An invitation to *p*-adic differential equations

Algebraicity and Transcendence for Singular Differential Equations
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This is an invitation to the theory of p-adic differential equations

It is addressed to an audience of students and of a wide range of non-specialists.

Large Introduction

Why p-adic numbers ?

Answer : Some phenomena can be regarded in both $\mathbb{R}$ and $\mathbb{Q}_p$.

- Some of them are complicated over $\mathbb{R}$, but rather easy over $\mathbb{Q}_p$.
- Unfortunately, some easy facts over $\mathbb{R}$ are complicated over $\mathbb{Q}_p$.

Examples of things that are going better :

- **Convergence of series :** in $\mathbb{Q}_p$ a series $\sum_{n\geq 0} a_n$ converges if, and only if, $a_n \to 0$;
- **Convergence of power series :** in $\mathbb{Q}_p[[T]]$ a power series $\sum_{n\geq 0} a_n T^n$ converges for $|T| \leq R$ if, and only if, $|a_n|R^n \to 0$;
- Every solution of a linear differential equation at a non singular point has a **non zero radius of convergence** ;
- Every finite Galois extension of $\mathbb{Q}_p$ has a **solvable Galois group ;**

Examples of things that are going worse :

- Topology of $\mathbb{Q}_p$ is **totally disconnected**.

- Solutions of linear DE **do not converge enough**, with a wide complexity of behaviors ;

- **Factorization** of linear differential operators has a different feature as we will see ;

- Over $\mathbb{C}$, the analytic de Rham cohomology of linear DE on curves is finite dimensional : the obstruction to the finiteness is given by the complexity of the curve (number of holes in your curve). In particular the index is independent on the equation ;

  Over a p-adic analytic curve as simple as an **open disk**, the majority of linear DE have **infinite dimensional de Rham cohomology** depending on the given DE.

The goal of this talk will be to **give criteria to have factorization of operators and finite dimensional de Rham cohomology.**

Setting : Let K be a field of characteristic 0, which is complete with respect to an ultrametric absolute value $|.|$.

Ultrametric means that the absolute value satisfies the strong triangular inequality

$$|x+y| \leq \max(|x|,|y|) \;, \qquad \text{for all } x,y \in K \;.$$

For simplicity, in this talk we will hide some complications and assume (maybe without mention) that K has good properties such that being algebraically close, spherically complete, ...

However, we want to maintain the discussion general, because we wish to take into account the case where K has trivial absolute value ($|0|_0 = 0$ and $|x|_0 = 1$, for all $x \neq 0$)

Remark : Every field, endowed with the trivial absolute value, is complete.

A couple of meaningful simple examples.

Example 1 : Consider the simplest non trivial differential equation

$$y(T)' = y(T)$$

It is **non singular at every point** $x_0 \in \mathbb{C}_p$ and its solution around x_0 is (up to a constant)

$$\exp(T - x_0) = \sum_{n\geq 0} \frac{(T-x_0)^n}{n!}$$

- The radius of convergence of $\exp(T - x_0)$ is not $+\infty$, it is

$$\mathrm{R}(x_0) = \omega \ := \begin{cases} |p|^{\frac{1}{p-1}} & \text{if} \quad |.| \text{ is } p\text{-adic} \\ 1 & \text{if} \quad |.| \text{ is trivial} \end{cases}$$

Why is the radius of convergence finite ?

Remark

Over the complex numbers we have

$$\text{Radius at } x_0 \; = \; \text{distance}(x_0 \; , \; \textit{Singularities})$$

But in our context, there are NO singularities .

- At least in appearance, there does not seem to be any obstruction to the convergence (such as a singularity). So what is stopping the convergence ?
- This is a deep question that has no answer up to today ;
- In the 60' B.Dwork and P.Robba observed that this lack of convergence **encodes deep information about our DE.**

Example 2 : Consider the differential equation

$$y(T)' = T \cdot y(T)$$

- Again, no singularities
- but now, for $x_0 \in \mathbb{C}_p$ we have

$$\omega = |p|^{\frac{1}{p-1}} , \qquad R(x_0) \ = \begin{cases} \omega^{1/2} & \text{for} \quad |x_0| \leq \omega^{1/2} \\ \omega |x_0|^{-1} & \text{for} \quad |x_0| \geq \omega^{1/2} \end{cases}$$

We have a new behavior here :

The radius of convergence depends on the point x_0.

