An uncountable Mittag-Leffler condition with an application to Ultrametric Locally Convex Spaces

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More specifically, we are interested here in a precise criterion, originally due to Mittag-Leffler [Bou07, II.19, *N*^{o}5, Exemple], ensuring that exactness of short exact sequences is preserved when passing to the limit.

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- A directed subset $I' \subset I$ is ***cofinal*** if for every $i \in I$ there exists $i' \in I'$ such that $i' \geq i$.
- An ***inverse system*** indexed on I is a collection of left R-modules $(S_i)_{i \in I}$ indexed by I, together with a family of maps

$$(\rho^S_{i,j} : S_i \to S_j)_{(i,j) \in I^2, i \geq j}$$

such that
  - for all $i \in I$ the map $\rho^S_{i,i}$ is the identity map of S_i,
  - for all $i, j, k \in I$ such that $i \geq j \geq k$ one has $\rho^S_{j,k} \circ \rho^S_{i,j} = \rho^S_{i,k}$.

- For any two systems $S = (S_i, \rho^S_{i,j})$ and $T = (T_i, \rho^T_{i,j})$ indexed on the same I a morphism $f : S \to T$ is a collection

$$(f_i : S_i \to T_i)_{i \in I}$$

of R-linear maps such that for every $i \geq j$ the following diagram commutes

$$\begin{array}{ccc} S_i & \xrightarrow{f_i} & T_i \\ \downarrow{\scriptstyle \rho^S_{i,j}} & & \downarrow{\scriptstyle \rho^T_{i,j}} \\ S_j & \xrightarrow{f_j} & T_j \end{array} \tag{3}$$

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- An inverse system $(S_i, \rho^S_{i,j})$ is nothing but a ***functor*** from the category $(I, \leq)$ to the category $R - Mod$ and a morphism $f : S \to T$ is just a morphism of functors.

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- Similarly, we can define the category of ***inverse systems of sets*** as functors from I to the category of sets.

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- Every operation in $R - Mod$ transports into the same operation in $R - Mod^I$. We have the notions of ***Kernels, coKernels, Images, coImages, direct sums, products, ...***.In particular $R - Mod^I$ is an ***abelian category***.
- In particular an ***exact sequence of inverse systems*** is a collection $(0 \to A_i \xrightarrow{g_i} B_i \xrightarrow{h_i} C_i \to 0)_{i \in I}$ of exact sequences such that for every $i \geq j$ we have commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_i & \xrightarrow{g_i} & B_i & \xrightarrow{h_i} & C_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho^A_{i,j}} & & \downarrow{\scriptstyle \rho^B_{i,j}} & & \downarrow{\scriptstyle \rho^C_{i,j}} & & \\ 0 & \longrightarrow & A_j & \xrightarrow{g_j} & B_j & \xrightarrow{h_j} & C_j & \longrightarrow & 0 \end{array}$$

- The ***inverse limit*** of a system $(S_i, \rho^S_{i,j})$ is the subset of the product $\prod_{i \in I} S_i$ formed by the vectors $(x_i)_{i \in I}$ that are compatible, in the sense that, for all $i \geq j$, one has

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$$\begin{array}{ccc} \varprojlim_{i\in I} S_i & \longrightarrow & S_i \\ & \searrow & \downarrow{\scriptstyle \rho^S_{i,j}} \\ & & S_j \end{array}$$

- For every morphism of inverse systems $S \to T$ we have a morphism $\varprojlim_{i\in I} S_i \to \varprojlim_{i\in I} T_i$.

- The inverse limit is a functor

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is an exact sequence of inverse systems in $R - Mod^I$, then

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In other words, the functor $\varprojlim_{i\in I}$ is ***left exact***.

