Dwork's congruences and p-adic cohomology

Masha Vlasenko (IMPAN, Warsaw)

joint work with Frits Beukers
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Setup

$g(\mathbf{x}) = \sum_{\mathbf{u}} g_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$supp(g) = \{\mathbf{u} \in \mathbb{Z}^n : g_{\mathbf{u}} \neq 0\}$

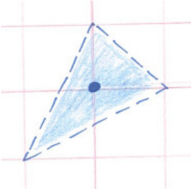
$\Delta \subset \mathbb{R}^n$ Newton polytope of g = convex hull of $supp(g)$

c_k = coefficient of $\mathbf{x}^{\mathbf{0}}$ (constant term) in $g(\mathbf{x})^k$, $k = 0, 1, 2, \ldots$

Example.

$$g(\mathbf{x}) = x_1 + x_2 + \frac{1}{x_1 x_2}$$

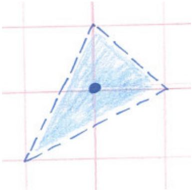
$$c_k = \begin{cases} 0, & 3 \nmid k \\ \frac{k!}{(k/3)!^3}, & 3 | k \end{cases}$$



Dwork's congruences

$$\gamma(t) = \sum_{k=0}^{\infty} c_k t^k \in \mathbb{Z}[\![t]\!], \qquad c_k = \text{constant term of } g(\mathbf{x})^k$$

$$\gamma_m(t) = \sum_{k=0}^{m-1} c_k t^k \qquad \text{truncations}$$



Theorem (Mellit-V, 2013). *Assume that $\mathbf{0} \in \Delta$ is the only internal integral point in the Newton polytope of $g(\mathbf{x})$. Then for any prime p and any integer $s \geq 1$*

$$\frac{\gamma(t)}{\gamma(t^p)} \equiv \frac{\gamma_{p^s}(t)}{\gamma_{p^{s-1}}(t^p)} \mod p^s.$$

$$s = 1: \quad \gamma(t) \equiv \gamma_p(t)\gamma(t^p) \mod p \equiv \gamma_p(t)\gamma_p(t^p)\gamma_p(t^{p^2})\ldots \mod p$$

$$\forall k = k_0 + k_1 p + k_2 p^2 + \ldots + k_m p^m, \quad 0 \leq k_i \leq p-1$$

$$c_k \equiv c_{k_0} c_{k_1} \ldots c_{k_m} \mod p \quad \text{(Lucas congruence)}$$

Towards a cohomological proof: initial observations

$$\gamma(t) := \sum_{k=0}^{\infty} c_k t^k \in \mathbb{Z}[\![t]\!], \qquad c_k = \text{ constant term of } g(\mathbf{x})^k$$

$$= \frac{1}{(2\pi i)^n} \oint \ldots \oint \frac{1}{1 - t\, g(\mathbf{x})} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}$$

$X = \mathbb{T}^n \setminus \{1 - t\, g(\mathbf{x}) = 0\}$ algebraic variety

$\omega = \frac{1}{1-tg(\mathbf{x})} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}$ differential n-form on X

de Rham cohomology

$H^n_{dR}(X) :=$ { closed differential n-forms $\omega : d\omega = 0$}
/ {exact differential n-forms: $\omega = d\eta$}

$\oint \ldots \oint : H^n_{dR}(X) \to \mathbb{C}$ period map

Cohomology of hypersurfaces

$f(\mathbf{x}) \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, R char 0 domain, e.g. $R = \mathbb{Z}$ or $R = \mathbb{Z}[t]$
$X_f = \{f(\mathbf{x}) = 0\} \subset \mathbb{T}^n$ toric hypersurface

$\Delta \subset \mathbb{R}^n$ Newton polytope of $f(\mathbf{x})$

$$\Omega_f^n := \left\{ (m-1)! \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} \;\middle|\; \begin{array}{l} m \geq 1, h \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \\ supp(h) \subset m\Delta \end{array} \right\}$$

$$\Omega_f^{n-1} := \left\{ \sum_{i=1}^{n} (m_i - 1)! \frac{h_i(\mathbf{x})}{f(\mathbf{x})^{m_i}} \frac{dx_1}{x_1} \ldots \frac{\check{dx_i}}{x_i} \ldots \frac{dx_n}{x_n} \;\middle|\; supp(h_i) \subset m_i\Delta \right\}$$