Link with irregularity : (Same DE as above)

$$y(T)' = T \cdot y(T)$$

- This DE has an irregularity equal to 2 at $+\infty$
- When $|x_0|$ approaches $+\infty$, the radius of convergence $R(x_0)$ is a function with slope -1,

As $x_0 \to \infty$ we have

(logarithmic slope of R) -1 = - irregularity at ∞ = 2

Spoiler

The radius of convergence contains the information about the finite dimensionality of the **De Rham cohomology** groups.

The above example may give you the impression that the radius depends only on the value $|x_0|$, but not on the chosen point x_0 itself.

In reality, the radius function may have a behavior similar to that of a function of type

$$x_0 \mapsto \min(\ c\ ,\ |f(x_0)|^{-1}\ )$$

where $c > 0$ is a constant, and $f(T)$ is a polynomial or, more generally, an analytic function.

The behavior of this function is meaningful :

- it is an invariant of the differential module associated with the differential equation ;
- it contains information about the (local and global) De Rham cohomology of our DE.

Example 3 : Consider a general first order DE over a (large) open disk

$$y(T)' = f(T) \cdot y(T)$$

- If $|f(x_0)|$ is "*large*", we have "*generically*" an explicit expression such as
$$R(x_0) \;=\; \frac{\omega|x_0|}{|f(x_0)|} \;=\; \frac{\omega\rho}{|f|_\rho}$$
with $\rho := |x_0|$ and $|f|_\rho := \sup_{|x|=\rho} |f(x)|$.

- If $|f(x_0)|$ is "*small*", there are (mysterious) changes in the behavior of $R(x_0)$ which is not directly related to the values $|f(x_0)|$.

- The de Rham cohomology groups of this DE are finite dimensional if, and only if, as $|x_0|$ approaches the radius of our disk, there are real constants C and s such that
$$R(x_0) \;=\; C|x_0|^s \text{ (log-affinity of the radius)}$$

The exact computation of the exact radius of a first order linear DE on the affine line is a rude challenge. It involves

- Artin-Hasse exponential power series

$$\exp(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots)$$

  whose radius of convergence is (surprisingly) equal to 1
- Witt vectors and their associated Artin-Hasse exponentials

$$\exp(\phi_0 T + \phi_1 \frac{T^p}{p} + \phi_2 \frac{T^{p^2}}{p^2} + \cdots)$$

  (here $(\phi_0, \phi_1, \ldots)$ is the ghost components of a Witt vector)
- Lubin-Tate formal groups

$$\exp(\pi_n \phi_0 T + \pi_{n-1}\phi_1 \frac{T^p}{p} + \cdots + \pi_0 \phi_n \frac{T^{p^n}}{p^n})$$

  (Here $(\pi_k)_k$ is a compatible sequence of torsion points of a Lubin-Tate group).

The first person doing similar computations was B.Dwork with his famous Dwork's exponential

$$\exp(\pi T)\ , \qquad \pi^p = -p\pi$$

After Dwork, others mathematicians like P.Robba, B.Chiarellotto, G.Christol, D.Chinellato, S.Matsuda, ... and myself worked on that problem.

I have computed entirely the radius of an exponential $\exp(f(t))$, by giving a closed formula in the case where $f(t)$ is a polynomial :

Formula : If f is a polynomial, there are (explicit) polynomials $f_1, \ldots, f_N$ depending on f and all its derivatives, such that

$$R(x_0) \;=\; \frac{1}{\max\limits_{n=1,\ldots,N} |f_n(x_0)|^{1/n}}$$

Before embarking into differential equations, allow me to do a step back and introduce you to Berkovich language, which is the most appropriate to describe the way the radii move.

Berkovich Language

We are going to deal with Berkovich **analytic curves over** K.

Why analytic curves ?

Because the more we have points, the more we have geometrical information.

In this talk we will see some phenomena showing up only on some "*hidden points*" (similar to "**generic points**" in algebraic geometry).

Algebraic geometry	Berkovich Analytic geometry
• K-algebras A	• Banach K-algebras B
• points are (equivalence classes of) morphisms from A to fields	• points are (equivalence classes of) **bounded** morphisms from B to complete valued fields
• points = prime ideals of A	• points = bounded (real valued) multiplicative semi-norms on B

Definition

The Berkovich space $\mathscr{M}(B)$ associated with B is the set of seminorms

$$x : B \to \mathbb{R}_{\geq 0}$$

that are bounded ($x(b) \leq C\|b\|$), multiplicative ($x(b_1 b_2) = x(b_1)x(b_2)$) and compatible with the absolute value of K.

