Exponents for irregular differential modules
A Tannakian approach to the theory of exponents

(beginning of a work in progress with C.Lazda and A.Pál)

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Introduction

Let K be an algebraically closed field, complete with respect to a non archimedean absolute value.
Exponents are invariants associated with differential equations of the form

$$t\frac{d}{dt}(y(t)) = a \cdot y(t) \, , \qquad a \in K$$

The solution is symbolically given by

$$y(t) \;=\; t^a \, .$$

Hence the terminology **exponent** associated with this equation.

There is a definition of the exponents only for **regular** differential modules.

We want to extend this definition to **irregular one**.

This is useful in p-adic cohomology.

Introduction

Over the complex numbers, exponents are useful to **classify regular** differential modules (Riemann-Hilbert as in the talk of Piotr Achinger).

Exponents are also a necessary tool to investigate on the **algebraicity of the solutions**. Indeed,

$$y(t) = t^a$$

is an algebraic function only when a is **rational**.

Introduction

For **regular** differential equations over $K((t))$, we find a definition in E.L. Ince [Inc39], A.H.M.Levelt [Lev61], P.Deligne [Del70], N.M.Katz [Kat70].

For **irregular** differential equations over $K((t))$, there is a definition in the Book by Y.André and F.Baldassarri [ABC20] which I discuss later on.

In p-adic we may have **infinite dimensional de Rham cohomology spaces.**

Lemma

*The differential equation $t\frac{d}{dt}(y(t)) = a \cdot y(t)$ has a finite dimensional de Rham cohomology over $\mathbb{G}_m$ if, and only if, a is not a p-adic **Liouville number**.*

We skip the definition of Liouville numbers which is technical. We only point out that

- the set of Liouville numbers is **not a group**
- Liouville numbers are p-adic numbers in $\mathbb{Z}_p$
- Liouville numbers are not algebraic over $\mathbb{Q}$

Introduction

In this context, exponents have been introduced by Christol and Mebkhout [CM97] mainly to deal with the problem of **finite dimensionality of the cohomology**.

As well as in the complex case, they are also related to the problem of the **algebraicity of solutions**.

The existing definition in *p*-adic is given by an iterative process which is quite **complicated**. A simplification was obtained by B.Dwork [Dwo97].

The two definitions are **potentially different** and nobody verified their equivalence. The definition of Dwork is the one adopted today.

Since then, important improvements have been obtained by K.S.Kedlaya and A.Shiho [Ked15, KS17].

Our aim is twofold, we wish to **simplify** the existing definition ; on the other hand, we wish to generalize it to **irregular** diff.eq.

Introduction

We (mainly) deal with two kind of rings :

- $R = K((t))$, with $K =$a field with discrete topology
- $R = \mathcal{R}$ the Robba ring over $K =$a field complete with respect to an ultrametric absolute value (see later for the definition).

In fact, this is the same ring : the Robba ring **equals** $K((t))$ when the absolute value is trivial.

Denote the category of differential modules over R by

$$d - \mathrm{Mod}(R)$$

Why exponents for irregular modules ?

Some inconvenient facts about the existing theory

”Hidden” exponents

Assume that $\mathrm{M} \in d - \mathrm{Mod}(R)$.

In several statements, one encounters condition for the exponents of the Regular part of M, but also of the Regular part of $End(\mathrm{M}) = \mathrm{M} \otimes \mathrm{M}^*$.

This is because we may have a large regular part of $\mathrm{M} \otimes \mathrm{M}^*$ even when the regular part of M is little or $= 0$.

The exponents of the regular part of $\mathrm{M} \otimes \mathrm{M}^*$ are supposed to be the differences of M, but, in the existing theory, M does not has “**enough exponents**”, because no definition exists for its irregular part.

Some inconvenient facts about the existing theory

In p-adic :

Lemma

Let $a \in K$. If M is the module associated with $t\frac{d}{dt} - a$, then

$$\mathrm{M} \textit{ is regular } \iff a \in \mathbb{Z}_p$$

$$\mathrm{M} \textit{ has a Frobenius } \iff a \in \mathbb{Z}_p \cap \mathbb{Q}$$

Problem

Exponents are not always defined for $t\frac{d}{dt} - a$ when $a \in K$ is general.