$R-$modules

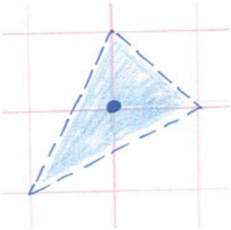
$\Omega_f^n / d\Omega_f^{n-1} \cong H_{dR}^n(\mathbb{T}^n \setminus X_f)$ (Griffiths–Batyrev)

$\cong R^d, \; d = n! \cdot vol(\Delta) + n$

under some conditions (f is Δ-*regular*),

and possibly after a localization of R

Example: $f(\mathbf{x}) = 1 - t(x_1 + x_2 + \frac{1}{x_1 x_2})$



$vol(\Delta) = \frac{3}{2}, \quad d = 2! \cdot \frac{3}{2} + 2 = 5$

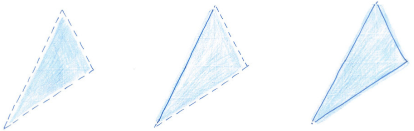
$R := \mathbb{Z}[t, \frac{1}{3t(1-27t^3)}] \rightsquigarrow \Omega_f^2 / d\Omega_f^1 \cong R^5$

Remark. We can cut out smaller modules

$\mu \subseteq \Delta$

$$\Omega_f^n(\mu) := \left\{ (m-1)! \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \;\middle|\; \begin{array}{l} m \geq 1, h \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \\ supp(h) \subset m\mu \end{array} \right\}$$

call $\mu \subseteq \Delta$ *open* if $\Delta \setminus \mu$ is a union of faces



e.g. in this example: $\Omega_f^2(\Delta^\circ)/d\Omega_f^1 \cong R^2$

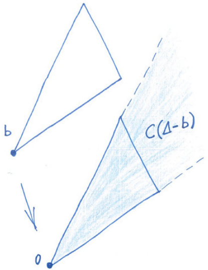
A tool: formal expansion

$f(\mathbf{x}) = \sum f_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$, fix a vertex $\mathbf{b} \in \Delta$

formal expansion of rational functions at $\mathbf{b}$:

$$\frac{h(\mathbf{x})}{f(\mathbf{x})^m} = \frac{h(\mathbf{x})}{f_{\mathbf{b}}^m \mathbf{x}^{m\mathbf{b}}(1+\ell(\mathbf{x}))^m}$$

$$= \frac{h(\mathbf{x})\mathbf{x}^{-m\mathbf{b}}}{f_{\mathbf{b}}^m} \sum_{s\geq 0} \binom{-m}{s} \ell(\mathbf{x})^s = \sum_{\mathbf{k}\in C(\Delta-\mathbf{b})} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$$



Example: $f(\mathbf{x}) = x_1 + x_2 + \frac{1}{x_1 x_2}$

$$\frac{1}{x_1 + x_2 + \frac{1}{x_1x_2}} = \frac{1}{\frac{1}{x_1x_2}(1 + x_1^2 x_2 + x_1 x_2^2)} = x_1 x_2 \sum_{k=0}^{\infty} (-1)^k (x_1^2 x_2 + x_1 x_2^2)^k$$

$$= x_1 x_2 - x_1^3 x_2^2 - x_1^2 x_2^3 + \ldots$$

$$\frac{1}{x_1 + x_2 + \frac{1}{x_1x_2}} = \frac{1}{x_2(1 + \frac{x_1}{x_2} + \frac{1}{x_1 x_2^2})} = \frac{1}{x_2} \sum_{k=0}^{\infty} (-1)^k (\frac{x_1}{x_2} + \frac{1}{x_1 x_2^2})^k$$

$$= \frac{1}{x_2} - \frac{x_1}{x_2^2} - \frac{1}{x_1 x_2^3} + \ldots$$

Cartier operator on differential forms

fix p prime

$$\mathcal{C}_p : \frac{h(\mathbf{x})}{f(\mathbf{x})^m}\frac{d\mathbf{x}}{\mathbf{x}} = \sum a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}\frac{d\mathbf{x}}{\mathbf{x}} \mapsto \sum a_{p\mathbf{u}}\mathbf{x}^{\mathbf{u}}\frac{d\mathbf{x}}{\mathbf{x}} \quad \notin \Omega_f^n$$

Def. A *Frobenius lift* $\sigma : R \to R$ is a ring endomorphism such that $\sigma(r) - r^p \in pR$ for all $r \in R$.