- The functor $\varprojlim_{i\in I} : R - Mod^I \to R - Mod$ ***can be derived***. We call

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- For every short exact sequence $0 \to (A_i) \to (B_i) \to (C_i) \to 0$ of systems in $R - Mod^I$ we have a ***long exact sequence*** in $R - Mod$

$$0 \quad \to \quad \varprojlim_{i\in I} A_i \to \varprojlim_{i\in I} B_i \to \varprojlim_{i\in I} C_i \stackrel{\delta_1}{\to}$$

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Theorem 1. (Classical Mittag-Leffler)

Let I be a directed poset. Let $0 \to (A_i) \to (B_i) \to (C_i) \to 0$ be an exact sequence of inverse systems in $R - Mod$. Assume that

1. There exists a cofinal subset of I which is at most ***countable*** ;
2. For all $i \in I$, there exists $j \geq i$ such that for all $r \geq j$ one has

$$\rho^A_{j,i}(A_j) = \rho^A_{r,i}(A_r)\ . \tag{4}$$

Then, the first derived functor $\varprojlim^{(1)}_{i\in I}$ of $\varprojlim_{i\in I}$ vanishes at $(A_i)_i$:

$$\varprojlim^{(1)}_{i\in I} A_i = 0\ .$$

In particular, the short sequence of limits

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Aim : we want to relax the countability assumption (1).

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- In particular, the claim implies $\varprojlim^{(n)}_{i\in I} A_i = 0$, for all integer $n \geq 0$, because this is true for $I = \mathbb{N}$.

Where is condition 1 used in the proof ?

- Let A, B, C be the limits. For all $i \geq j$ we have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{h} & C & & \\
 & & \downarrow{\scriptstyle \pi_i^A} & & \downarrow{\scriptstyle \pi_i^B} & & \downarrow{\scriptstyle \pi_i^C} & & \\
0 & \longrightarrow & A_i & \xrightarrow{g_i} & B_i & \xrightarrow{h_i} & C_i & \longrightarrow & 0 \\
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- The inverse images $S_i := h_i^{-1}(c_i)$ are stable by the map $\rho_{i,j}^B$ and form an ***inverse system of non empty sets*** and we have

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h is surjective if, and only if, for every $c \in C$ this limit is ***not empty***.

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This is another inverse system with ***surjective maps*** and s.t.

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$$\varprojlim_i S_i = \varprojlim_i S'_i .$$

- At this point, an ***induction on*** $\tau(\mathbb{N}) \subset I$ permits to construct step by step a sequence in $\lim_i S'_i$.

- Is $\varprojlim_i S_i$ empty ?
- If $b_i \in S_i$, then

$$S_i = b_i + A_i$$

and for every $j \leq i$ the map $\rho^B_{i,j} : S_i \to S_j$ can be identified (composing with the addition of b_i) with $\rho^A_{i,j} : A_i \to A_j$. We will see that, in the language of sheaves, this is a ***LOCAL*** isomorphism of systems.
- In particular, the system $(S_i)_i$ satisfies Mittag-Leffler condition.
- For every $i \in I$ let

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- At this point, an ***induction on*** $\tau(\mathbb{N}) \subset I$ permits to construct step by step a sequence in $\lim_i S'_i$.

More precisely, Mittag-Leffler condition implies that $(S'_i)_i$ is

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  These are strong condition on the objects of the system.

- The limit $\lim^{(1)}_{i\in I} A_i$ may not vanish, even if the system has surjective maps.

Cohomological dimension of a directed poset

- For any ring R there exists an (enormous) directed poset I and an inverse system $(A_i)_{i \in I}$ such that ***for all*** $n \geq 0$ one has (cf. [Jen72])

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- If $\aleph_k$ is the smallest ordinal of a cofinal directed subset of I, then we have

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for all inverse systems $(A_i)_{i\in I}$ and all $n \geq k+2$ (cf. [Mit73, Roo61, Gob70, Jen72]).

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  for all inverse systems $(A_i)_{i \in I}$ and all $n \geq k + 2$ (cf. [Mit73, Roo61, Gob70, Jen72]).
- Several other specific criteria exists under the assumption that R is Noetherian and the modules A_i satisfy specific conditions ...

In particular, this little panorama shows that for the vanishing of $\lim^{(1)}_{i\in I} A_i$ in Theorem 1, some ***finiteness*** condition is needed either on the set I, or on the objects, or on the transition maps.

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- On the other hand, the quoted statements of Bourbaki, or their consequence for Artinian R-modules, can be considered as finiteness condition on the nature of the objects A_i.

Surprisingly enough, if I does not contain any cofinal countable subset and if no condition about on R and the modules A_i are made, then in our knowledge ***no statement ensuring the vanishing of*** $\varprojlim^{(1)}_{i \in I} A_i$ ***exists in literature***.