We have a map

$$\mathscr{M}(B) \times B \to \mathbb{R}_{\geq 0}$$

$$(x, b) \mapsto x(b)$$

As well as in algebraic geometry, we may interpret the elements of B as functions on $\mathscr{M}(B)$:

$$b \ : \ \mathscr{M}(B) \to \mathbb{R}_{\geq 0}$$

$$x \mapsto x(b)$$

Definition

The topology of $\mathscr{M}(B)$ is the minimal one that makes continuous all the functions $b : \mathscr{M}(B) \to \mathbb{R}$.

Similarly as in algebraic geometry, Berkovich analytic spaces are obtained by gluing these local objects.

Why Berkovich spaces ?

There are several languages for analytic spaces, Berkovich's one furnishes spaces with good properties such as :

- (locally) **Archwise connectedness**
- (locally) **Hausdorff**
- **Continuity** has similarities with that over the real line.
- They have **enough points** (no need of Grothendieck topologies : sheaves are true classical sheaves)

Other types of geometries (Tate Geometry, Raynaud, Rigid Geometry, Meredith,...) do need Grohendieck topologies to describe the sheaves. Moreover, continuity of real valued functions is hard to be described.

On the other hand, Huber Geometry has "*too many points*", continuity is not similar to that over the real line.

Open and closed disks

E=closed unit disk, D=open unit disk

$$\mathcal{O}(E) \cong \{\sum_{n\geq 0} a_n T^n, s.t. \quad a_n \to 0\} \text{ Banach}$$

$$\mathcal{O}(D) \cong \{\sum_{n\geq 0} a_n T^n, s.t. \quad \forall \rho < 1, |a_n|\rho^n \to 0\} \text{ Fréchet}$$

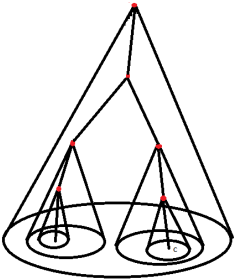
Proposition

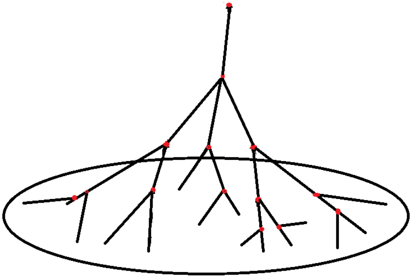
- All Berkovich points of $E := (\mathcal{O}(E))$ are sup-norms on a closed sub-disk (possibly defined over a larger field extensions L/K).
- All Berkovich points of D are of same form (sup-norms on a closed sub-disk of D).

Notice that D is union of its closed sub-disks : it is an analytic space .

Open disk

As an analytic space, an open disk is the union of its closed sub-disks.

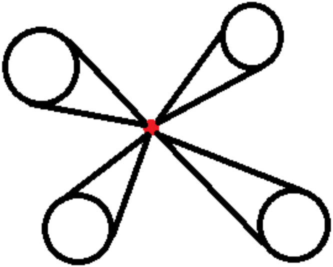




It is an arcwise connected space

The topology induced on a segment is that of a real interval

Closed disk



1. The union of all open sub-disks is an open, but **not a covering** (the red point is not in the covering)
2. The space is **connected**
3. The red-point is the **boundary** of the closed disk

Open annuli

$A_{]r,s[}$ = open annulus $= \{r < |T| < s\}$

$$\mathcal{O}(A_{]r,s[}) \;=\; \{\sum_{n\in\mathbb{Z}} a_n T^n,\;\; s.t.\;\; \forall\rho \in]r,s[,\; |a_n|\rho^n \to 0 \text{ for } n \to \pm\infty\}$$

Proposition

Two types of Berkovich points

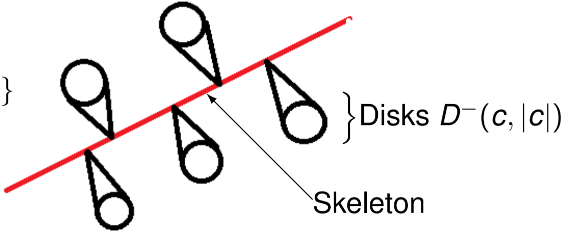
- Sup-norms on closed sub-disks
- ρ-Gauss points, with $\rho \in]r,s[$

$$x_\rho(\sum a_n T^n) \;=\; \sup_{n\in\mathbb{Z}} |a_n|\rho^n$$

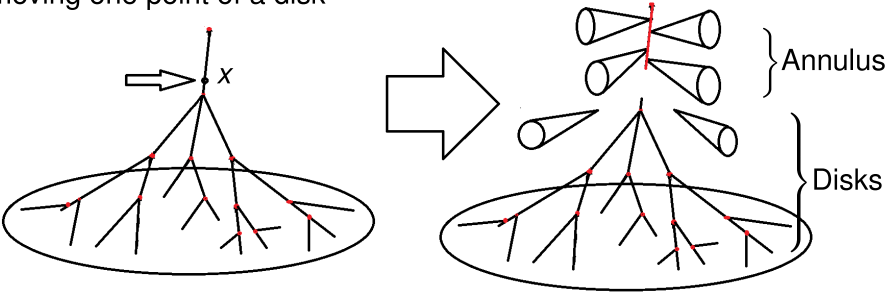
Skeleton := ρ-Gauss points $\xrightarrow[\cong]{\text{homeomorphic}}$]r,s[

