwo82], applying non cohomological methods introduced by D.Barsky [Bar80], was able to compute the exact radius of convergence of $\Gamma_p(T)$ in a neighborhood of the points $0,\dots,p-1$, which is $\omega|p|^{1/p}$. Denote by $\Gamma^i_p(T) := 1 + \sum_{n\geqslant 1}\gamma^i_n \cdot (T-i)^n$ the Taylor expansion of Γ_p at $T = i \in \{0,\dots,p-1\}$. Clearly $\Gamma^i_p(T+i) = (-1)^i(T+1)(T+2)\cdots(T+i-1)\Gamma^0_p(T)$. From the functional equation, for all $n \geqslant 1$, we have

$$\sigma_{1,p^n}(\Gamma^0_p(T)) \;=\; A(1,p^n;T)\cdot\Gamma^0_p(T)\;, \qquad\qquad A(1,np;T) \;=\; -\prod_{i=1,(i,p)=1}^{np-1}(T+i)\;. \tag{7.2}$$

Theorem 7.2.1. *The function $\Gamma^0_p(T)$ is the Taylor solution at $T = 0$ of a rank one differential equation $Y' = g_0(T)\cdot Y$ such that $g_0(T) \in \mathcal{O}(D^-(0,1)) \subset \mathbb{Q}_p[[T]]$. If $\mathscr{F}$ is the associated differential module, and if $D^-(0,1)$ has the empty weak triangulation, then:*

$$\mathcal{R}^{\mathscr{F}}(x_{0,\rho}) = \begin{cases} \omega|p|^{1/p} & \text{if } \quad 0 \;\leqslant \rho \leqslant\; r_0 \\ \frac{\omega|p|^n}{\rho p^{n-1}(p-1)} & \text{if } \; r_{n-1} \leqslant \rho \leqslant r_n\,, \qquad n\geqslant 1. \end{cases} \tag{7.3}$$

[17]By Lemma 6.1.2 this means $|(q-1)t_x + h| < r(x)$.

where $r_0 = |p|^{1/p}$, *and* $r_n = \omega^{\frac{1}{p^{n-1}(p-1)}}$, *for all* $n \geqslant 1$. *Moreover, the Small Radius Lemma 7.1.1 gives*

$$|g_0(T)|(x_{0,\rho}) = \begin{cases} |g_0(T)|(x_{0,\rho}) \leqslant \rho^{-1} & \text{if} \quad 0 \leqslant \rho \leqslant r_0 \\ \rho^{p^{n-1}(p-1)}/|p|^n & \text{if} \quad r_{n-1} \leqslant \rho \leqslant r_n\ , \quad n\geqslant 1. \end{cases} \tag{7.4}$$

Proof. Let $(\mathscr{F}, \sigma_{1,p^n})$ be the difference module associated to (7.2) in the basis $\mathbf{e} \in \mathscr{F}$. Every $A(1,p^n;T)$ converges everywhere, because it is a polynomial. We may think that $(\mathscr{F}, \sigma_{1,p^n})$ is defined over a conveniently large disc with empty weak triangulation, so $\rho_{S,T}$ does not play any role in (6.9), which is always computed by the infimum limit. So, for all $\rho \geqslant 0$ we set $R(n,\rho) := \liminf_s(|G_{[s]}(1,p^n;T)|_\rho/|[s]^!_q|)^{-1/s}$. By Remark 6.3.6, the locus of points where (6.10) holds is an open disc $D^-(0,r_n)$. And r_n is the supremum value of ρ satisfying $R(n,\rho) > \mathcal{R}^{\sigma_{1,p^n}}(x_{0,\rho})$. Over that disc Corollary 6.3.3 applies, and we have a differential equation $(\mathscr{F}, \nabla_n)$, whose σ_{1,p^n}-deformation is $(\mathscr{F}, \sigma_{1,p^n})$, satisfying $\mathcal{R}^{(\mathscr{F},\nabla_n)}(x_{0,\rho}) = R(n,\rho)$ for all $\rho < r_n$. By definition of the deformation, the $\sigma_{1,p^{n+1}}$-deformation of $(\mathscr{F}, \nabla_n)$ is $(\sigma_{1,p^n})^p : \mathscr{F} \xrightarrow{\sim} \mathscr{F}$. Hence $R(n+1,\rho) = R(n,\rho)$ for all $\rho < r_n$, and by concavity of $R(n+1,\rho)$ we have $r_n < r_{n+1}$. Now, for $n+1$, the range of application of Corollary 6.3.3 is the disc $D^-(0,r_{n+1})$, and it is clear that $(\mathscr{F}, \nabla_{n+1})_{|D^-(0,r_n)} = (\mathscr{F}, \nabla_n)$. Since $\Gamma^0_p(T)$ is a solution of σ_{1,p^n} and of $\sigma_{1,p^{n+1}}$, it is also a solution of ∇_n and of ∇_{n+1}, hence the matrix of the two connections in the basis $\mathbf{e}$ of $\mathscr{F}$ coincide. This proves that the matrix of ∇_n in the basis $\mathbf{e}$ actually lies over $D^-(0,r_{n+1})$. We now prove that $\lim_n r_n = 1$, and inductively compute the function $R(n,\rho)$. The proof consists in computing inductively the small values of the radii $R(n,\rho)$ by Lemma 7.1.1, and also r_n, and $|g_0|(x_{0,\rho})$ by the same Lemma.