Nevertheless, in this general context, there are interesting cases of inverse systems behaving very similarly to Mittag-Leffler ones just because much part of the restriction maps $\rho^{A}_{i,j}$ are isomorphisms and their limit is then ***"controlled" by some countable subset of maps***.

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Situations of this type show up for instance in sheaf theory as pull-back of some sheaf on a stain space which actually inspired our approach to this problem. Another interesting example is provided by the theory of ultrametric locally convex topological vector spaces as we will see in the last part of this talk.

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In this situation, any a direct set-theoretical attempt of the proof of Mittag-Leffler theorem based on the lifting non vanishing of the system $(S_i)_{i\in I}$ is unhelpful, as one can easily see.

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We provide here two generalizations of Theorem 1 to the case of an uncountable I without countable cofinal subsets that only involve ***a finiteness condition on the transition maps*** of the system $(A_i)_{i \in I}$ and no conditions on I nor on the objects.

Theorem 2. (pull-back)

Let $(\rho^A_{i,j} : A_i \to A_j)_{i,j\in I}$ be an inverse systems of left R-modules indexed on I. Assume that there exists another directed partially ordered set $(J,\leq)$ and an inverse system of R-modules $(\rho^S_{i,j} : S_i \to S_j)_{i,j\in J}$ s.t.

1. There exist cofinal directed subsets $I' \subseteq I$ and $J' \subseteq J$ and a surjective map $p : I' \to J'$ preserving the order relation ;
2. There exists a system of R-linear isomorphisms $(\psi_i : A_i \xrightarrow{\sim} S_{p(i)})_{i\in I'}$ such that for all $i,j \in I'$ with $i \geq j$ one has a commutative diagram

$$\begin{array}{ccc} A_i & \xrightarrow[\sim]{\psi_i} & S_{p(i)} \\ {\scriptstyle \rho^A_{i,j}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho^S_{p(i),p(j)}} \\ A_j & \xrightarrow[\sim]{\psi_j} & S_{p(j)} \end{array}$$

Then, for all integer $n \geq 0$, we have a canonical isomorphism

$$\varprojlim_{i\in I}{}^{(n)} A_i \xrightarrow{\sim} \varprojlim_{j\in J}{}^{(n)} S_j .$$

In particular, if J and $(S_j)_{j\in J}$ satisfy Theorem 1, then $\varprojlim^{(n)}_{i\in I} A_i = 0$ for all $n \geq 1$.

- Theorem 2 implies Theorem 1. Indeed, if $I' \subseteq I$ is a countable cofinal directed subset in Theorem 1, then $I' = J$, $S = A$, $\psi = id$ satisfies the assumptions of Theorem 2.

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  1. inverse limits of sets indexed on an uncountable poset may be empty even with surjective transition maps ;

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  1. inverse limits of sets indexed on an uncountable poset may be empty even with surjective transition maps ;
  2. the proof of Bourbaki provides only a ***LOCAL*** isomorphism of the ***systems of sets***, which actually does not preserve the non vanishing of the limit.

Theorem 3. (Push-forward)

Let $(\rho^A_{i,j} : A_i \to A_j)_{i,j \in I}$ be an inverse systems of left R-modules indexed on I.

Assume that there exists a *directed* partially ordered set $(J, \leq)$ together with an inverse system of R-modules $(\rho^T_{i,j} : T_i \to T_j)_{i,j \in J}$ such that

(i) There exists cofinal directed subset $I' \subseteq I$ and $J' \subseteq J$ and a map $q : J' \to I'$ preserving the order relation such that for all $i \in I'$, the set

$$U_i := \{j \in J', q(j) \leq i\},$$

endowed with the partial order induced by J', satisfies at least one of the following conditions

1. U_i is empty ;
2. U_i has a unique maximal element $r(i)$;
3. U_i is directed, it has countable cofinal directed poset J'_i and the system $(\rho^T_{j,k} : T_j \to T_k)_{j,k \in J'_i}$ satisfies Mittag-Leffler Theorem.

(ii) For all $i \in I'$ there exists an R-linear isomorphisms $\phi_i : A_i \xrightarrow{\sim} \varprojlim_{j \in U_i} T_j$ such that for all $k \in I'$ with $k \geq i$ one has a commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow[\sim]{\phi_i} & \varprojlim_{j \in U_k} T_j \\ \rho^A_{k,i} \downarrow & \circlearrowleft & \downarrow \alpha_{k,i} \\ A_i & \xrightarrow[\sim]{\phi_j} & \varprojlim_{j \in U_i} T_j \end{array}$$

where the right hand vertical arrow $\alpha_{k,i}$ is deduced by the universal properties of the limits as $U_i \subset U_k$.