Open annuli

Annulus $\{r < |T| < 1\}$



Removing one point of a disk

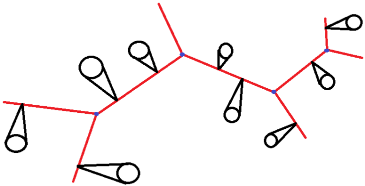


$D = \{|T| < 1\}$ $\qquad$ $D - \{x\}$

Theorem (V.Berkovich - A.Ducros)

Let X be a quasi-smooth curve. There exists a ***locally finite*** *subset $S \subseteq X$ formed by convenient "non rational points" such that $X - S$ is a* *disjoint union* *of open* ***disks*** *and* ***annuli***.

We call S a ***weak triangulation.***



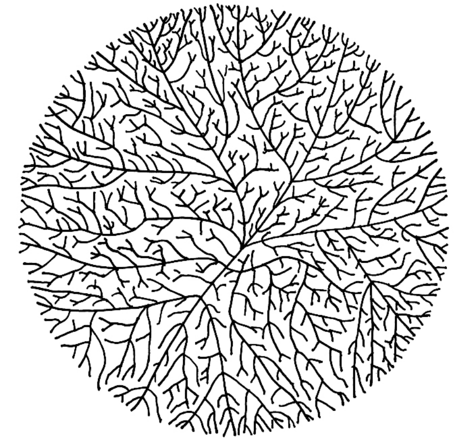
Skeleton

The union Γ_S of the skeletons of the annuli that are connected components of $X - S$ together with the points of S is a locally finite graph in X.

Γ_S := **skeleton** of S

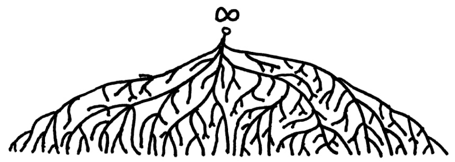
Projective line

Analytic Projective line $\mathbb{P}^{1,\mathrm{an}}_K$



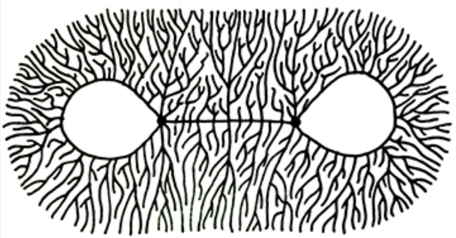
The affine line

Analytic affine line $\mathbb{A}_K^{1,\mathrm{an}}$



A curve

A Berkovich analytic curve

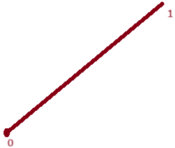


Open disk over trivially valued field

Assume for a moment that the valuation of K is trivial. Then

$$\mathcal{O}(D) = K[[T]]$$

The Berkovich space in this case "is" an individual (real) segment $[0, 1[$



All bounded continuous semi-norms on $K[[T]]$ are of the form

$$|\sum a_n T^n|_{0,\rho} = \sup_n |a_n|\rho^n$$

This is the sup-norm on the sub-disk $\{|T| \leq \rho\}$.

Notice that the K-rational points of D are reduced to the point 0, because K is trivially valued. "Sup-norm" means the sup-norm on the L-valued points of the disk for every L/K.

For every $0 \leq r < s \leq 1$

$$\mathcal{O}(A_{]r,s[}) = K((T))$$

Notice that what changes here from $]r,s[$ to $]r',s'[$ is the topology :

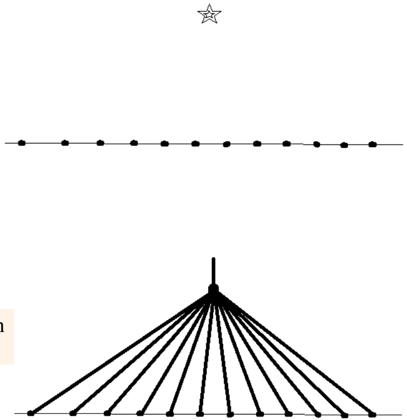
- We endow $\mathcal{O}(A_{]r,s[})$ with the family of ρ-Gauss norms $\{x_\rho\}_{\rho\in]r,s[}$;
- Then, the restrictions are bounded, but not homeomorphisms ;

The datum of the norms on the ring describes its geometry.