Examples:

- $R = \mathbb{Z}$ with $\sigma = id$
- $R = \mathbb{Z}[t]$ with $\sigma(r(t)) = r(t^p)$

Lemma. For $\frac{h(\mathbf{x})}{f(\mathbf{x})^m} = \sum a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$, the series $\sum a_{p\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ can be approximated p-adically by rational functions with powers of $f^\sigma(\mathbf{x})$ in the denominator:

$$\sum a_{p\mathbf{u}}\mathbf{x}^{\mathbf{u}} \equiv \frac{h_s(\mathbf{x})}{f^\sigma(\mathbf{x})^{m_s}} \mod p^s, \quad s = 1, 2, \ldots$$

$$\exists m_s, h_s(\mathbf{x}) \text{ with } supp(h_s) \in m_s\Delta$$

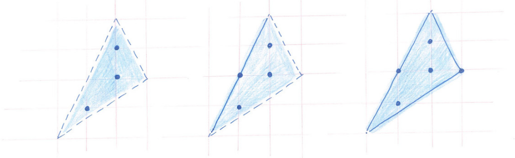
Properties of the Cartier operator

$$\mathcal{C}_p : \sum a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \frac{d\mathbf{x}}{\mathbf{x}} \mapsto \sum a_{p\mathbf{u}} \mathbf{x}^{\mathbf{u}} \frac{d\mathbf{x}}{\mathbf{x}}$$

- $\mathcal{C}_p(\Omega_f^n) \subset \widehat{\Omega}_{f^\sigma}^n$, p-adic completion of $\Omega_{f^\sigma}^n$
- moreover, the map $\mathcal{C}_p : \widehat{\Omega}_f^n \to \widehat{\Omega}_{f^\sigma}^n$ is independent of the choice of vertex $\mathbf{b} \in \Delta$ at which the formal expansion is done
- $\mathcal{C}_p$ maps exact forms into exact forms
  $\mathcal{C}_p \circ x_i \frac{\partial}{\partial x_i} = p\, x_i \frac{\partial}{\partial x_i} \circ \mathcal{C}_p \Rightarrow \quad \mathcal{C}_p(d\widehat{\Omega}_f^{n-1}) \subset d\widehat{\Omega}_{f^\sigma}^{n-1}$
- when $R = \mathbb{Z}_p$, trace of $\mathcal{C}_p^s$ counts points on $\mathbb{T}^n \setminus X_f$ over $\mathbb{F}_{p^s}$ for $s \geq 1$
- $\mathcal{C}_p : \widehat{\Omega}_f^n(\mu) \to \widehat{\Omega}_{f^\sigma}^n(\mu)$ when μ is *open*

Key theorem

$\mu_{\mathbb{Z}} := \mu \cap \mathbb{Z}^n$ integral points in μ



Hasse–Witt matrix $\beta_p = \beta_p(\mu) \in R^{h \times h}$, $h = \#\mu_{\mathbb{Z}}$

$$(\beta_p)_{\mathbf{u},\mathbf{v} \in \mu_{\mathbb{Z}}} = \text{ coefficient of } \mathbf{x}^{p\mathbf{v}-\mathbf{u}} \text{ in } f(\mathbf{x})^{p-1}$$

Theorem (Beukers-V, 2019). Assume R is p-adically complete, $\mu \subset \Delta$ is *open* and the Hasse–Witt matrix $\beta_p(\mu)$ is invertible. Then

$$\widehat{\Omega}_f^n(\mu)/\{\text{formally exact forms}\}$$

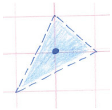
is a free R-module of rank h where

$$\omega_{\mathbf{u}} := \frac{\mathbf{x}^{\mathbf{u}}}{f(\mathbf{x})}\frac{d\mathbf{x}}{\mathbf{x}}, \qquad \mathbf{u} \in \mu_{\mathbb{Z}}$$

is a basis.

An application: Dwork's congruences

Let $g(\mathbf{x}) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be such that $\mathbf{0} \in \mathbb{Z}^n$ is the only internal integral point of its Newton polytope Δ.
$\gamma(t) = \sum_{k=0}^{\infty} c_k t^k, \quad c_k = \text{const. term of } g(\mathbf{x})^k$



Take $f(\mathbf{x}) = 1 - t\,g(\mathbf{x}),\ \mu = \Delta^\circ$.
Then $\mu_{\mathbb{Z}} = \{\mathbf{0}\}$, the 1×1 Hasse–Witt matrix is

$$\beta_p(t) = \text{ const. term of } (1 - tg(\mathbf{x}))^{p-1} = \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} c_k t^k.$$

Take $R := \mathbb{Z}[t, \beta_p(t)^{-1}]^{\widehat{}}$. Take $\sigma(t) = t^p$.

