On p-adic vector bundles and local systems on diamonds

Annette Werner

Report on joint works with Marvin Anas Hahn (Leipzig) and Lucas Mann (Bonn)

Tropical Geometry, Berkovich Spaces, Arithmetic $\mathcal{D}$-Modules and p-adic Local Systems
December 8-10, 2020

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Goal: Relate p-adic vector bundles to p-adic local systems.

Motivation:

Narasimhan-Seshadri correspondence (1965) on a compact Riemann surface X, relating irreducible unitary representations of $\pi_1^{top}(X, x)$ to stable vector bundles on X of degree 0.

Simpson's correspondence (1992) on smooth projective complex varieties X, relating finite dimensional complex representations of $\pi_1^{top}(X, x)$ to semistable Higgs bundles on X with vanishing Chern classes.

p-adic results

Deninger/W. (2005): p-adic analog of Narasimhan-Seshadri correspondence relating vector bundles on smooth proper curves over $\overline{\mathbb{Q}}_p$ with nice (potentially strongly semistable) reductions to continuous p-adic representations of the étale fundamental group.

p-adic results

Faltings' p-adic Simpson corrrespondence (2005) on smooth proper curves X over $\overline{\mathbb{Q}}_p$, which provides an equivalence of categories between p-adic Higgs bundles on $X_{\mathbb{C}_p}$ and generalized representations of the étale fundamental group.

See also Abbes, Gros, Tsuji (Annals of Math Studies 2016) for an alternative and more general approach.

Xu (2017): Both approaches are compatible in the case of trivial Higgs field.

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Liu and Zhu's p-adic Riemann Hilbert correspondence (2016) associates on a smooth rigid variety X over finite extension K of $\mathbb{Q}_p$ to every p-adic étale local system a Higgs bundle on $X_{\mathbb{C}_p}$ with nilpotent Higgs field.

Vector bundles with numerically flat reduction

Let $\mathfrak{o}_p \subset \mathbb{C}_p$ with residue field $k \simeq \overline{\mathbb{F}}_p$.

Let X be a smooth proper connected variety over $\overline{\mathbb{Q}}_p$.

Denote by $\mathcal{B}_X$ the (full) category of all vector bundles E on $X_{\mathbb{C}_p}$ with numerically flat reduction, i.e. such that

- there exists a flat, proper scheme $\mathcal{X}$ of finite presentation over $\overline{\mathbb{Z}}_p$ with generic fiber X, and
- there exists a vector bundle $\mathcal{E}$ on $\mathcal{X} \otimes \mathfrak{o}_p$ with generic fiber E such that the special fiber $\mathcal{E}_k = \mathcal{E} \otimes k$ is numerically flat on $\mathcal{X}_k$.

Vector bundles with numerically flat reduction

We call the bundle $\mathcal{E}_k$ on $\mathcal{X}_k$ numerically flat if both $\mathcal{E}_k$ and its dual $\mathcal{E}_k^*$ are numerically effective.

Langer: This is equivalent to the fact that for all k-morphisms $f : C \longrightarrow \mathcal{X}_k$ from a smooth projective curve C the bundle $f^*\mathcal{E}_k$ is semistable of degree 0 on C (Nori).

Note that if X is projective, then the numerically flat line bundles are precisely the ones in the torsion component $\mathrm{Pic}^\tau(X)$.

$\mathcal{E}_k$ numerically flat implies that E is numerically flat. If X is projective, this means that E is semistable with vanishing Chern classes.

Theorem 1 (Deninger / W. 2020)

Let X be a proper, smooth, connected variety over $\overline{\mathbb{Q}}_p$.
Let E be a vector bundle on $X_{\mathbb{C}_p}$. If $\alpha : Y \longrightarrow X$ is finite étale surjective with $\alpha^* E$ in $\mathcal{B}_Y$, i.e. if $\alpha^* E$ has numerically flat reduction on some model of Y, then E admits p-adic parallel transport.

The functor of p-adic parallel transport is a continuous functor

$$\rho_E : \Pi_1(X_{\mathbb{C}_p}) \longrightarrow \mathrm{Vec}_{\mathbb{C}_p}$$

from the étale fundamental groupoid on $X_{\mathbb{C}_p}$ to the category of finite dimensional $\mathbb{C}_p$-vector spaces which is $x \mapsto E_x$ on objects, i.e. $\mathbb{C}_p$-rational points.