Affine line over trivially valued K

Scheme $\mathbb{A}^1_K$

Analytic line $\mathbb{A}^{1,\mathrm{an}}_K$



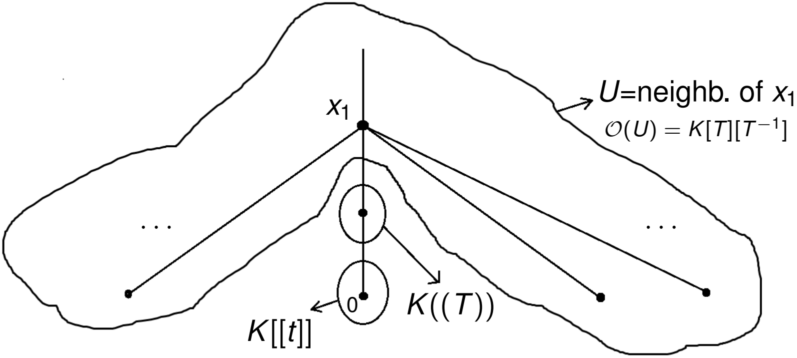
generic point

rational points

generic points

rational points

Berkovich Topology of $\mathbb{A}_K^{1,\mathrm{an}}$ over trivially valued K



A neighborhood of the 1-Gauss point x_1 has to cover almost all, but finitely many, branches out of it, and it has to contain non empty segments out of x_1 of the other branches.

If now K is general and x is any point of a curve, the situation is similar.

p-adic Differential Equations

(including the case of a trivially valued base field)

Why *p*-adic differential equations ?

De Rham cohomology is used in general to investigate the local and global nature of algebraic or analytic varieties.

Moreover, the theory has applications to several other domains

- *p*-Adic Cohomologies and Index Theorems ;
- Counting points of varieties over finite fields, Zeta functions ;
- *G*-functions , periods , Algebraicity of solutions ;
- *p*-curvature ;
- Deformation and p-Adic q-Differences Equations ;
- *p*-Adic Representations , *p*-Adic Hodge Theory ;
- Ramification Theory ;
- ...

Differential equations

We are working over a **Berkovich analytic curve** X over p-adic numbers, which will be introduced later on.

A **differential equation** over X is a locally free sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ of finite rank, together with a connection

$$\nabla \ : \ \mathscr{F} \to \mathscr{F} \otimes \Omega^1_X \ ,$$

that is a map satisfying the Leibniz rule

$$\nabla(g \cdot f) = f \otimes d(g) + g \cdot \nabla(f) \ , \qquad \forall\, g \in \mathcal{O}_X, f \in \mathscr{F}$$

Locally, this is given by a genuine **linear differential system**

$$d\big(Y(x)\big) = G(x) \cdot Y(x) \ , \qquad x \in X$$

where d is a derivation generating the $\mathcal{O}_X$-module of continuous derivations.

De Rham cohomology

The **de Rham cohomology of** $(\mathscr{F}, \nabla)$ is the complex (38) : for every quasi-stein open $U \subset X$ we have

$$\begin{aligned} \mathrm{H}^0_{\mathrm{dR}}(U, \mathscr{F}) &= \mathrm{Ker}(\nabla : \mathscr{F}(U) \to \mathscr{F}(U)) \\ \mathrm{H}^1_{\mathrm{dR}}(U, \mathscr{F}) &= \mathrm{coKer}(\nabla : \mathscr{F}(U) \to \mathscr{F}(U)) \end{aligned}$$

For non-stein curves (e.g. projective ones) we have to consider the hyper-cohomology. We can have in this case a non zero $\mathrm{H}^2_{\mathrm{dR}}(X, \mathscr{F})$, but in general $\mathrm{H}^i_{\mathrm{dR}}(U, \mathscr{F}) = 0$ for $i \neq 0, 1, 2$.