We expect from any reasonable definition that, for all $a \in K$

$$a = \text{exponent of } (t\frac{d}{dt} - a).$$

But, if $a \notin \mathbb{Z}_p$ there is no definition at all.

Why exponents for irregular modules ?

In the original spirit of Robba-Christol-Mebkhout, exponents are useful to **test the finite dimensionality of de Rham cohomology**.

Namely, for a **regular** module $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{R})$, they prove that we have the following implications

Frobenius structure $\Rightarrow$ Rational exponents $\Rightarrow$ **DNL** condition $\Rightarrow$ M is extension of rank one sub-quotients whose exponents are NL

$\Downarrow$

$H^i_{dR}(\mathrm{M})$ Finite dimensional

DNL=Exp. and their differences non Liouville

Some inconvenient facts about the existing theory

Let $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{O}(I))$ be a Regular (=Robba) module.

In order to have finite dimensionality of de Rham cohomology, the assumption M **is extension of rank one sub-quotients** is enough.

So, there is no need of Frobenius, nor rationality, nor **DNL** condition.

In this case **our definition will coincide with the classical one** :

Tannakian exponents = exponents

From a cohomological point of view, do we really need the complicate definition ? $\rightarrow$ Not really.

Incompatibility with the formal definition

Assume that $I =]0, \varepsilon[$ and M has a **meromorphic singularity at $t = 0$.** In this case we may consider two modules

$$\widehat{\mathrm{M}} \;:=\; \mathrm{M} \otimes K((t)) = \text{the formal completion of } \mathrm{M}$$

$$\mathrm{M}_0 := \mathrm{M} \otimes \mathcal{R}_0$$

where

$$\mathcal{R}_0 \;=\; \cup_{\varepsilon > 0} \mathcal{O}(]0, \varepsilon[) \;.$$

Then, it is possible to extend the definition of Christol-Mebkhout-Dwork-Kedlaya-Shiho to this case. But

Problem

The regular part of $\widehat{\mathrm{M}}(0)$ and the regular part of M_0 possibly **do not have the same dimension.** How to compare their exponents ?

Aim

Want a definition which gives a multi-set of exponents with $r = \mathit{rank}(\mathrm{M})$ **elements in both cases**, and which makes the two definitions compatible.

Introduction

AIM :

If $r = \mathrm{rank}(\mathrm{M})$, we want exponents to be

- a multi-set of r constants $\{a_1, \ldots, a_r\}$, with $a_i \in K/\mathbb{Z}$
- invariant by isomorphisms of differential modules
- compatible with direct sums, tensor products, internal homs and duals.

Moreover, we want a **uniform definition** working over K((t)) and over the Robba ring $\mathcal{R}$.

FACT :

The definition will not be intrinsic. However, the existing definition in p-adic, is relatively not intrinsic neither (it depends on some choices).

Remind of exponents over a field of power series

The formal Newton polygon

Let $\mathrm{M} \in d-\mathrm{Mod}(K((t)))$ of rank r. It is possible to use **cyclic vector theorem** to obtain an operator

$$L = \sum_{i=0}^{r} f_i \nabla^i$$

associated with M.

From the t-adic valuations of the coefficients f_i we may recover a polygon called **formal newton polygon.**

Problem : In p-adic we do not have cyclic vector.

The formal Newton polygon

There is another interpretation of this polygon, which will be relevant in p-adic.

Let us consider on K the trivial absolute value, then we may interpret every power series $f(t) = \sum_{i \geq n} a_i t^i \in K((t))$ as a **bounded function on the punctured open unit disk**.

For all $\rho \in]0,1[$, we may find a complete non trivially valued field $(K', |.|')/(K, |.|)$ whose absolute value extends that of K and such that there exists a point $x_\rho \in K'$ with $|x_\rho|' = \rho$.

The series $f(t) \in K((t))$ converge at x_ρ as $|a_i x_\rho^i|' \leq \rho^i \to 0$ (indeed $|a_n|' = |a_n|$ is either equal to 0 or 1, because the valuation of K is trivial).

The formal Newton polygon

This means that, over K', we can find solutions of our differential module M around x_ρ.
More precisely, we can consider the largest open disk $D_i := D(x_\rho, R_i) \subset K'$ over which $\mathrm{Ker}(\nabla \otimes 1 : \mathrm{M} \otimes_K K' \to \mathrm{M} \otimes_K K')$ has at least $r - i + 1$ linearly independent solutions.