In particular, E gives rise to a p-adic étale local system on $X_{\mathbb{C}_p}$.

p-adic parallel transport

The functor $E \mapsto \rho_E$ from Theorem 1 is functorial in E, exact and well-behaved with respect to tensor products, pullbacks, internal homomorphisms and Galois-conjugation.

Any numerically flat line bundle L satisfies condition i) of Theorem 1, i.e. $\alpha^* L$ has numerically flat reduction on some model of a finite étale cover $\alpha : Y \to X$.

Theorem 2 (Deninger / W. 2005, 2007)

If X is a curve, then Theorem 1 holds for arbitrary finite coverings $\alpha : Y \to X$ (not necessarily étale ones).

Würthen (2019)

Let X be a proper, connected seminormal rigid analytic variety over $\overline{\mathbb{Q}}_p$. Then the results of Theorem 1 hold for all analytic vector bundles E on X with numerically flat reduction on a proper flat formal model of X.

This allows us to drop smoothness.

Moreover, this approach works with locally free sheaves on the pro-étale site. Using adic spaces and their integral structure sheaves makes some cumbersome arguments with models of schemes superfluous.

Fundamental open question

Find a category equivalence between genuine $\mathbb{C}_p$-representions of the étale fundamental group and a suitable category of Higgs bundles.

In the case of vanishing Higgs field, we know that vector bundles with numerically flat reductions give genuine p-adic representations.

Assume we are given a smooth projective variety X over $\overline{\mathbb{Q}}_p$ and a numerically flat vector bundle E on X. How can we produce a finite étale cover $\alpha : Y \longrightarrow X$ and a numerically flat degeneration of $\alpha^* E$? We may find many models of X and E, but how do we control the behaviour of the reductions of these models?

If X is a curve, an arbitrary finite cover α suffices (which makes our choices even larger), but how do we control the pullbacks of the vector bundle in the special fiber under Frobenius powers?

Concrete example

Let $j : C \hookrightarrow \mathbb{P}^2_K$ be the Fermat curve $x^d + y^d + z^d = 0$ over some finite extension K of $\mathbb{Q}_p$.

Consider the stable, rank 2, degree 0 bundle

$$E = j^*\mathrm{Syz}(x^2, y^2, z^2)(3) \text{ on } C,$$

where $\mathrm{Syz}(x^2, y^2, z^2) = \ker(\mathcal{O}(-2)^3 \xrightarrow{(x^2,y^2,z^2)} \mathcal{O})$ on $\mathbb{P}^2_K$.

We have the obvious reduction, given by the Fermat curve over $\mathcal{O}_K$ and the pullback $\mathcal{E}$ of the analogous syzygy bundle on $\mathbb{P}^2_{\mathcal{O}_K}$. Its special fiber $\mathcal{E}_k$ is semistable, but not numerically flat: Brenner (2005) shows that certain Frobenius pullbacks of $\mathcal{E}_k$ are no longer semistable.

So we need reductions with "bad" singularities.
Idea: Use **Mustafin varieties**, which are degenerations of projective space $\mathbb{P}^n_K$ depending on a choice of finitely many lattices in K^{n+1}.

Theorem (Cartwright, Häbich, Sturmfels, W. 2011)

If the configuration Γ of lattice classes is contained in one apartment of the Bruhat-Tits building associated to $PGL_{n+1,K}$, then the special fiber of the associated Mustafin variety $\mathcal{M}(\Gamma)$ is determined by the regular mixed subdivision of the scaled simplex associated to the tropical convex hull of Γ.

Recall $E = j^*\mathrm{Syz}(x^2, y^2, z^2)(3)$ on the Fermat curve $j : C \hookrightarrow \mathbb{P}^2_K$, where $\mathrm{Syz}(x^2, y^2, z^2) = \ker(\mathcal{O}(-2)^3 \xrightarrow{(x^2,y^2,z^2)} \mathcal{O})$.