Introduction : some major historical landmarks

1960 B.Dwork,
Rationality of the Zeta function *of a variety of positive characteristic.*

1980 B.Dwork-P.Robba,
Affine line, ***neighborhood of a point*** *(of Berkovich),* ***Robba ring***.

2000 G.Christol-Z.Mebkhout, (André, Crew, Kedlaya, ...) *Differential Equations over the* ***Robba ring****, application over* ***rigid curves****, link with p*-***adic representations.***

2010 F.Baldassarri, K.S.Kedlaya, J.Poineau, A.P.
Differential Equations over ***Berkovich curves*** *(global theory).*

The view point of our recent works is somehow **orthogonal** to that of rigid cohomology (where the Frobenius structure plays a key role).

- We do not start from a problem in positive characteristic. Instead, we directly consider
  1. a quasi-smooth **Berkovich curve** X,
  2. a **differential equation** $\mathscr{F}$ **over** X,
  3. with **no restrictions**.

- The study of differential equations in such a degree of generality, in particular the finiteness of their de Rham cohomology, was a relatively **unexplored problem until 2013**.

  For instance, even for a curve as simple as an open disk or annulus, there was ***no criteria*** *describing the finiteness of the cohomology.*

- Results in this direction are essentially due (among other actors) to Dwork and Robba, then Christol and Mebkhout, and are (up to some exceptions) of **local nature** in the sense of Berkovich.

Radii of convergence

How to define the radii of conv. of the solutions at a Berkovich point ?

The problem is the fact only rational points have a basis of neighborhoods isomorphic to open disks (want Cauchy existence thm)

Idea

We extend the scalars to a large field L/K over which the points becomes rational and consider solutions there. We prove that this is well defined (independent on the choices).

Filtration on the space of solutions at $x \in X$ by the radii

The space of solutions at a point x is filtered by the radii of convergence of the solutions (and this depends on the choice of the triangulation S).

This filtration gives n-radii associated with $x \in X$

$$0 < \mathcal{R}_{S,1}(x, \mathscr{F}) \leq \mathcal{R}_{S,2}(x, \mathscr{F}) \leq \cdots \leq \mathcal{R}_{S,n}(x, \mathscr{F}) .$$

where n is the dimension of our differential equation $\mathscr{F}$.

The convergence Newton polygon

We have $n = \mathrm{rank}(\mathscr{F})$ radii functions

$$\mathcal{R}_{S,i}(-,\mathscr{F}) \;:\; X \longrightarrow \mathbb{R}_{>0}.$$

$$x \mapsto \mathcal{R}_{S,i}(x,\mathscr{F})$$

The convergence Newton polygon is the ordered sequence

$$\ln(\mathcal{R}_{S,1}(x,\mathscr{F})) \;\leq\; \ln(\mathcal{R}_{S,2}(x,\mathscr{F})) \;\leq \cdots \leq\; \ln(\mathcal{R}_{S,n}(x,\mathscr{F}))$$

In order to understand the terminology, we have to do a step back to the trivially valued case.

Assume K is trivially valued.

- Every differential module $\mathscr{F}$ over $K((T))$ can be seen as a differential equation over the open annulus $A_{]0,1[}$ $(S = \emptyset)$;
- The radii can be naturally identified with functions over $]0,1[$:

$$\mathcal{R}_{S,i}(-,\mathscr{F}) \ :]0,1[\longrightarrow \mathbb{R}_{>0}$$

Theorem (reinterpretation of Malgrange, Ramis, Robba, ...)

- For every $i = 1, \ldots, n$ there exists $s_i \geq 0$ such that for every $\rho \in]0,1[$ we have

$$\mathcal{R}_{S,i}(x_\rho, \mathscr{F}) \ = \ \rho^{s_i}$$

- The polygon whose slopes are $s_1 \leq \ldots \leq s_n$ coincides with the classical **formal Newton polygon** (as defined by Malgrange-Ramis-etc...).
- If for some i we have $s_{i-1} < s_i$, then $\mathscr{F}$ breaks as

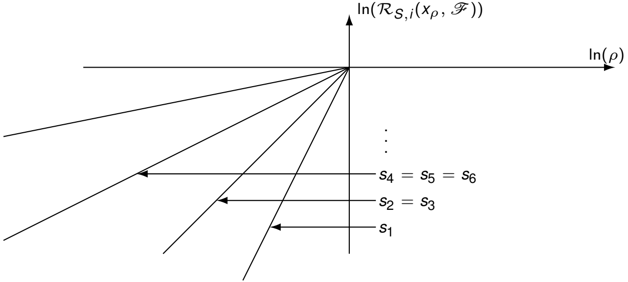
$$\mathscr{F} \ = \ \mathscr{F}_{<i} \oplus \mathscr{F}_{\geq i}$$

  where the radii functions of $\mathscr{F}_{<i}$ (resp. $\mathscr{F}_{\geq i}$) are those of $\mathscr{F}$ that are $< \mathcal{R}_{S,i}(-,\mathscr{F})$ (resp. $\geq \mathcal{R}_{S,i}(-,\mathscr{F})$).