Lemma 7.2.2. *Let* $G_{[1]}(1,p^n;T) := \frac{A(1,p^n;T)-1}{p^n}$. *For all* $n \geqslant 1$ *one has*

$$|G_{[1]}(1,p^n;T)|(x_{0,\rho}) \ = \ \frac{\rho^{\deg(G_{[1]}(1,p^n;T))}}{|p|^n} = \frac{\rho^{p^{n-1}(p-1)}}{|p|^n}\ , \qquad \text{for all } \rho \geqslant 1\ , \tag{7.5}$$

and for $n = 1$ *the equality holds for all* $\rho \geqslant \omega$.

Proof. If $\sum a_i T^i \in \mathbb{Z}[T]$ is a polynomial of degree n, with $|a_n| = 1$, then $x_{0,\rho}(f) = \rho^n$ for all $\rho \geqslant 1$. Then (7.5) follows from the fact that the degree of $G_{[1]}$ is $p^{n-1}(p-1)$. For $n = 1$, the reduction of $A(1,p;T)$ in $\mathbb{F}_p[T]$ is the cyclotomic polynomial $1 - T^{p-1}$. Then $A(1,p;T) - 1 = -T^{p-1} + a_{p-2}T^{p-2} + \cdots + a_0$, with $|a_i| \leqslant |p|$, for all $i = 0,\ldots,p-2$. Hence $|A(1,p;T) - 1|_\rho = \max(\rho^{p-1}, |a_{p-2}|\rho^{p-2},\ldots,|a_0|)$, and $\rho^{p-1} \geqslant |p|\rho^i$ for all $i = 0,\ldots,p-2$ if and only if $\rho \geqslant \omega = |p|^{1/(p-1)}$. □

Lemma 7.2.3. *We have* $R(1,\rho) = \omega|p|^{1/p}$ *for* $\rho \leqslant |p|^{1/p}$, *and* $R(1,\rho) = \omega|p|/\rho^{p-1}$, *for all* $\rho \geqslant |p|^{1/p}$.

Proof. For all $n \geqslant 1$, the function $R(p^n,\rho) = \mathcal{R}^{(\mathscr{F},\nabla)}(x_{0,\rho})$ is constant over $D^-(0,\omega|p|^{1/p})$, with value $\omega|p|^{1/p}$, because this is the disc of convergence of $\Gamma^0_p(T)$. By Lemmas 7.2.2 and 7.1.1 we have $R(1,\rho) = \omega|p|^{1/p}/\rho^{p-1}$, for all $\rho > |p|^{1/p}$. Now, since $\ln(\rho) \mapsto \ln(R(1,\rho))$ is concave and continuous, it must be constant for all $\rho \leqslant |p|^{1/p}$ since $R(1,|p|^{1/p}) = \lim_{\rho\to(|p|^{1/p})^+} R(1,\rho) = \omega|p|^{1/p} = R(1,0)$. □

As explained $r_1 = \sup(\rho \text{ such that } R(1,\rho) > |p|) = \omega^{\frac{1}{p-1}}$.

We now inductively assume that, for $n \geqslant 1$, $R(n,\rho) = \omega|p|^n/\rho^{p^{n-1}(p-1)}$, for all $r_{n-1} \leqslant \rho \leqslant r_n$.