One has (recall that $r = rank(\mathrm{M})$)

$$\begin{array}{ccccccc} D_1 & \subseteq & D_2 & \subseteq & \cdots & \subseteq & D_r \\ R_1 & \leq & R_2 & \leq & \cdots & \leq & R_r \end{array}$$

Now, it is a theorem that these numbers do not depend on the choice of x_ρ nor K'. The interesting fact is that

Theorem

Let s_i be the i-th slope of the ***formal Newton polygon of the operator*** *L. Then, for all $\rho \in]0,1[$*

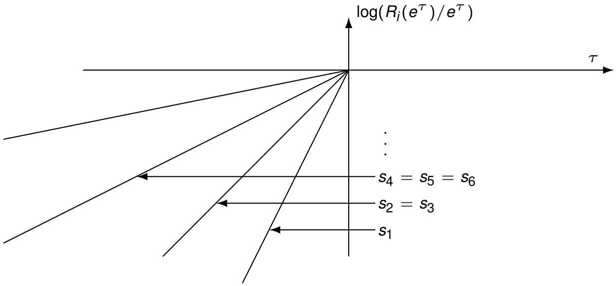
$$R_i(\rho) = \rho^{1+s_i}$$

The above equality can be written as

$$\log(R_i(\rho)/\rho) = s_i \cdot \log(\rho) \;, \qquad \rho \in]0,1[$$

which means that with respect to the coordinate $\tau := \log(\rho)$ we have

- The function $\tau \mapsto \log(R_i(e^\tau)/e^\tau) = s_i \cdot \tau$ is **a line passing through the origin** (=Solvability property)
- the slope of that line is $s_i =$**the *i*th slope of the Formal Newton polygon of *L*.**



Decomposition by the slopes

A differential module over $K((t))$ is **pure of slope** $s \in [0, +\infty[$ if we have $s_i = s$ for all $i = 1, \dots, \mathrm{rank}(\mathrm{M})$.

Theorem (Formal decomposition)

We have a decomposition of M by the slopes

$$M = \bigoplus_s M(s)$$

where $M(s)$ is a submodule of M which is pure of slope s.

If M is pure of slope 0, we say that it is a **REGULAR** module.

Decomposition of the slope zero part by the exponents

Theorem (Fuchs decomposition theorem)

The slope zero part $\mathrm{M}(0)$ *is successive extension (in the sense of exact sequences) of* ***rank one*** *differential modules isomorphic to some modules* $\mathrm{N}(a)$ *defined by the operator* $(t\frac{d}{dt} - a)$.

Every $a \in K$ *such that* $\mathrm{N}(a)$ *appears in a Jordan-Hölder sequence of* $\mathrm{M}(0)$ *is called* ***exponent*** *of* $\mathrm{M}(0)$. *The class* $a \in K/\mathbb{Z}$ *is an invariant.*

Let $r' := \mathrm{rank}\,\mathrm{M}(0)$ and $a_1, \ldots, a_{r'}$ be the multi-set of the exponents associated with $\mathrm{M}(0)$.

Then, the image in $K/\mathbb{Z}$ of the multi-set $\{a_1, \ldots, a_{r'}\}$ is an invariant of the isomorphism class of $\mathrm{M}(0)$.

The theory classifying the isomorphism classes of slope zero modules is similar to Jordan classification of endomorphisms of finite dimensional vector spaces.

Theorem

Let $\mathrm{Reg}(K((t)))$ *be the category of regular differential modules. Then*

$$\mathrm{Reg}(K((t))) \;\cong\; \mathrm{Rep}_K(\mathbb{Z}^{env})$$

where $\mathbb{Z}^{env}$ *is the algebraic envelop of* $\mathbb{Z}$*.*

We have

$$\mathbb{Z}^{env} \;=\; \mathbb{G}_a \times Z$$

with

$$Z \;:=\; \mathrm{Hom}(K/\mathbb{Z}, K^{\times}) \;=\; Gal^{diff}(\; K((t))[t^a, a \in K]/K((t)) \;)$$

Slogan

Diff. eq.	Solution	Tannakian Group	Character group
$(t\frac{d}{dt})(y) = ay$	t^a	$Z = Hom(K/\mathbb{Z}, K^{\times})$	$K/\mathbb{Z} = Hom(Z, K^{\times})$

The group $\mathbb{G}_a$ "describes" the extensions of such rank one diff.eq.