The above equality can be written as

$$\ln(\mathcal{R}_{S,i}(x_\rho,\mathscr{F})) = s_i \cdot \ln(\rho)\ , \qquad \rho \in]0,1[$$

- $\ln(\rho) \mapsto \ln(\mathcal{R}_{S,i}(x_\rho,\mathscr{F}))$ is **a line passing through the origin** (= Solvability property)
- Fact : $s_i =$**ith slope of the Formal Newton polygon of $\mathscr{F}$.**



Decomposition theorems

How can we extend the above theorem in p-adic ?

- Problem 1 : Differential modules are possibly not cyclic over X ;
- Problem 2 : Even when they are cyclic over X, the valuations of the coefficients of the operator are not intrinsic (they are not stable by base change in the module) ;

Good substitute : Radii functions.

- Problem 3 : The space X is not $]0,1[$, but a graph ;
- Problem 4 : The radii function moves in a wide way. Can they possibly have infinitely many changes in their behavior along the paths in X ? (see below for the answer)
- Problem 5 : The radii are possibly not separated everywhere.

Theorem

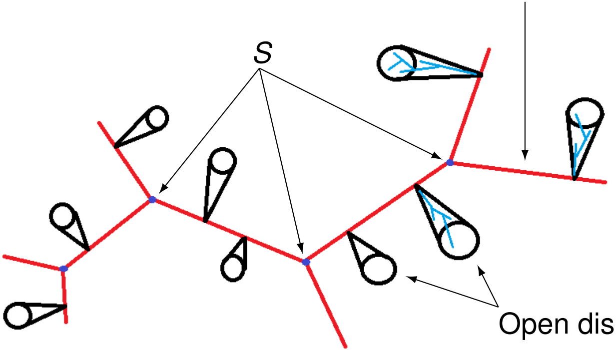
- *The radii are continuous functions on X for the Berkovich topology.*
- *They are locally finite functions : for every i, there exists a locally finite graph $\Gamma_{S,i}(\mathscr{F})$ in X on which the radius function $\mathcal{R}_{S,i}(-,\mathscr{F})$ factorize.*

  In other words, $\mathcal{R}_{S,i}(-,\mathscr{F})$ is locally constant outside $\Gamma_{S,i}(\mathscr{F})$ and along it they are $\log$-affine with a locally finite number of breaks.

- Christol-Dwork : continuity for the restriction of $\mathcal{R}_{S,1}$ to segment $]r,s[$ in $A_{]r,s[}$ (a convex function is continuous)
- Kedlaya : continuity for $\mathcal{R}_{S,i}$ on $]r,s[$
- Baldassarri-Di Vizio : continuity for $\mathcal{R}_{S,1}$ on X (extend the proof of Christol-Dwork)
- Poineau-P. : in general (different proof) + finiteness property
- Kedlaya : second proof, with similar methods

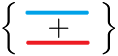
The locally finite graph $\Gamma_{S,i}(\mathscr{F})$

$X :=$ Curve



S = weak triangulation

$\Gamma_S \subseteq \Gamma_{S,i}(\mathscr{F})$



= Controlling graph

Global decomposition/factorization by the radii

Theorem (Poineau-P.)

Let $i \in \{1, \ldots, r\}$ be a fixed index. Assume that for all $x \in X$ we have

$$\mathcal{R}_{S,i-1}(x, \mathscr{F}) \; < \; \mathcal{R}_{S,i}(x, \mathscr{F}) \; .$$

Then $\mathscr{F}$ decomposes as

$$0 \to \mathscr{F}_{\geq i} \to \mathscr{F} \to \mathscr{F}_{<i} \to 0 \; ,$$

where $\mathrm{rank}(\mathscr{F}_{<i}) = i - 1$ and for all $x \in X$ we have

$$\mathcal{R}_{S,k}(-, \mathscr{F}) \; = \; \begin{cases} \mathcal{R}_{S,k}(x, \mathscr{F}_{<i}) & \text{if} \quad k < i \\ \mathcal{R}_{S,k-i+1}(x, \mathscr{F}_{\geq i}) & \text{if} \quad k \geq i \end{cases}$$

This extends the existing decomposition results

- For differential equations over $\mathbb{C}((T))$ this "***is***" the classical decomposition of **B.Malgrange-Ramis-etc** ...