Now we know that $R(n+1,\rho) = R(n,\rho)$ for all $\rho \leqslant r_n$, and by Lemmas 7.2.2 and 7.1.1 we have $R(n+1,\rho) = \omega|p|^{n+1}/\rho^{p^n(p-1)}$, for all $\rho \geqslant 1$. The values for $\rho \in [r_n, 1]$ are deduced by continuity and concavity. Indeed the function $\rho \mapsto \omega|p|^{n+1}/\rho^{p^n(p-1)}$ is logarithmically a line, and its value at $\rho = r_n$ is $|p|^n = R(n+1,r_n)$. So we have $\mathcal{R}(n+1,\rho) = \omega|p|^{n+1}/\rho^{p^n(p-1)}$, for all $\rho \geqslant r_n$. Again Corollary 6.3.3 implies $r_{n+1} = \sup(\rho \text{ such that } R(n+1,\rho) > |p|^{n+1}) = \omega^{\frac{1}{p^n(p-1)}}$. This concludes the computation of the radius. Now (7.4) is an immediate consequence of Lemma 7.1.1. □

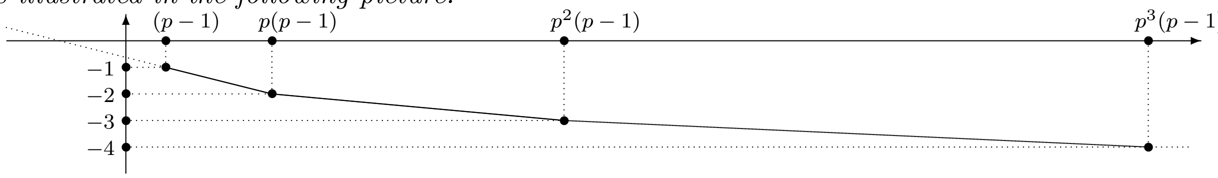
Define the Newton polygon of $g_0(T) := \sum_{n\geqslant 0} a_n T^n \in \mathbb{Q}_p[[T]]$ as the convex hull of the points $\{(n, v_p(a_n))\}_{n\geqslant 0} \cup \{(0,+\infty)\}$, where v_p is the p-adic valuation normalized by $v_p(p) = 1$.

Corollary 7.2.4. *The wedges (i, x_i) of the Newton polygon of $g_0(T)$ such that $i \geqslant p-1$ are the points $\{(p^{n-1}(p-1), -n)\}_{n\geqslant 1}$. In particular $v_p(a_{p^{n-1}(p-1)}) = -n$, for all $n \geqslant 1$.* □

Remark 7.2.5. *For all $k \geqslant 0$ one has*

$$v_p(a_k) \quad\geqslant\quad \begin{cases} 0 & \text{if} \qquad\quad 0 \;\leqslant\; k \;\leqslant\; p-2\,, \\ -n & \text{if} \quad p^{n-1}(p-1) \;\leqslant\; k \;<\; p^n(p-1)\,, \quad n\geqslant 1\,. \end{cases} \tag{7.6}$$

as illustrated in the following picture:



7.3. Applications to Kubota-Leopoldt's p-adic L-functions

We preserve the assumption $p \neq 2$. It has been known since Y.Morita [Mor75] and J.Diamond [Dia77, Theorem 10] (see also [Rob00, p.376]) that $\log(\Gamma_p)$ admits the following Taylor expansion for $|T| \leqslant |p|$:

$$\log(\Gamma_p^0(T)) = \lambda_0 T - \sum_{m\geqslant 1} L_p(1+2m, \overline{\omega}_p^{2m}) \frac{T^{1+2m}}{1+2m} \tag{7.7}$$

where $\overline{\omega}_p$ is the inverse of the Teichmüller Dirichlet character corresponding to the prime p and where $L_p(1+2m, \overline{\omega}_p^{2m})$ is the value at $s = 1+2m$ of the p-adic Kubota-Leopoldt's L-function corresponding to the character $\overline{\omega}_p^{2m}$. The constant λ_0 is the constant coefficient appearing in the Taylor expansion of the Zeta function $\zeta_p(s)$ at $s=1$: $\zeta_p(s) = \sum_{n\geqslant -1} \lambda_n (s-1)^n$. We note that

$$g_0(T) = \frac{d}{dT}(\log(\Gamma_p^0(T))) = \lambda_0 - \sum_{m\geqslant 1} L_p(1+2m, \overline{\omega}_p^{2m}) T^{2m}\,. \tag{7.8}$$

The Newton polygon of $g_0(T)$ have been computed in Corollary 7.2.4. It gives the following estimate on the values of the L-functions appearing in (7.8):