Turrittin-Hukuhara-Levelt theorem

For all $n \geq 1$ we can pull-back M over $K((t^{\frac{1}{n}}))$:

$$\mathrm{M}_n := \mathrm{M} \otimes_{K((t))} K((t^{\frac{1}{n}}))$$

It is a differential module over $K((t^{\frac{1}{n}}))$.

Theorem

*There exists $n \geq 1$ such that M_n is successive extension of **rank one** differential modules. Moreover*

- *The slopes are multiplied by n :*

$$M(s)_n = M_n(ns)$$

- *The exponents of $\mathrm{M}(0)$ are also multiplied by n.*
- M_n ***is trivial*** *if and only if M is direct sum of rank one modules of the form $\mathrm{N}(a)$ with $a \in \frac{1}{n}\mathbb{Z}$.*

Tannakian group

Theorem

There is an equivalence of categories

$$d - \mathrm{Mod}(K((t))) \;\cong\; \mathrm{Rep}_K(G) \qquad (1)$$

where G is a pro-algebraic group satisfying

- *there is an exact sequence*

$$1 \to \mathcal{T} \to G \to \mathbb{Z}^{env} \to 1$$

  where $\mathcal{T}$ is a ***pro-torus****.*
- *G is a* ***semi-direct product***

$$G \;=\; \mathcal{T} \rtimes \mathbb{Z}^{env}$$

Description of $\mathcal{T}$

The group $\mathcal{T}$ is

- **abelian** ;
- It is the dual of the group

$$\mathcal{Q} = \bigcup_{n\geq 1} t^{-1/n}K[t^{-1/n}]\ .$$

$$\mathcal{T} = Hom(\mathcal{Q}, K^{\times})$$

Slogan

Diff. eq.	Solution	Tannakian Group	Character group
$q \in \mathcal{Q}$ $y' = q'(t)y$	$\exp(q(t))$	$\mathcal{T} = Hom(\mathcal{Q}, K^{\times})$	$\mathcal{Q} = Hom(\mathcal{T}, K^{\times})$

The filtration of $\mathcal{Q}$ by the degree induces a filtration on G.

Exponents for irregular differential equations

The existing definition of exponents is given only for **regular** (i.e. slope zero) differential modules.

Idea :

- Exponents for **irregular** differential modules are well defined in rank one
- Use Turrittin's theorem plus Galois descent to reduce to rank one case.

In the end, we will obtain a direct Tannakian definition bypassing this process.

Exponents for irregular differential equations

Rank one irregular differential modules

Let

$$y' = (q'(t) + \frac{a}{t}) \cdot y \; , \qquad q' \in t^{-1}K[t^{-1}]$$

be a rank one diff. equation.

We have

$$Pic(K((t))) \cong t^{-1}K[t^{-1}] \oplus \frac{K}{\mathbb{Z}} \; ,$$

so the class of a in $K/\mathbb{Z}$ is an invariant of the isomorphism class.

Definition (Exponents in rank one)

We call the image of a in $K/\mathbb{Z}$ **the exponent** of this differential module.

The definition extends trivially to successive **extensions** (in the sense of exact sequences) of rank one differential modules.

Exponents for irregular differential equations

Reduction to Rank one case

Let $K((t^{1/n}))/K((t))$ be a finite étale extension such that $\mathrm{M}_n = \mathrm{M} \otimes_{K((t))} K((t^{1/n}))$ is extension of rank one differential modules.

How do we define the exponents of M ?

We know that the pull-back $\mathrm{M} \mapsto \mathrm{M}_n$ sends a rank one regular differential module $\mathrm{N}(a) = (t\frac{d}{dt} - a)$ into

$$\mathrm{N}(a)_n \;=\; \mathrm{N}(na)\;.$$

Therefore, it would be natural to define the exponents of M as those of M_n divided by n. This is actually the approach of [ABC20].

However, **this kills rational exponents** as we have the exact sequence

$$0 \to \mathbb{Z}[1/n]/\mathbb{Z} \to K/\mathbb{Z} \xrightarrow{a \mapsto na} K/\mathbb{Z} \to 0$$

In particular, this will be a problem in p-adic, as ***all** the exponents are rational in presence of a Frobenius structure*.