- **Robba** 's decomposition by the radii over $\mathscr{H}(x_\rho)$ the differential field of the ρ-Gauss point (completion of $K(T)$ by the ρ-Gauss norm) ;

- **Dwork-Robba** 's decomposition by the radii over $\mathcal{O}_{X,x_\rho}$ the differential field of the ρ-Gauss point ;

- **Christol-Mebkhout** 's decomposition by the radii over the Robba ring.

- All these decompositions are of local nature .

- All of them give direct sum decompositions. In contrast, we have explicit examples of non-split exact sequence .

Direct sum decomposition

Theorem

The exact sequence

$$0 \to \mathscr{F}_{\geq i} \to \mathscr{F} \to \mathscr{F}_{<i} \to 0 ,$$

splits in the following cases

1. For all $x \in X$ one has
$$\mathcal{R}_{S,i-1}(x, \mathscr{F}^*) \; < \; \mathcal{R}_{S,i}(x, \mathscr{F}^*);$$
2. We have the following inclusion of graphs
$$\Big(\Gamma_{S,1}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F})\Big) \; \subseteq \Gamma_{S,i}(\mathscr{F}) \; .$$

It result important to **measure** the size of these graphs.

A measure of the complexity of the graphs

A measure of the complexity of the graphs

- We have been able to provide an explicit **upper bound** for the number of **vertex** and **edges** of the controlling graphs.
- The bound is given in term of the **slopes of the radii** at the boundary of X.
- Over a **projective curve**, the bound **only depends on the rank** of the differential equation.
- Under appropriate conditions on the p-adic exponents at the boundary of the curve, the bound is related to the de Rham **index**.

Global irregularity and index theorem

Index theorem

Definition

A **quasi-smooth** Berkovich curve X is **finite** if it has finite dimensional de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(X, \mathcal{O}_X)$.

We obtain a topological characterization of finite curves.

Roughly speaking, X is a projective curve with finitely many open or closed disks, or points removed. In particular, this implies that X has

- finite **genus**,
- finite **boundary**,
- finite **skeleton** Γ_S.

Global irregularity

Let X be a **finite** curve. Notations : $r = \mathrm{rank}(\mathscr{F})$

1. b = **germ of segment** in X ;
2. $\partial_b H_{S,r} = \sum_i \partial_b \mathcal{R}_{S,i}$= **slope** along b of the **total height** $H_{S,r}$ of the convergence Newton polygon ;
3. $\partial^o X$ = **open boundary** of X (open germs of segments at infinity) ;
4. ∂X=**boundary** of X ;
5. $dd^c H_{S,r}(x, \mathscr{F}) = \sum \partial_b H_{S,r}(x, \mathscr{F})$, where the sum is taken over all germs of segments b out of x ;
6. $\chi(x, S) := 2 - 2g(x) - N_S(x)$, where $g(x)$ is the genus of x and $N_S(x)$ is the number of directions of Γ_S out of x. It is a certain **characteristic** related to the residual curve of x.

We define the **global irregularity** of $\mathscr{F}$ as

$$\mathrm{Irr}_X(\mathscr{F}) \ := \ \sum_{b \in \partial^o X} \partial_b H_{S,r}(-, \mathscr{F}) + \Big[\sum_{x \in \partial X} dd^c H_{S,r}(x, \mathscr{F}) + \chi(x, S) \Big]$$

Index theorem

Let X be a **finite quasi-smooth** Berkovich curve.

Theorem (Poineau-P.)

Let $\mathscr{F}$ be a differential equation over X, such that

(1) *$\mathscr{F}$ is free of **Liouville** numbers (technical assumption) ;*

(2) *The radii are **not maximal** at the **boundary** of X.*

Then the following are equivalent :

1. *We have*

$$\dim \mathrm{H}^{\bullet}_{\mathrm{dR}}(X, \mathscr{F}) \ < \ +\infty$$

2. *The total height $\mathscr{H}_{S,r}(-, \mathscr{F})$ is **log-affine** at the **open boundary** of X ;*

In this case, we have the following index formula

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) \ = \ \chi_c(X) \cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_X(\mathscr{F}) \ .$$

Index Theorem

- The assumption about the log-affinity of $\mathscr{H}_{S,r}$ is **automatically satisfied** in the following situations

  1. Around a **meromorphic singularity** ;
  2. If X is **relatively compact** in a larger curve Y and $\mathscr{F}$ is the restriction of an equation defined over Y ;
  3. In particular, in the **overconvergent** case (our result implies the finite dimensionality of rigid cohomology for quasi-smooth curves) ; We have similar statement for

- "**meromorphic cohomology**".
- **overconvergent cohomology**".
- a mixed of these two.

Thank you