By our theorem, there exists an element $\Lambda \in R$ such that

$$\mathcal{C}_p \frac{1}{1 - tg(\mathbf{x})} \frac{d\mathbf{x}}{\mathbf{x}} = \Lambda \frac{1}{1 - t^p g(\mathbf{x})} \frac{d\mathbf{x}}{\mathbf{x}} + \text{ formally exact form}$$

The residue map

$$\omega \mapsto \frac{1}{(2\pi i)^n} \oint \omega =: Res_{\mathbf{0}}(\omega)$$

is R-linear, vanishes on exact (and formally exact) forms and is $\mathcal{C}_p$-invariant. We have

$$\mathcal{C}_p \frac{1}{1 - tg(\mathbf{x})} \frac{d\mathbf{x}}{\mathbf{x}} = \Lambda \frac{1}{1 - t^p g(\mathbf{x})} \frac{d\mathbf{x}}{\mathbf{x}} + \text{ formally exact form}$$

$$Res_{\mathbf{0}}(\frac{1}{1 - tg(\mathbf{x})} \frac{d\mathbf{x}}{\mathbf{x}}) = \sum_{m \geq 0} t^m Res_{\mathbf{0}}(g(\mathbf{x})^m \frac{d\mathbf{x}}{\mathbf{x}}) = \gamma(t)$$

$$Res_{\mathbf{0}}(\frac{1}{1 - t^p g(\mathbf{x})} \frac{d\mathbf{x}}{\mathbf{x}}) = \sum_{m \geq 0} t^{pm} Res_{\mathbf{0}}(g(\mathbf{x})^m \frac{d\mathbf{x}}{\mathbf{x}}) = \gamma(t^p)$$

$$\Rightarrow \Lambda = \frac{\gamma(t)}{\gamma(t^p)} \in R.$$

Recall: $R = \mathbb{Z}[t, \beta_p(t)^{-1}]^{\frown}$, $\quad \beta_p(t) = \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} c_k t^k$.

Dwork's congruences ?

$$\frac{\gamma(t)}{\gamma(t^p)} \equiv \frac{\gamma_{p^s}(t)}{\gamma_{p^{s-1}}(t^p)} \mod p^s$$

A tool: period maps modulo m

For any $h(\mathbf{x}) \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $m \geq 1$ the maps

$$\rho_{m,h} : \omega \mapsto Res_{\mathbf{0}}\left(h(\mathbf{x})^m \omega\right)$$

satisfy

- $\rho_{m,h}(d\eta) = Res_{\mathbf{0}}(h(\mathbf{x})^m d\eta) \equiv Res_{\mathbf{0}}(d(h(\mathbf{x})^m \eta)) \equiv 0 \mod m$
- $\rho_{m,h} \equiv \rho_{m/p,\, h^{\sigma}} \circ \mathcal{C}_p \mod p^{ord_p(m)}$

In our application: $f(\mathbf{x}) = 1 - tg(\mathbf{x})$, and we consider

$$\gamma_m := \rho_{m,1}\left(\frac{1}{1-tg(\mathbf{x})}\frac{d\mathbf{x}}{\mathbf{x}}\right) - \rho_{m,tg(\mathbf{x})}\left(\frac{1}{1-tg(\mathbf{x})}\frac{d\mathbf{x}}{\mathbf{x}}\right)$$

$$= Res_{\mathbf{0}}\left(\frac{1-t^m g(\mathbf{x})^m}{1-tg(\mathbf{x})}\frac{d\mathbf{x}}{\mathbf{x}}\right) = \sum_{k=0}^{m-1} t^k Res_{\mathbf{0}}(g(\mathbf{x})^m \frac{d\mathbf{x}}{\mathbf{x}}) = \sum_{k=0}^{m-1} c_k t^k.$$

$$\rightsquigarrow \quad \Lambda \equiv \frac{\gamma_m(t)}{\gamma_{m/p}(t^p)} \mod p^{ord_p(m)}$$

$$\frac{\gamma(t)}{\gamma(t^p)} \equiv \frac{\gamma_m(t)}{\gamma_{m/p}(t^p)} \mod p^{ord_p(m)}$$

Thank you!