Corollary 7.3.1. *For all $n \geqslant 1$ one has*

$$v_p(\, L_p(1+p^{n-1}(p-1), \overline{\omega}_p^{p^{n-1}(p-1)})\,) \quad=\quad v_p(\,\zeta_p(1+p^{n-1}(p-1))\,) \quad=\quad -n\,. \tag{7.9}$$

Moreover for all $m \geqslant 0$

$$v_p(\, L_p(1+2m, \overline{\omega}_p^{2m})\,) \quad\geqslant\quad \begin{cases} 0 & \text{if} \qquad\quad 0 \;\leqslant\; 2m \;\leqslant\; (p-1) \\ -n & \text{if} \quad p^{n-1}(p-1) \;\leqslant\; 2m \;<\; p^n(p-1)\,, \quad n\geqslant 1\,. \end{cases}$$

Indeed, this constitutes a proof of the existence of a pole of ζ_p at $s=1$.

7.4. An application to sums of powers.

We now apply the above computations to some sums of powers. The following computations have been obtained in collaboration with Daniel Barsky.

For all integers $\ell, k \geqslant 1$, set

$$S_\ell(k) \quad:=\quad \sum_{i=1,(i,p)=1}^{k-1} \frac{1}{i^\ell}\,. \tag{7.10}$$

This and similar sums have been studied by several authors modulo powers of p [Dic52, pp. 95-103]. A result of L.Washington [Was98] expresses it as sum of values at certain positives integers of some Kubota-Leopoldt's p-adic L-functions. Similar expressions have been found by D.Barsky [Bar83].

The following corollary gives another proof of [Was98, Theorem 1,(a)] (cf. Remark 7.4.2).

Corollary 7.4.1. *For all integers $n,\ell \geqslant 1$ we have*

$$\frac{(-1)^{\ell-1}}{\ell}\cdot S_\ell(np) \;=\; -\sum_{m\geqslant \ell/2}\binom{1+2m}{\ell}(np)^{(1+2m-\ell)}\cdot\frac{L_p(1+2m,\overline{\omega}_p^{2m})}{1+2m}\;. \tag{7.11}$$

Moreover for $\ell=1$ we have $S_1(np)=g_0(np)-g_0(0)$. In particular, for $n=p^{k-1}$, we recover the relation $\lim_{k\to\infty}p^{-k}\sum_{i=0,(i,p)=1}^{p^k-1}i^{-1}=0$ because $g_0'(0)=0$.

Proof. We have $\Gamma_p^0(T)=\exp\Big(\lambda_0 T-\sum_{m\geqslant 1}L_p(1+2m,\overline{\omega}_p^{2m})\cdot\frac{T^{1+2m}}{1+2m}\Big)$. The functional equation gives

$$\Gamma_p^0(T+np)/\Gamma_p^0(T)\;=\;A(1,np;T)\;=\;-\prod_{\substack{i=1,\\(i,p)=1}}^{np-1}(T+i)\;. \tag{7.12}$$

On the left hand side one finds

$$\Gamma_p^0(T+np)/\Gamma_p^0(T)=\exp\Big(\lambda_0 np-\sum_{m\geqslant 1}L_p(1+2m,\overline{\omega}_p^{2m})\cdot\frac{(T+np)^{1+2m}-T^{1+2m}}{1+2m}\Big)\;. \tag{7.13}$$

We now compute the argument of the exponential. To simplify the notations let $b_{1+2m}:=L_p(1+2m,\overline{\omega}_p^{2m})/(1+2m)$, then

$$\sum_{m\geqslant 1}b_{1+2m}\cdot((T+np)^{1+2m}-T^{1+2m})=\sum_{m\geqslant 1}b_{1+2m}\cdot\sum_{k=1}^{1+2m}\binom{1+2m}{k}(np)^kT^{1+2m-k} \tag{7.14}$$

$$=\sum_{\ell\geqslant 0}\Big(\sum_{m\geqslant\max(1,\ell/2)}\binom{1+2m}{\ell}(np)^{(1+2m-\ell)}\cdot b_{1+2m}\Big)T^\ell$$

Now taking log of both sides of (7.12) one finds

$$\lambda_0 np-\sum_{\ell\geqslant 0}\Big(\sum_{m\geqslant\max(\ell/2,1)}\binom{1+2m}{\ell}(np)^{(1+2m-\ell)}\cdot b_{1+2m}\Big)T^\ell=\log\Big(-\prod_{\substack{i=1,\\(i,p)=1}}^{np-1}(T+i)\Big)\;. \tag{7.15}$$