Then, for all integer $n \geq 0$, we have a canonical isomorphism

$$\varprojlim_{i \in I}{}^{(n)} A_i \xrightarrow{\sim} \varprojlim_{j \in J}{}^{(n)} T_j. \tag{5}$$

In particular, if J and $(T_j)_{j \in J}$ satisfy Theorem 1, then $\varprojlim^{(n)}_{i \in I} A_i = 0$ for all $n \geq 1$.

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- If $J = \mathbb{N}$, and T satisfies Mittag-Leffler, then conditions 1, 2, 3 on U_i are automatic.

- Moreover, if $J = \mathbb{N}$ and if we impose that the image of J in I is never contained in some $\Lambda(i)$ for all $i \in I$, then 3 is impossible, while 1 and 2 are automatically verified.

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- If $J = \mathbb{N}$, and T satisfies Mittag-Leffler, then conditions 1, 2, 3 on U_i are automatic.
- Moreover, if $J = \mathbb{N}$ and if we impose that the image of J in I is never contained in some $\Lambda(i)$ for all $i \in I$, then 3 is impossible, while 1 and 2 are automatically verified.
- Again, it is possible to prove that $\varprojlim_{i\in I} A_i = \varprojlim_{j\in J} T_j$, but the proof of Bourbaki doesn't permit to prove the equality of $\varprojlim^{(n)}_{i\in I}$ for $n \geq 1$.

Idea of the proof

The idea of the proof is based on a (well known) ***coincidence of theories*** :

$$\{\text{Inverse systems of } R\text{-modules over } I\}$$
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- The above Theorems relate the cohomology of a sheaf on $X(J)$ with that of its pull-back and its push-forward on $X(I)$.

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- The above Theorems relate the cohomology of a sheaf on $X(J)$ with that of its pull-back and its push-forward on $X(I)$.
- These results do not have an analogous for general topological spaces as we use properties that are specific of posets.

Inverse systems and sheaves

Topological space associated to a poset

Let I be a poset.

- Remind : for all $i \in I$ we set

$$\Lambda(i) \quad = \quad \{j \in I, j \leq i\} \ ,$$

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$$\begin{aligned} \Lambda(i) &= \{j \in I, j \leq i\} \,, \\ V(i) &= \{j \in I, j \geq i\} \,. \end{aligned} \tag{6}$$

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$$V(i) = \{j \in I, j \geq i\} . \quad (7)$$

- The family $\{\Lambda(i)\}_{i \in I}$ is a ***basis*** for a topology on the set I.
- Let $X(I)$ be this topological space :
  - points of $X(I)$ =points of I ;
  - A subset $U \subseteq I$ is ***open*** if, and only if, for all $i \in U$ we have $\Lambda(i) \subseteq U$;
  - A subset $C \subseteq I$ is ***closed*** if, and only if, for all $i \in C$ we have $V(i) \subseteq C$;
- Arbitrary intersections of opens are opens ;
- Arbitrary unions of closed are closed ;
- $\Lambda(i)$ =smallest open containing i ;
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  - If we have a family of sections $s_i \in S(U_i)$ such that for all i, j we have $s_i = s_j$ on $U_i \cap U_j$, then they ***glue*** and there exists $s \in S(U)$ such that $s = s_i$ on U_i.

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From now on inverse system on I = sheaf on $X(I)$

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- For any short exact sequence of sheaves $0 \to S' \to S'' \to S''' \to 0$ we have a long exact sequence

$$0 \quad \to \quad S'(X) \to S''(X) \to S'''(X) \xrightarrow{\delta_1} \tag{8}$$

$$\xrightarrow{\delta_1} \quad H^n(X, S') \to H^1(X, S'') \to H^1(S''', X) \xrightarrow{\delta_2} \cdots \tag{9}$$

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- If $0 \to S \to F_1 \to F_2 \to \cdots$ is a long exact sequence where F_k is ***flabby/injective/acyclic***, then we have a long sequence

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- There is a class of sheaves called ***weakly flabby*** that are acyclic and f^* preserves weakly flabby resolutions. ***END of PROOF***

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