Exponents for irregular differential equations

This permits to associate to any differential module M an **intrinsic exponent** as a multi-set in

$$K/\mathbb{Q}\ .$$

We now propose a definition which is not canonical, but furnishes naturally exponents in $K/\mathbb{Z}$.

Remember that

$$Reg(K((t))) \cong Rep_K(\mathbb{Z}^{env})$$

We need to start from a representation of G and find back a representation of $\mathbb{Z}^{env}$.

Now, $G = \mathcal{T} \rtimes \mathbb{Z}^{env}$. We do not have knowledge of a theory capable to produce a **canonical** functor

$$\mathbf{R} \;:\; Rep_K(G) \longrightarrow Rep_K(\mathbb{Z}^{env}) .$$

There is a non canonical solution : consider the exact sequence

$$1 \to \mathcal{T} \to G \to \mathbb{Z}^{env} \to 1$$

Chose a section $$s : \mathbb{Z}^{env} \to G .$$

Then, the functor **R** can be just the restriction to $s(\mathbb{Z}^{env})$.

Definition (Tannakian exponents over $K((t))$)

Let M be a differential module over $K((t))$.

See M as a representation of G.

The restriction of M to $s(\mathbb{Z}^{env}) \subset G$ is a representation in

$$\mathrm{M}_{|s(\mathbb{Z}^{env})} \in Rep(\mathbb{Z}^{env})$$

It corresponds to a **regular** differential module in $Reg(K((t)))$ whose exponent multi-set is called the **Tannakian exponent** of M.

In a suitable way, the above definition is compatible with Turrittin result.

It is clear that the exponent multi-set so obtained is

- invariant by isomorphisms of differential modules
- compatible with
  - direct sums,
  - exact sequences,
  - tensor products,
  - internal homs,
  - duals

p-adic exponents for irregular differential modules over the Robba ring

The Robba ring as a counterpart of K((t))

Recall that $(K, |.|)$ is an algebraically closed complete valued field w.r.t. a non archimedean absolute value $|.|$.

Definition

Let $I \subset \mathbb{R}_{>0}$ be an interval. Define the ring

$$\mathcal{O}(I)$$

of analytic functions on the annulus $\{|t| \in I\}$ as the ring of series

$$f(t) = \sum_{i \in \mathbb{Z}} a_i t^i$$

where $a_i \in K$ and such that f converges for all t satisfying $|t| \in I$. Define the Robba ring $\mathcal{R}$ as

$$\mathcal{R} \ := \bigcup_{0<e<1} \mathcal{O}(]e, 1[) \ .$$

The Robba ring as a *p*-adic counterpart of K((t))

- Within rigid cohomology, $\mathcal{R}$ is an appropriate lifting in characteristic 0 of a field of power series in characteristic p.

- If the valuation of K is trivial, we actually find

$$\mathcal{R} \ = \ K((t)) \ .$$

Frobenius structure on differential modules over $\mathcal{R}$

In the sequel, we consider the categories

$$d - Mod(\mathcal{O}(I)) \quad \text{and} \quad d - Mod(\mathcal{R})$$

Moreover, we pay a particular attention to the category

$$d - Mod(\mathcal{R})^{(\varphi)}$$

of differential modules over $\mathcal{R}$ with an **unspecified Frobenius structure.**

This means that we chose a Frobenius map $\varphi : \mathcal{R} \to \mathcal{R}$ and we assume that our differential modules M have a unspecified action of φ commuting with the connection. However, we do not ask the morphisms of the category to commute with the Frobenius action. (Please forgive the abuse here : in some theorems K needs to be discretely valued, but this abuse can be easily fixed ...)

What is a **regular** differential module over these rings ?

Recall that over $K((t))$: **regular = slope zero**

What is the slope in p-adic ?