Theorem (Hahn, W. 2019)

For the example E on the projectively embedded Fermat curve C we find

- a (ramified) degree 2 cover $\alpha : D \to C$
- a configuration Γ of three lattices such that the closure $\mathcal{D}$ of D in the Mustafin variety $\mathcal{M}(\Gamma)$ satisfies: the special fiber consists of irreducible components C_i with each $C_i^{red} \simeq \mathbb{P}^1_k$
- a vector bundle $\mathcal{E}$ on $\mathcal{D}$ with generic fiber α^*E such that $\mathcal{E}_k|_{C_i^{red}}$ is trivial for all i.

This proves that the bundle E admits p-adic parallel transport.

Local systems on diamonds: joint work with Lucas Mann

Let X be a proper adic space of finite type over $\mathbb{C}_p$.

Scholze's theory of diamonds provides a fascinating tool to disguise characteristic zero objects as characteristic p. Diamonds are certain sheaves on the big pro-étale site on the category Perf of perfectoid characteristic p spaces, which are quotients of perfectoid spaces by pro-étale equivalence relations.

There is a functor $X \mapsto X^\diamond$ mapping analytic adic spaces over $\mathbb{Z}_p$ to (locally spatial) diamonds, such that

- $X_{et} \simeq X^\diamond_{et}$
- If K is a non-archimedean extension of $\mathbb{Q}_p$, the diamond functor from seminormal rigid analytic K-spaces to diamonds over $\mathrm{Spd}(K)$ is fully faithful

We use the v-topology on Perf: Coverings of Z are given by collections of maps to Z such that every quasi-compact open subset of Z can be covered by finitely many quasi-compact open subsets of our collection.

The v-site on a diamond X is then defined as the category of small v-sheaves mapping to X, with coverings given by jointly surjective maps. There is a natural v-structure sheaf $\check{\mathcal{O}}_X$ with integral structure sheaf $\check{\mathcal{O}}_X^+$.

For every condensed ring Λ (e.g. every metric ring), there is a notion of the asscoiated constant sheaf on a diamond, and hence a notion of local systems (locally free sheaf of Λ-modules) for the v-topology.

Let X be a proper adic space of finite type over $\mathbb{C}_p$. Let $\mathcal{M}^+$ be the full subcategory of all $\check{\mathcal{O}}^+_{X^\diamond}$-modules $\mathcal{E}$ on $X_v^\diamond$ such that for all n the $\check{\mathcal{O}}^+_{X^\diamond}/p^n$-module $\mathcal{E}/p^n$ becomes trivial on a proper cover of X.

There is a fully faithful functor Δ^+ from $\mathcal{M}^+$ to the category of $\mathfrak{o}_p = \mathcal{O}_{\mathbb{C}_p}$-local systems on X such that for every $\mathcal{E}$ in $\mathcal{M}^+$ we have

$$\Delta^+(\mathcal{E}) \otimes_{\mathfrak{o}_p} \check{\mathcal{O}}^+_{X^\diamond} = \mathcal{E}.$$

Let $\mathcal{M}$ be the (abelian) category of all $\check{\mathcal{O}}_{X^\diamond}$-modules of the form $E = \mathcal{E}[p^{-1}]$ for some $\mathcal{E}$ in $\mathcal{M}^+$.

Theorem 3 (Mann, W. 2020)

- Inverting p, we get a functor Δ from $\mathcal{M}$ to the category $\mathrm{ILoc}_{\mathbb{C}_p}(X)$ of $\mathbb{C}_p$-local systems with an integral model.
- The functor Δ is an equivalence of categories, compatible with direct sums, tensor products, internal homs, exterior powers and pullback.
- For every $E \in \mathcal{M}$ there is a natural isomorphism $\Delta(E) \otimes_{\mathbb{C}_p} \check{\mathcal{O}}_{X^\diamond} = E$

Theorem 4 (Mann, W. 2020)

Let $f\colon Y \to X$ be a surjective morphism of finite type proper adic spaces over $\mathbb{C}_p$ and assume that X is normal. Let E be an $\check{\mathcal{O}}_{X^\diamond}$-module. If $f^*E \in \mathcal{M}_{Y^\diamond}$, then $E \in \mathcal{M}_{X^\diamond}$. In other words, the property of having properly trivializable reduction descends along f.

For adic spaces associated to proper smooth algebraic curves, this generalizes the previously explained result for finite pullbacks.