Then write $-\prod_{i=1,(i,p)=1}^{np-1}(T+i)=\Gamma_p(np)\cdot\prod_{i=1,(i,p)=1}^{np-1}(1+\frac{T}{i})$. Since $|\Gamma_p(np)-1|\leqslant|p|$, it has a meaning to consider $\log(\Gamma_p(np))$. Then

$$\log\Big(-\prod_{\substack{i=1,\\(i,p)=1}}^{np-1}(T+i)\Big)\;=\;\log(\Gamma_p(np))+\sum_{i=1,(i,p)=1}^{np-1}\log(1+\frac{T}{i}) \tag{7.16}$$

$$=\;\log(\Gamma_p(np))+\sum_{\ell\geqslant 1}\frac{(-1)^{\ell-1}}{\ell}S_\ell(np)T^\ell \tag{7.17}$$

This proves the corollary. □

Remark 7.4.2. *Equality (7.11) is equivalent to the following relation given in [Was98, Theorem*

1,(a)]: for all integers $n,\ell \geqslant 1$ one has

$$S_\ell(np+1) \;=\; \sum_{i=1,(i,p)=1}^{np} \frac{1}{i^\ell} \;=\; -\sum_{k\geqslant 1}\binom{-\ell}{k}(np)^k\cdot L_p(\ell+k,\omega_p^{1-k-\ell})\;. \tag{7.18}$$

We now prove the equivalence with (7.11). First notice that $S_\ell(np+1)=S_\ell(np)$ because it is a sum over natural numbers that are prime to p. Moreover observe that $(-1)^k\cdot\binom{-\ell}{k}=(-1)^{\ell-1}\cdot\binom{k-1}{\ell-1}$, that $L_p(s,\overline{\omega}_p^k)=0$ if k is odd, and that by definition $\omega_p^{1-k-\ell}=\overline{\omega}_p^{\ell-k-1}$. Equation (7.18) is then equivalent to

$$S_\ell(np) \;=\; -\sum_{k\geqslant 1}(-1)^k\binom{\ell+k-1}{k}(np)^k\cdot L_p(\ell+k,\overline{\omega}_p^{\ell+k-1})\;. \tag{7.19}$$

Since $L_p(\ell+k,\overline{\omega}_p^{\ell+k-1})=0$ if $\ell+k-1$ is odd, we have $L_p(\ell+k,\overline{\omega}_p^{\ell+k-1})=(-1)^{\ell+k-1}L_p(\ell+k,\overline{\omega}_p^{\ell+k-1})$, moreover $\binom{\ell+k-1}{k}=\binom{\ell+k-1}{\ell-1}=\binom{\ell+k}{\ell}\cdot\frac{\ell}{\ell+k}$. Equation (7.19) is then equivalent to

$$S_\ell(np) \;=\; -\sum_{k\geqslant 1}(-1)^{2k+\ell-1}\ell\binom{\ell+k}{\ell}(np)^k\cdot\frac{L_p(\ell+k,\overline{\omega}_p^{\ell+k-1})}{\ell+k} \tag{7.20}$$

$$\;=\; -\frac{\ell}{(-1)^{\ell-1}}\cdot\sum_{k\geqslant 1}\binom{\ell+k}{\ell}(np)^k\cdot\frac{L_p(\ell+k,\overline{\omega}_p^{\ell+k-1})}{\ell+k} \tag{7.21}$$

$$\;=\; -\frac{\ell}{(-1)^{\ell-1}}\cdot\sum_{s\geqslant \ell}\binom{s+1}{\ell}(pn)^{s-\ell+1}\frac{L_p(s+1,\overline{\omega}_p^{s})}{1+s}\;. \tag{7.22}$$

Since $L_p(s+1,\overline{\omega}_p^s)=0$ for s odd, this is equivalent to (7.11).

7.5. Note

Examples of Σ-deformation appear in several process. Here are some examples:

The deformation appears in [CM02, 7.1] to show the independence to the Frobenius.

The $\sigma_{q,0}$-deformation, with $q\in\boldsymbol{\mu}_p$ a p-th root of unity, appears in [Ked10, 10.4.2] under the name of "*Taylor series*" to show the existence of the antecedent by Frobenius of a differential equation $\mathscr{F}$, which is the sub-space of $\mathscr{F}$ fixed points under the action of $\boldsymbol{\mu}_p$ on $\mathscr{F}$ obtained by deformation.

It also appears in [Ked13, 3.2.2, 3.2.6] to show the existence of rank one submodules.

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