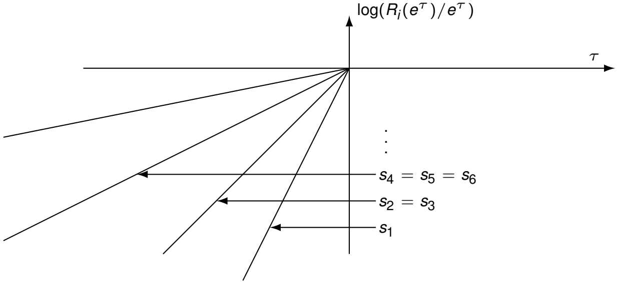
No cyclic vector nor slopes in p-adic (not always)

Over the above rings, we have **no cyclic vector theorem.**

For general differential modules over $\mathcal{O}(I)$ there is no notion of **slopes**. Not even if you use the definition with the radii o convergence.

Slope theory in presence of Frobenius

However, for differential modules in $d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$, we have the same **solvability property** as in the formal case.



Regular modules in p-adic

Moreover, for differential modules M in $d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$, we have a decomposition by the slopes

$$\mathrm{M} = \oplus_{s \geq 0} \mathrm{M}(s)$$

Exponents for Regular differential modules.

Christol-Mebkhout, after Robba, proposed then the following definition

Definition (Robba)

A module $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{O}(]a, b[))$ (possibly with no Frobenius) is **regular** (or **Robba**) if all its solutions converge with **maximal radius.** If $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$, this means $\mathrm{M} = \mathrm{M}(0)$.

The following is due to [CM97, Dwo97, Ked15, KS17].

Theorem (Existence of exponents)

*For every **regular** $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{O}(]a, b[))$, there exists an exponent.*

The exponent is an element of a **certain quotient of** $(\mathbb{Z}_p/\mathbb{Z})^{rank(\mathrm{M})}$.

Theorem (p-adic Fuchs theorem)

*Let $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{O}(]a, b[))$ be a **regular** module. If the exponent satisfy the **DNL** condition (=the Exponents and their differences are non Liouville), then M is extension of **rank one** sub-quotients.*

A p-adic Turrittin holds for $d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$

We also have an analogous of Turrittin theorem.

- Let k be the residual field of K.
- Let $\mathcal{I} := Gal(k((t))^{sep}/k((t)))$.
- Finite separable extensions of $k((t))$ lift canonically into finite étale extensions of $\mathcal{R}$.

Theorem (Y.André-K.S.Kedlaya-Z.Mebkhout)

For every $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$, there is a finite étale extension $\mathcal{R}'$ of $\mathcal{R}$ coming from a finite separable extension of $k((t))$ such that $\mathrm{M} \otimes_{\mathcal{R}} \mathcal{R}'$ is unipotent (i.e. it is extension of rank one trivial sub-quotient).

The Tannakian group G_p of $d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$ is

$$G_p \ = \ \mathcal{I} \times \mathbb{G}_a$$

Analogies and differences with respect to $K((t))$

	$d-Mod(K((t)))$	$d-Mod(\mathcal{R})$	$d-Mod(\mathcal{R})^{(\varphi)}$
Existence of cyclic vector	Yes	?	Yes
Slopes	Yes	?	Yes
Decomposition by the slopes	Yes	?	Yes
Notion of Regular diff. mod.	Yes	Yes (?)	Yes
Decomposition of $\mathrm{M}(0)$ by the exponents	Yes	?	Yes
Tannakian group	$G = \mathcal{T} \rtimes \mathbb{Z}^{env}$	$\mathbf{G} = ?$	$G_p = \mathcal{I} \times \mathbb{G}_a$
Regular Tannakian group	$\mathbb{Z}^{env} = Z \times \mathbb{G}_a$	?	$\widehat{\mathbb{Z}}' \times \mathbb{G}_a$

Remind that

$$G = \mathcal{T} \rtimes \mathbb{Z}^{env}, \qquad \mathbb{Z}^{env} = Z \times \mathbb{G}_a.$$

We actually have for $d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$ a similar situation.

$$\mathcal{I} = \mathcal{P} \rtimes \widehat{\mathbb{Z}}'$$

where $\widehat{\mathbb{Z}}' = \prod_{\ell \neq p} \mathbb{Z}_\ell$ and $\mathcal{P}$ is the wild inertia subgroup of $\mathcal{I}$.

So that

$$G_p = \mathcal{I} \times \mathbb{G}_a = \mathcal{P} \rtimes (\widehat{\mathbb{Z}}' \times \mathbb{G}_a).$$

Regular differential modules

A differential module $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{R})^{(\varphi)}$ is **regular** if, and only if, it corresponds to a representation of the group

$$G_p/\mathcal{P} = \widehat{\mathbb{Z}}' \times \mathbb{G}_a.$$

p-adic Tannakian exponents for irregular modules in $d-\mathrm{Mod}(\mathcal{R})^{(\varphi)}$

Definition (Tannakian Exponents for irregular modules in $d-\mathrm{Mod}(\mathcal{R})^{(\varphi)}$)

Consider a section $\widehat{\mathbb{Z}}' \to \mathcal{I}$. This produces a section

$$s \;:\; \widehat{\mathbb{Z}}' \times \mathbb{G}_a \to G_p$$

Let $\mathrm{M} \in d-\mathrm{Mod}(\mathcal{R})^{(\varphi)}$, which we see as a representation of G_p.

The restriction

$$\mathrm{M}_{|s(\widehat{\mathbb{Z}}' \times \mathbb{G}_a)}$$

corresponds to a regular differential module (in the sense of Robba), which has a well defined exponent. We call it the **Tannakian exponent of M** .

Compatibility with the standard operations

It is clear that, this is compatible with all the standard operations such as

- direct sum
- tensor product
- dual
- internal Hom
- ...

p-adic Tannakian exponents without Frobenius

Really regular modules

Let $R \in \{ K((t)) , \mathcal{R} , \mathcal{O}(I) \}$.

Definition (Really regular modules)

A differential module is **really regular** if it is **extension of rank one** sub-quotients of the form $(t\frac{d}{dt} - a)$.

Theorem

Let

$$RReg(R)$$

be the full sub-category of $d - \mathrm{Mod}(R)$ *whose objects are isomorphism classes of* ***really regular modules.***

Then the Tannakian group of $RReg(R)$ *is*

$$\mathbb{Z}^{env} .$$

p-adic Tannakian exponents in general

Universal property of $\mathbb{Z}^{env}$

The group $\mathbb{Z}^{env}$ is a **projective object** in the category of pro-algebraic groups.

In particular, for every surjective map $\mathbf{G} \to \mathbb{Z}^{env}$ there is a section

$$s \ : \ \mathbb{Z}^{env} \to \mathbf{G}$$

Definition (Tannakian exponents in general)

Let $R \in \{K((t)), \mathcal{R}, \mathcal{O}(I)\}$. Let $\mathrm{M} \in d - \mathrm{Mod}(R)$.

Let **G** be the Tannakian group of the category $d - \mathrm{Mod}(R)$.

We have a natural surjective projection

$$\mathbf{G} \to \mathbb{Z}^{env}$$

associated with the inclusion of categories

$$d - \mathrm{Mod}(R) \quad \supset \quad RReg(R)$$

Chose a section (which exists because $\mathbb{Z}^{env}$ is aprojective object)

$$s \ : \ \mathbb{Z}^{env} \to \mathbf{G}$$

Define the **Tannakian exponent** of M as that associated with the representation

$$\mathrm{M}_{|s(\mathbb{Z}^{env})} \ .$$

Again we have immediately the nice properties with respect to

- direct sum
- tensor product
- dual
- internal Hom
- ...

Compatibility

If M is Regular (in the sense of Robba), **and if it is extension of rank one modules**, the above definition coincides with that of Christol-Mebkhout-Dwork-Kedlaya-Shiho :

Tannakian exponent = exponent

In particular, this is the case in presence of a Frobenius structure on M.

We plan to deal with the following open questions :

- Compare the two definitions when M does not split into rank one pieces.
- Assume that $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{O}(I))$ is regular in the sense of Robba (i.e. maximal solutions), and it satisfies the **DNL** condition on the Tannakian exponents. Then, is M extension of rank one sub-quotients ? (i.e. ***p*-adic Fuchs theorem using Tannakian exponents**)
- If we have a Frobenius structure on a module $\mathrm{M} \in d - \mathrm{Mod}(\mathcal{R})$, we know that the two definitions coincide. Is there another Tannakian proof of the fact that we have **rational exponents** ?
- How much is this Tannakian definition of exponents intrinsic ? It actually depends on the choice of the coordinate t.
- Can we control the behavior of these Tannakian exponents by pull-back and push-forward by finite étale morphisms ? (this is essentially unknown for classical exponents).

Thank you !

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